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A NOTE ON A FORMAL MANIPULATION OF DIVERGENT SERIES AND INTEGRALS

1. Preliminary comments

The problem of manipulation of divergent series has been considered by almost all 19th century analysists. Even now it remains a subject of considerable interest. As a sample of recent papers see [1], [2] or [3].

It is surprising that the new techniques of model theory and non-standard analysis (as outlined in [4], [5] and [6]) have not been applied to this problem up to the present time, despite an almost natural fit of the Robinson-Luxemburg ultraproduct approach to the representation of any infinite series in a non-standard model $^*\mathbf{R}$ of the real line $\mathbf{R}$. In the present note only elementary extensions of $\mathbf{R}$, derived by the ultraproduct construction, of non-standard saturated models of $\mathbf{R}$ are considered (since these are isomorphic to the ultrapower models). (See [8]).

Each such model $^*\mathbf{R}$ posesses infinitessimal elements and contains a copy of $\mathbf{R}$ isomorphically imbeded.

This note intends to point out that certain purely formal manipulations lead to results which are valid in each such non-standard model.

2. The $\mathbf{R}$-uniqueness results for infinite series

Only real series will be considered in this note for the sake of simplicity.

With each infinite series of real numbers $\sum_{i=1}^{\infty} C_i$ we associate the corresponding sequence of partial sums $\{S_k\}$, where $S_m = \sum_{i=1}^{m} C_i$. Each (infinite) sequence is regarded in turn as an element of the ultraproduct $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots$, which has been given a ring structure. Let U denote a free ultrafilter. Following the usual ultraproduct arguments (see [4]) and applying the theorem of Łoś, we associate with $\{S_k\}/U$ a unique element $r_k \in {}^*\mathbb{R}$ where ${}^*\mathbb{R}_U$ is a non-standard model of $\mathbb{R}$, (depending on U) that is with each equivalence class of sequences, which are in effect the cosets of U we associate a unique element of ${}^*\mathbb{R}_U$.

We denote this correspondence by a map $\{s_k\} \xrightarrow{U} r_k$ $(\in {}^*\mathbb{R}_U)$. It follows from the ring homomorphism theorem of intermediate algebra (see for example Birkhoff and MacLane [7]) that all elementary operations performed on sequences of partial sums are valid for the corresponding elements of ${}^*\mathbb{R}$, if U (and therefore ${}^*\mathbb{R}$) are regarded as fixed.

For example if $\{S_i\} \xrightarrow{U} r_i$ and $\{S_j\} \xrightarrow{U} r_j$, and $r_j \neq 0 (\in {}^*\mathbb{R})$, then $\{S_i\} \odot \{S_j\} \xrightarrow{U} r_i r_j$, and $\left(\frac{\{S_i\}}{\{S_j\}}\right) \xrightarrow{U} \frac{r_i}{r_j}$ $(\in {}^*\mathbb{R})$, where $\odot$ is the coordinate or pointwise multiplication of sequences, and $\frac{\{S_i\}}{\{S_j\}}$ denotes the sequence obtained by pointwise division of sequences with zero elements of the sequence $\{S_j\}$ replaced by an arbitrary non-zero number. The numbers $r_i, r_j \in {}^*\mathbb{R}$ generally depend on the choice of the ultrafilter U. However it may be true (in some trivial cases) that for some internal function $f : {}^*\mathbb{R} \times {}^*\mathbb{R} \to {}^*\mathbb{R}$, $f(r_i, r_j) = r \in {}^*\mathbb{R}$ and r is unique and independent of U. Or it may be true that r does depend on the choice of the ultrafilter U, but it is near standard, and Std(r) is the same number (in $\mathbb{R}$) for every choice of an ultrafilter U.

