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Tadeusz Gerstenkorn

MOMENT RECURRENCE RELATIONS FOR THE INFLATED NEGATIVE BINOMIAL, POISSON AND GEOMETRIC DISTRIBUTIONS

Introduction

In Section 1 of this paper we have dealt with the so-called discrete inflated distributions and shortly presented the results obtained for them in the field of estimating the parameters, of moments and the distribution of the sums of random variables. In Section 2 we have given a few theorems concerning recurrence relations for the moments about the origin and about the mean of the negative binomial distribution. The correspondent relations for the moments of the Poisson and geometric distributions have been shown in the form of corollaries in Section 3.

1. Discrete inflated distributions

Let us denote by X a discrete random variable with a distribution $P(X = k;\theta)$ (θ - a certain parameter) and by Y a variable with the inflated distribution $P_1(Y = k;\theta)$ defined as follows:

$$(1.1)\qquad P_1(Y = k;\theta) = \begin{cases} \beta + \alpha P(X = 0;\theta) & \text{for } k = 0, \\ \alpha P(X = k;\theta) & \text{for } k \neq 0, \end{cases}$$

where $\alpha \in (0,1]$ and $\beta = 1-\alpha$.

The probability distribution of this form was introduced into statistical studies by S.N.Singh [18] in 1963 in the case of the function $P(X = k;\theta)$ being the Poisson law. The distribution was to serve as the probabilistic description of such experiments that were virtually well described by the above-mentioned law, with a slight inflation, however, of the probability at zero point.

In the paper of 1965/66 M.P.Singh [17] returned to this problem having taken the binomial distribution to be a basis of his investigations. He pointed out that there exist such situations that can be described by the binomial distribution almost well, i.e., one can however perceive some deformation of theoretically expected values of frequency, consisting in a distinct increase of the frequency of the observed event at zero point as well as a respective decrease of its value at the remaining points. The object for this author's studies was a population of four-person families. The set of the families considered was a mixture in the sense that some of them were in the environment exposed to the danger of being afflicted with a certain disease, while other families were not. The quantity of sick individuals in each family of the population considered was examined and the results recorded. The binomial distribution was adopted as a statistical model of the population and on the ground of this model there were calculated theoretically expected frequencies of the occurrence of values 0,1,2,3,4, that is, of the numbers representing all individuals in a family being healthy (no person sick), one person sick, etc. The results of the observations had been compared with the theoretical frequencies and it turned out that the frequency of the occurrence of zero was actually a little greater than one could expect on the ground of the model adopted. The case described above was after all quite well modelled by (1.1) if $P(X = k)$ constituted the very binomial distribution. It was due to the fact that distribution (1.1) is a special case of the mixture of distributions $P_1(X = k)$ and $P_2(X = k)$, i.e., the distribution given by the formula

$$P(X=k) = \beta P_1(X=k) + \alpha P_2(X=k),$$

where P_1 is the so-called degenerate distribution

$$P_1(X = k) = \begin{cases} 1 & \text{when } k = 0, \\ 0 & \text{when } k \neq 0 \end{cases}$$

and P_2 - an arbitrary discrete distribution.

The authors in question did not analyze this fact but it is completely evident if one takes into account that the coefficient of inflation α under consideration represents, in reality, a share (fraction) of the distribution P_2 in the mixture. Therefore, the studies of the inflated distributions are a particular case of the researches of the mixed distributions and hence their great importance in statistical problems.

Soon after the afore-quoted paper M.P.Singh published in 1966 a next one [16] in which he generalized his former considerations for the case of inflation at an arbitrary point of the binomial distribution. The proposals of such generalizations were presented a little earlier (1964/65) by K.N.Pandey [12] for the case of the Poisson distribution.

Generally speaking, we may say that the random variable Y has the so-called generalized discrete inflated distribution with parameter θ if its probability function is expressed by the formula

$$(1.2) \qquad P_1(Y=k;\theta) = \begin{cases} \beta + \alpha P(X=1;\theta) & \text{for } k = 1 \\ \alpha P(X=k;\theta) & \text{for } k \neq 1, \end{cases}$$

where $\alpha \in (0,1]$ and $\beta = 1-\alpha$, and $P(X=k;\theta)$ is a distribution of the random variable X.

So, the generalized inflated distribution is such a mixture of the afore-mentioned distributions P_1 and P_2 in which the former is a degenerate distribution of the variable X at the point $k = 1$.

The studies of the Indian investigators quoted here had respect to the questions of estimating the coefficient α and to those of estimating parameters of the binomial and Poisson distributions under examination (the parameters p and λ, respectively).

The problem of the moments for the inflated distributions was taken up by L.Grzegórska (Sobich) in paper [9] of 1973. She occupied herself with the question of calculating the simple and the central moments for the class of power series distributions (PSD), including to her considerations the case of the truncated distribution as well.

In paper [19] published a year later the same authoress dealt with the establishment of the recurrence relations for the moments of the inflated binomial and Poisson distribution.

The present writer in [5] has reported various problems connected with the inflated distributions.

In paper [6] one can find the recurrence relations for the moments about an arbitrary point of a class of the discrete inflated distributions.

In the studies concerning the inflated distributions there has also been taken into account a problem of the distribution of the sum of random variables with inflated distributions. There is a reference to it in papers [8] and [20]. In the former the binomial variables have been treated of, while in the latter - variables having a distribution of type PSD.

2. Moments of the inflated negative binomial distribution

D e f i n i t i o n . The random variable Y is said to obey a generalized inflated negative binomial distribution if its probability function is expressed by the formula

$$(2.1)\quad P(Y=k) = \begin{cases} \beta+\alpha(-1)^k\binom{-n}{k}p^kq^n & \text{for } k = l \\ \alpha(-1)^k\binom{-n}{k}p^kq^n & \text{for } k = 0,1,\ldots,l-1,l+1,\ldots, \end{cases}$$

where $0 < \alpha \leq 1$, $\alpha+\beta = 1$, $0 < p < 1$, $p+q = 1$.

If $\alpha = 1$, the above distribution is reduced to the negative binomial one without inflation.

If, in formula (2.1), we accept n equal to 1, then we get the so-called geometric inflated distributions

$$(2.2)\qquad P(Y=k) = \begin{cases} \beta + p^k q & \text{for } k = l \\ \alpha p^k q & \text{for } k = 0,1,\dots,l-1,l+1,\dots \end{cases}$$

or in the case of $\alpha = 1$ - the one without inflation.

We shall derive the recurrence relations for the simple and the central moments of the inflated negative binomial distribution and give, in particular, formulae for the inflated geometric and Poisson distributions. To prove, we shall make use of the following, almost obvious, lemma.

L e m m a . Let $\varphi_x(t)$ and $\varphi_y(t)$ denote characteristic functions of the random variables X and Y with and without inflation, respectively. Then

$$(2.3)\qquad \varphi_y(t) = \beta e^{itl} + \alpha\varphi_x(t).$$

T h e o r e m 1. If Y is a random variable with inflated negative binomial distribution (2.1), then the simple moments $\bar{m}_r$ of this distribution satisfy the recurrence relation

$$(2.4)\qquad \bar{m}_r = \frac{1}{q}\Big\{\beta l^r - \beta p(l+n)(l+1)^{r-1} + p\sum_{k=0}^{r-1}\Big[n\binom{r-1}{k} + \binom{r-1}{k+1}\Big]\bar{m}_{r-1-k}\Big\},\quad r = 1,2,\dots$$

P r o o f . On the ground of formula (2.3) of the lemma and the well-known form of the characteristic function of the negative binomial distribution ([3], p.179, (5.13.9), German ed., p.151) we have

$$(2.5)\qquad \varphi_y(t) = \beta e^{itl} + \alpha q^n(1-pe^{it})^{-n}.$$

The characteristic function of the random variable possessing the moments of an arbitrary order can be expanded in the MacLaurin series

$$\varphi_y(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \bar{m}_j,$$

where $\bar{m}_j$ is the jth simple moment of the random variable with inflated distribution. Thus putting $it = \theta$, we obtain the equality

$$\sum_{j=0}^{\infty} \frac{\theta^j}{j!} \bar{m}_j = \beta e^{\theta l} + \alpha q^n (1-pe^{\theta})^{-n}. \tag{2.6}$$