D e f i n i t i o n A. Let U be any ultrafilter, $\{S_1\}, \{S_2\} \ldots \{S_n\}$ an n-tuple of sequences $(N \to \mathbb{R})$ and $r_1, r_2, \ldots r_n$ the corresponding numbers in ${}^*\mathbb{R}_U$. Let

$f(x_1, x_2, \ldots, x_n)$ be a standard function $f : \mathbf{R}^n \to \mathbf{R}$. Interpreting f as a function from ${}^*\mathbf{R}^n_U \to {}^*\mathbf{R}_U$, let $r = f(r_1, r_2, \ldots, r_n)$, suppose that $f(r_1, r_2, \ldots, r_n) = r \in {}^*\mathbf{R}_U$ is not necessarily unique (that is, it does depend on the choice of the ultrafilter U), but r is near-standard and $\mathrm{Std}(r) = r_0 \in \mathbf{R}$ does not depend on the choice of U. We shall say that $f(S_1, S_2, \ldots, S_n)$ has an ultrafilter-independent representation in $\mathbf{R}$.

The remainder of this note points out that certain formal manipulations in classical analysis have in fact an ultrafilter-independent representation $\mathbf{R}$.
For example if $\{S^j_k\} \xrightarrow{U} r_j$

$$\{S^i_k\} \xrightarrow{U} r_i$$

then

$$\{S^i_k\} \cdot \{S^j_k\} \xrightarrow{U} r_i r_j,$$

where $\cdot$ denotes coordinate (or pointwise) multiplication of sequences. Clearly the numbers $r_i, r_j \in {}^*\mathbf{R}$ depend on the choice of the ultrafilter U. However, if it can be shown that a formula

$$\Phi(S^1_k, S^2_k, \ldots, S^m_k)$$

can be interpreted in ${}^*\mathbf{R}$ independently of the choice of the ultrafilter U, then by Łoś' theorem, this interpretation (if it can be restated in $\mathbf{R}$) gives a unique result in $\mathbf{R}$.

As an example of an application we consider the $(-\log(1-x))$ series: $\sum_{n=1}^{\infty} \frac{x^n}{n}$ which is valid in $\mathbf{R}$ if $x < 1$. For $x > 1$ the divergent series $\sum^{\infty} \frac{x^n}{n}$ is replaced by the corresponding sequence of partial sums $\{S_k(x)\} = \left\{\sum_{n=1}^{k} \frac{x^n}{n}\right\}$. Let U be a fixed ultrafilter, $S_k(x) \xrightarrow{U} r(x)$ the corresponding map into $^*\mathbb{R}$. Denoting by $\left\{e^{S_k(x)}\right\}$ the corresponding exponential sequence i.e.

$$\left\{e^{S_k(x)}\right\} = \left\{e^{\sum_1^k \frac{x^n}{n}}\right\} = \left\{\sum_{n=1}^{k} e^{\frac{x^n}{n}}\right\},$$

we check the corresponding element of $^*\mathbb{R}$ is given by

$$\left\{e^{S_k(x)}\right\} \underset{\rightarrow}{U} e^{r(x)} \qquad (x > 1).$$

Since $r(x) > 0$, $\left\{S_k(x)\right\} > 0 \; \forall \; k = 1,2,\ldots)$, the following formula obtained in $^*\mathbb{R}$

$$\log\left(e^{r(x)}\right) = r(x). \tag{a}$$

However the formula (a) can be derived by considering the sequence of partial sums, and the corresponding map "U"

$$\left\{\log\, e^{S_k(x)}\right\} = \log\left\{\prod_{n=1}^{k} e^{\frac{x^n}{n}}\right\} = \left\{S_k(x)\right\} \underset{\rightarrow}{U} r(x),$$

for any choice or an ultrafilter U. Now it follows easily from similar arguments that the following formulas are unique in $\mathbb{R}$.

$$\log\; e^{-\log(1-x)} = -\log\,(1-x),$$

$$\log\, e^{\log(1-x)} = \log(1-x),$$

and

$$e^{\log(1-x)} = (1-x) \;\forall\; x \in \mathbb{R},$$

despite the fact that $\log(1-x)$ is undefined for $x \geq 1$.

No such claim can be made for any summability scheme for divergent series such as $\sum_{n=1}^{\infty} (-1)^n$. Clearly the non-standard number r, derived as before,

$$\{S_k\} = \{\sum_{n=1}^{k} (-1)^n\} \underset{\to}{U} r$$

will depend on the choice of the ultrafilter U, and attempting to interpret Std(r) in $\mathbf{R}$ independently of U is not possible.