We differentiate both sides of (2.6) with respect to θ. After some suitable calculations we get

$$(1-pe^{\theta}) \sum_{j=1}^{\infty} \frac{\theta^{j-1}}{(j-1)!} \bar{m}_j = (1-pe^{\theta}) \beta l e^{\theta l} + \tag{2.7}$$

$$+ npe^{\theta} \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \bar{m}_j - \beta npe^{\theta(l+1)}.$$

Paying respect to the transformation

$$1 - pe^{\theta} = -p(e^{\theta}-1) + q$$

and the possibility of representing e^{θ} in the form of a series for all θ, i.e., taking into account the formula

$$1 - pe^{\theta} = q - p \sum_{k=1}^{\infty} \frac{\theta^k}{k!}, \tag{2.8}$$

we shall have for the left side of (2.7):

$$-p \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\theta^{k+j-1}}{k!(j-1)!} \bar{m}_j + q \sum_{j=1}^{\infty} \frac{\theta^{j-1}}{(j-1)!} \bar{m}_j, \tag{2.9}$$

while the right side of (2.7), for the convenience of further transformations, will be written as follows

$$\beta l e^{\theta l} - \beta p(l+n) e^{\theta(l+1)} = npe^{\theta} \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \bar{m}_j.$$

After having expanded $e^{\theta(l+1)}$ and $e^{\theta}-1$ in a series, we then obtain:

$$(2.10) \qquad \beta l \sum_{k=0}^{\infty} \frac{\theta^k l^k}{k!} - \beta p(l+n) \sum_{k=0}^{\infty} \frac{\theta^k (l+1)^k}{k!} + $$

$$+ np \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\theta^{j+k}}{k!j!} \bar{m}_j.$$

By combining formulae (2.9) and (2.10) obtained, we get

$$(2.11) \qquad -p \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\theta^{k+j-1}}{k!(j-1)!} \bar{m}_j + q \sum_{j=1}^{\infty} \frac{\theta^{j-1}}{(j-1)!} \bar{m}_j =$$

$$= \beta l \sum_{k=0}^{\infty} \frac{\theta^k l^k}{k!} - \beta p(l+n) \sum_{k=0}^{\infty} \frac{\theta^k (l+1)^k}{k!} +$$

$$+ np \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\theta^{k+j}}{k!j!} \bar{m}_j.$$

Successive addends appearing in relation (2.11) will be denoted by: $-a$, qb, c, $-d$, e. We shall then write this relation down briefly as

$$b = \frac{1}{q} (c - d + e + a).$$

Considering the coefficients at θ^{r-1} obtained in this relation for b, c, d, e, a, we may write

$$\frac{\bar{m}_r}{(r-1)!} = \frac{1}{q}\left\{-\frac{\beta p(1+n)(1+1)^{r-1}}{(r-1)!} + \frac{\beta 1^r}{(r-1)!} + \right.$$

$$\left. + \frac{p}{(r-1)!}\sum_{k=0}^{r-1}\left[n\binom{r-1}{k} + \binom{r-1}{k+1}\right]\bar{m}_{r-1-k}\right\},$$

from which we immediately get formula (2.4) representing the proposition.

Let us calculate a few simple moments, according to recurrence relation (2.4) for inflated negative binomial distribution (2.1)

$$(2.12)\quad \begin{cases} \bar{m}_1 = \beta 1 + \alpha n \frac{p}{q}, \\ \bar{m}_2 = \beta 1^2 + \alpha n \frac{p}{q}\left[(n+1)\frac{p}{q} + 1\right] = \beta 1^2 + \alpha n \frac{p}{q^2}(np+1), \\ \bar{m}_3 = \beta 1^3 + \alpha n \frac{p}{q}\left[(n^2 + 3n + 2)\left(\frac{p}{q}\right)^2 + 3(n+1)\frac{p}{q} + 1\right], \\ \bar{m}_4 = \beta 1^4 + \alpha n \frac{p}{q}\left[(n^3 + 6n^2 + 11n + 6)\left(\frac{p}{q}\right)^3 + \right. \\ \qquad \left. + 6(n^2 + 3n + 2)\left(\frac{p}{q}\right)^2 + 7(n+1)\frac{p}{q} + 1\right]. \end{cases}$$

At the same time, the given examples provide an illustration of the application of the simple formula

$$(2.13)\qquad \bar{m}_r = \beta 1^r + \alpha m_r, \quad r = 1,2,\ldots$$

which determines the relationship between the simple moments of the inflated distribution and the one without inflation. The moments m_r can be attained by putting $\beta = 0$ in formula (2.4). We then get the recurrence relation offered, for the first time, by R.Risser and C.E.Traynard in [14] (p.323, 2nd ed. p.94). The same relation has been achieved by the present writer as a result of considerations on the particular cases of the so-called power series distributions ([7], p.21, (2.8)).

We shall show one more recurrence relation concerning the simple moments of the inflated negative binomial distribution.

Theorem 2. If the random variable Y has inflated negative binomial distribution (2.1), then the following recurrence relation holds

$$\bar{m}_{r+1} = \beta 1^r(1-m) + n\bar{m}_r + p\frac{d\bar{m}_r}{dp}, \quad r = 0,1,2,\ldots, \tag{2.14}$$

where $m = n\frac{p}{q}$ denotes the first simple moment of the negative binomial distribution without inflation.

Proof. In virtue of formula (2.13) we will have for (2.1)

$$\bar{m}_r = \beta 1^r + \alpha \sum_{k=0}^{\infty} (-1)^k k^r \binom{-n}{k} p^k q^n. \tag{2.15}$$

By differentiating both sides of (2.15) with respect to p, we shall obtain, after some appropriate calculations,

$$\frac{d\bar{m}_r}{dp} = \frac{\alpha}{p}\sum_{k=0}^{\infty} k^{r+1} (-1)^k \binom{-n}{k} p^k q^n -$$
$$- \alpha\frac{n}{q}\sum_{k=0}^{\infty} k^r (-1)^k \binom{-n}{k} p^k q^n.$$

Adding up and subtracting $\frac{\beta 1^{r+1}}{p}$ and $n\beta\frac{1^r}{q}$, we have

$$\frac{d\bar{m}_r}{dp} = \frac{1}{p}\left[\beta 1^{r+1} + \alpha\sum_{k=0}^{\infty} k^{r+1}(-1)^k\binom{-n}{k}p^k q^n\right] -$$
$$- \frac{n}{q}\left[\beta 1^r + \alpha\sum_{k=0}^{\infty} k^r(-1)^k\binom{-n}{k}p^k q^n\right] +$$
$$+ \beta 1^r\left(\frac{n}{q} - \frac{1}{p}\right) = \frac{1}{p}\bar{m}_{r+1} - \frac{n}{q}\bar{m}_r + \beta 1^r\frac{np - 1q}{pq},$$

whence, after some simple transformations, we get formula (2.14) representing the proposition.

The technique of calculating the moments according to (2.14) is simple. They result in exactly the same formulae for consecutive moments as those mentioned above. We disregard full particulars of the calculations.

Formula (2.14) with $\beta = 0$ gives the formula for the moments of the negative binomial distribution without inflation

$$(2.16) \qquad m_{r+1} = mm_{r+1} + p\frac{dm_r}{dp}, \qquad r = 1,2,3,\ldots$$

almost identical in its structure with that offered by A.R. Crathorne [2] (p.1202, (3)) for the binomial distribution. The method of proving the formula used by this author was completely different. It was based on the exploitation of a certain recurrence relation given by R.Frisch [4] for the cumulants of the very distribution. (The discussion of this matter can be found in [7], pp.9-10). The calculation of moments by means of (2.16) seems to be simpler than by applying the characteristic function as it happens, for example, in [3] (p.179, German ed., p.151).