Let the symbol log (1-x) stand for the formal series $\sum_{n=1}^{\infty} \frac{x^n}{n}$, whether the series converges, or not. Then

$$\left(\frac{\log(1-x_1)}{\log(1-x_2)}\right) = 0,$$

for all $x_1 < x_2$, is an ultrafilter independent representation (in $\mathbf{R}$). This follows immediately from an observation that if k is an infinite integer, then

$$A = \left[\left(\sum_{n=1}^{k} \frac{x_1^n}{n}\right)\Big/\left(\sum_{n=1}^{k} \frac{x_2^n}{n}\right)\right]^{-1}$$

is an infinite integer in any ${}^*\mathbf{R}_U$, independently of the choice of the ultrafilter U.

3. A treatment of divergent integrals

The definition given above can be extended to higher order nonstandard models if necessary. Considering for example a divergent integral $\int_a^{\infty} f(x)\,dx$, we replace it by a truncated integral $\int_a^{m} f(x)\,dx$ (in $\mathbf{R}$). In turn the truncated integral is considered to be a pseudo-finite sum (in ${}^*\mathbf{R}$). $\int_a^{m} f(x)\,dx \approx$

$\approx \sum_{i=1}^{n} f(x_i)\Delta x_i$, $\Delta x_i \approx 0$, where $a \approx b$ implies that $|a - b|$ is an infinitesimal.

Let M be an infinite number in $^*\mathbf{R}$. Then there exists Δx_i (sufficiently small) such that $I_M = \int_a^m f(x)\, dx \approx$ $\approx \sum_{i=1}^{N} f(x_i)\Delta x_i$, assigning to each infinite integer M a number I_M m $^*\mathbf{R}$. Considering the sequence $\{I_M\} \in {*R} \times {*R} \times \ldots$, and factoring out an ultrafilter U, we obtain a unique element of $^{**}\mathbf{R}$. Since $\mathbf{R}$ is embedded isomorphically in $^{**}\mathbf{R}$, we may be able to define a corresponding ultrafilter independent representation (in $\mathbf{R}$).

We indicate a possible application of this observation.

Consider the Fourier law of heat transfer or the law of diffusion:

$$\frac{\partial u}{\partial t} = D(u)\, \frac{\partial^2 u}{\partial x^2}\ . \tag{a}$$

This equation arises from well known heuristic steps concerning random walk property of diffision (or heat transfer) which first leads to a differential-difference equation

$$W_t(x,t)\Delta t + O(\Delta t)^2 - \frac{1}{2}\, W_{xx}(x,t)(\Delta x)^2 = O(\Delta x)^3.$$

Assuming that

$$\frac{(\Delta x)^2}{2\Delta t} \approx D(u), \qquad 2u\Delta x \approx W,$$

we can derive the equation (a) as the continuous version of the random walk, with $u(x,t)$ being the probability density function for the random walk process. (See for example [9]). In terms of the random walk process the probability of finding a particle in the interval $n_1\Delta x \le x \le n_2\Delta x$ is given by

$$2 \sum_{i=n_1}^{i=n_2} (u(i\Delta x,t) \cdot \Delta x)\ .$$

Suppose the strength of sources is $f(x_i)$ distributed at points $x_1, x_2, \dots, x_N$. Then, in the diffusion process, the total mass M at time t and at a point x is given by

$$M(x,t) = \sum_{i=-k}^{+k} u_0(x - \xi_i, t)\, f(\xi_i)\Delta\xi_i .$$

Suppose $f(x)$ is in turn given by $f(x) = \int_{-\infty}^{+\infty} K(x,\xi)\Phi(\xi)\, d\xi$. It is clear that $f(x_i)$ does not have to be finite, i.e. an element of $^*\mathbf{R}_{bd}$ (or $^{**}\mathbf{R}_{bd}$) in order for $M(x,t)$ to be finite (i.e. near-standard, and Std $M(x,t)$ can be regarded as an $\mathbf{R}$ ultrafilter independent solution of the integral equation, despite the fact that the corresponding integrals diverge.

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