We now proceed to the recurrence relations concerning the central moments of the inflated negative binomial distribution.

T h e o r e m 3. If the random variable Y has inflated negative binomial distribution (2.1), then there arises a recurrence relation for the central moments $\bar{\mu}_r$ of this distribution

$$(2.17) \qquad \bar{\mu}_{r+1} = \alpha^r\beta(1-m)^{r+1} - \beta(1-m)\bar{\mu}_r + p\left(\frac{d\bar{\mu}_r}{dp} + \frac{\alpha n r\bar{\mu}_{r-1}}{q^2}\right), \qquad r = 1,2,\ldots,$$

where $m = n\frac{p}{q}$ denotes the first moment (expected value) of the negative binomial distribution without inflation.

P r o o f . From the definition of the central moments of the rth order we have:

$$(2.18)\quad \bar{\mu}_r = \beta(1-\bar{m}_1)^r + \alpha \sum_{k=0}^{\infty} (k-\bar{m}_1)^r (-1)^k \binom{-n}{k} p^k (1-p)^n.$$

By substituting (2.12) in (2.18), we obtain

$$(2.19)\quad \bar{\mu}_r = \beta\alpha^r \left(1 - n \frac{p}{1-p}\right)^r +$$

$$+ \alpha \sum_{k=0}^{\infty} \left(k-\beta 1-\alpha n \frac{p}{1-p}\right)^r (-1)^k \binom{-n}{k} p^k (1-p)^n.$$

By differentiating the above formula with respect to p, we get

$$\frac{d\bar{\mu}_r}{dp} = -\alpha \frac{nr}{q^2} \bar{\mu}_{r-1} + \frac{\alpha}{pq} \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} (k-\beta 1-\alpha m)^r p^k q^n (kq-np).$$

The second addend of this formula is transformed into

$$\frac{\alpha}{p} \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} k(k-\beta 1-\alpha m)^r p^k q^n - \alpha \frac{n}{q} \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} (k-\beta 1-\alpha m)^r p^k q^n.$$

Substituting $k = (k-\beta-1-\alpha m) + (\beta 1+\alpha m)$ in the subtrahend of the above difference, we obtain, after simple transformations,

$$(2.20)\quad \frac{d\bar{\mu}_r}{dp} = -\alpha \frac{nr}{q^2} \bar{\mu}_{r-1} + \frac{\alpha}{p} \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} (k-\beta 1-\alpha m)^{r+1} p^k q^n +$$

$$+ \frac{\alpha\beta}{pq} (1q-np) \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} (k-\beta 1-\alpha m)^r p^k q^n.$$

It follows from (2.19) that

$$\alpha \sum_{k=0}^{\infty} (k-\beta l-\alpha m)^i \, (-1)^k \binom{-n}{k} p^k q^n = \bar{\mu}_i - \beta \alpha^i \, (1-m)^i,$$

where $i = r, r+1$.

Considering this relation in (2.20), we attain

$$\frac{d\bar{\mu}_r}{dp} = -\alpha \frac{nr}{q^2} \bar{\mu}_{r-1} + \frac{1}{p} \left[\bar{\mu}_{r+1} - \beta \alpha^{r+1} \, (1-m)^{r+1}\right] +$$

$$+ \frac{\beta}{p} (1-m) \left[\bar{\mu}_r - \beta \alpha^r \, (1-m)^r\right],$$

from which

$$\bar{\mu}_{r+1} = p \frac{d\bar{\mu}_r}{dp} + \alpha \frac{nrp}{q^2} \bar{\mu}_{r-1} + \alpha^r \beta (1-m)^{r+1} - \beta (1-m) \bar{\mu}_r,$$

and finally - the required formula (2.17). The very formula with $\beta = 0$ gives the relation for the central moments of the negative binomial distribution without inflation

$$\mu_{r+1} = p \left(\frac{d\mu_r}{dp} + \frac{nr}{q^2} \mu_{r-1}\right), \tag{2.21}$$

analogous to the one for the binomial distribution offered by V.Romanovsky [15] (p.410, (2)). To obtain this relation, the above author made use of a method of differentiating the moment-generating function. (The discussion of the problem - see: [7], p.6). Formula (2.17) was given earlier in [9] ((13), p.24) as a special case of formula ((3), p.20) for the moments of power series distribution (PSD).

Let us calculate two central moments by using (2.17):

$$\bar{\mu}_2 = \alpha \left[\beta (1-m)^2 + \frac{m}{q}\right],$$

$$\bar{\mu}_3 = \alpha \beta (\alpha - \beta)(1-m)^3 - 3\alpha\beta \frac{m}{q} (1-m) + \alpha \frac{m}{q^2} (1+p).$$

We shall give another formula for the central moments of distribution (2.1).

Theorem 4. The central moments $\bar{\mu}_r$ of inflated negative binomial distribution (2.1) satisfy the recurrence relation

$$(2.22)\quad \bar{\mu}_r = \beta l\,\frac{\alpha^{r-1}}{q}(l-m)^{r-1} - \beta\frac{p}{q}(n+l)(1+\alpha l-\alpha m)^{r-1} - (\alpha m+\beta l)\,\bar{\mu}_{r-1} + m\sum_{j=0}^{r-1}\binom{r-1}{j}\bar{\mu}_j + \frac{p}{q}\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_{j+1} + \frac{p}{q}(\alpha m+\beta l)\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_j,\quad r = 1,2,\ldots,$$

where $m = n\frac{p}{q}$ denotes the first simple moment of the negative binomial distribution without inflation.

Assuming that the variable Y has distribution (2.1) with characteristic function (2.5), we calculate the characteristic function of the centred variable $Y_1 = Y - \bar{m}_1$ and represent it in the form of the MacLaurin series

$$(2.23)\qquad \varphi_{y_1}(\theta) = \sum_{j=0}^{\infty}\frac{\theta^j}{j!}\,\bar{\mu}_j.$$

We shall then have

$$(2.24)\qquad \sum_{j=0}^{\infty}\frac{\theta^j}{j!}\,\bar{\mu}_j = \beta e^{\theta\alpha(l-m)} + \alpha q^n e^{-\theta(\beta l+\alpha m)}(1-pe^{\theta})^{-n}.$$

Let us differentiate both sides of (2.24) with respect to θ. The derivative L' of the left-hand side is expressed by

$$L' = \sum_{j=1}^{\infty}\frac{\theta^{j-1}}{(j-1)!}\,\bar{\mu}_j$$

and the derivative P' of the right-hand side of (2.24) has the form

$$P' = \alpha\beta(l-m)e^{\theta\alpha(l-m)} - (\beta l+\alpha m)\alpha q^n e^{-\theta(\beta l+\alpha m)}(1-pe^{\theta})^{-n} +$$

$$+ npe^{\theta}\alpha q^n e^{-\theta(\beta l+\alpha m)}(1-pe^{\theta})^{-n-1}.$$

It follows from (2.24) that

$$\alpha q^n e^{-\theta(\beta l+\alpha m)}(1-pe^{\theta})^{-n} = \varphi_{y_1}(\theta) - \beta e^{\theta\alpha(l-m)}.$$

Substituting this relation in the derivative of the right-hand side, we obtain, after having equated L' to P':

$$(2.25) \qquad (1-pe^{\theta}) \sum_{j=1}^{\infty} \frac{\theta^{j-1}}{(j-1)!} \bar{\mu}_j = \alpha\beta(l-m)(1-pe^{\theta})e^{\theta\alpha(l-m)} -$$

$$- (1-pe^{\theta})(\beta l+\alpha m)\left(\varphi_{y_1}(\theta) - \beta e^{\theta\alpha(l-m)} + \right.$$

$$\left. + npe^{\theta} \varphi_{y_1}(\theta) - \beta e^{\theta\alpha(l-m)}\right) .$$

After (2.8) has been made use of, the left-hand side of the above formula takes the form

$$q \sum_{j=1}^{\infty} \frac{\theta^{j-1}}{(j-1)!} \bar{\mu}_j - p \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\theta^{k+j-1}}{k!(j-1)!} \bar{\mu}_j .$$

The right-hand side of (2.25) is transformed into

$$\beta l(1-pe^{\theta})e^{\theta\alpha(l-m)} - (1-pe^{\theta})(\beta l+\alpha m)\varphi_{y_1}(\theta) +$$

$$+ npe^{\theta}\varphi_{y_1}(\theta) - \beta npe^{\theta(1+\alpha l-\alpha m)}.$$

Using (2.8) as well as (2.23) once more, we get further transformations:

$$\beta l e^{\theta\alpha(l-m)} - \beta l p e^{\theta(1+\alpha l-\alpha m)} - q(\beta l+\alpha m)\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\bar{\mu}_j +$$

$$+ p(\beta l+\alpha m)\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j + np\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j -$$

$$- \beta npe^{\theta(1+\alpha l-\alpha m)} = \beta l\sum_{k=0}^{\infty}\frac{\theta^k\alpha^k(l-m)^k}{k!} - \beta p(1+n)\sum_{k=0}^{\infty}\frac{\theta^k(1+\alpha l-\alpha m)^k}{k!} +$$

$$+ np\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j + p(\beta l+\alpha m)\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j -$$

$$- q(\beta l+\alpha m)\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\bar{\mu}_j.$$

Putting the left-hand side of (2.25) and its right-hand side together, we eventually obtain

$$(2.26)\quad q\sum_{j=1}^{\infty}\frac{\theta^{j-1}}{(j-1)!}\bar{\mu}_j - p\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\theta^{k+j-1}}{k!(j-1)!}\bar{\mu}_j =$$

$$= -q(\beta l+\alpha m)\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\bar{\mu}_j + p(\beta l+\alpha m)\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j +$$

$$+ np\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j + \beta l\sum_{k=0}^{\infty}\frac{\theta^k\alpha^k(l-m)^k}{k!} -$$

$$- \beta p(l+n)\sum_{k=0}^{\infty}\frac{\theta^k(1+\alpha l-\alpha m)^k}{k!}.$$

Let us compare the coefficients at θ^{r-1} in the above formula. In the first addend of the left-hand side the very coefficient amounts to $\frac{q\bar{\mu}_r}{(r-1)!}$.

In the addend

$$p\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\theta^{k+j-1}}{k!(j-1)!}\bar{\mu}_j = p\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j+1}}{(k+1)!j!}\bar{\mu}_{j+1}$$

there will be θ^{r-1} when $k = r-2-j$; if $k = 0$, then $j = r-2$. Therefore

$$p\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{j+k+1}}{(k+1)!j!}\bar{\mu}_{j+1} = p\sum_{r=2}^{\infty}\frac{1}{(r-1)!}\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_{j+1} .$$

Thus we have the coefficient at θ^{r-1}

$$\frac{p}{(r-1)!}\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_{j+1} .$$

In the first addend of the right-hand side of (2.26) we have the coefficient at θ^{r-1} equal to

$$-q(\beta l+\alpha m)\frac{\bar{\mu}_{r-1}}{(r-1)!} .$$

In the addend

$$p(\beta l+\alpha m)\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j}}{k!j!}\bar{\mu}_j = p(\beta l+\alpha m)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{k+j+1}}{(k+1)!j!}\bar{\mu}_j$$

there will still be θ^{r-1} when $k = r-j-2$; so we have

$$p(\beta l + \alpha m)\sum_{r=2}^{\infty}\frac{\theta^{r-1}}{(r-1)!}\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_j.$$

With θ^{r-1} we then have

$$\frac{p(\beta l + \alpha m)}{(r-1)!}\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_j.$$

Analogically, in the subsequent addend we have:

$$np \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\theta^{j+k}}{j!k!} \bar{\mu}_j = np \sum_{r=1}^{\infty} \cdot \frac{\theta^{r-1}}{(r-1)!} \sum_{j=0}^{r-1} \binom{r-1}{j} \bar{\mu}_j .$$

The coefficient is then as follows:

$$\frac{np}{(r-1)!} \sum_{j=0}^{r-1} \binom{r-1}{j} \bar{\mu}_j .$$

The coefficients at θ^{r-1} in the last two addends on the right-hand side are as follows

$$\frac{\beta l \alpha^{r-1}(l-m)^{r-1}}{(r-1)!} , \qquad - \frac{\beta p(l+n)(1+\alpha l-\alpha m)^{r-1}}{(r-1)!} .$$

Paying respect to the results obtained, we immediately attain (2.22) which represents the proposition of Theorem 4.

Relation (2.22) with $\beta = 0$ gives the formula for the central moments of the negative binomial distribution without inflation

$$(2.27) \qquad \mu_r = n \frac{p}{q} \sum_{j=0}^{r-2} \binom{r-1}{j} \mu_j + \frac{p}{q} \sum_{j=0}^{r-2} \binom{r-1}{j} \mu_{j+1}$$

or the one in the form of

$$(2.28) \qquad \mu_r = \frac{p}{q} \sum_{j=0}^{r-2} \binom{r-1}{j} (n\mu_j + \mu_{j+1}) .$$

Relation (2.27) is analogous to the one for the binomial distribution offered by R.Frisch [4] (p.171, (20)) or to the formula equivalent to it (but of another form, cf. [4], p.165, (1)) which was also obtained for the binomial distribution by K.Pearson [13] (p.160, (XII)) a year before. The formula of the Pearsonian structure for the negative binomial distribution equivalent to (2.28) was given in [7], (p.37, (3.13)). More through discussion of those matters can be found in [7], pp.6-8.

We shall give here the formula for $\bar{\mu}_2$, calculated according to (2.22), for the inflated negative binomial distribution. The details of calculations have been omitted. We have

$$\bar{\mu}_2 = \alpha\left[\beta(1-m)^2 + \frac{m}{q}\right].$$

3. Moments of the inflated Poisson and geometric distributions

From the recurrence relations for the simple and the central moments of the inflated negative binomial distribution we can obtain in the limit proceeding

$$\lim_{n\to\infty} n\frac{p}{q} = \lambda > 0$$

(cf. Theorem 5.13.1, p.181 in [3]; German ed., p.151, and Lemma 1, p.462 in [19]) the recurrence relations for the same moments of the inflated Poisson distribution. Hence we have

C o r o l l a r y 1. The simple and the central moments of the inflated Poisson distribution are expressed by the following recurrence relations:

$$(3.1)\qquad \bar{m}_r = \beta l^r - \beta\lambda(l+1)^{r-1} + \lambda\sum_{j=0}^{r-1}\binom{r-1}{k}\bar{m}_{r-1-j},$$

$$(3.2)\qquad \bar{m}_{r+1} = \beta l^r(1-\lambda) + \lambda\left(\bar{m}_r + \frac{d\bar{m}_r}{d\lambda}\right),$$

$$(3.3)\qquad \bar{\mu}_{r+1} = \alpha^r\beta(1-\lambda)^{r+1} - \beta(1-\lambda)\bar{\mu}_r + \lambda\left(\alpha r\bar{\mu}_{r-1} + \frac{d\bar{\mu}_r}{d\lambda}\right),$$

$$(3.4)\qquad \bar{\mu}_r = \beta l\left[\alpha(1-\lambda)\right]^{r-1} - \beta\lambda\left[1+\alpha(1-\lambda)\right]^{r-1} - (\alpha\lambda+\beta l)\bar{\mu}_{r-1} + \lambda\sum_{j=0}^{r-1}\binom{r-1}{j}\bar{\mu}_j$$

for $r = 1,2,\ldots$.

The relations mentioned above were also obtained by L.Sobich [19] (p.463, (11) - (14)) as corollaries from formulae for the inflated binomial distribution in the limit proceeding.

Putting the value 1 instead of n in the recurrence relations for the simple and the central moments of the inflated negative binomial distribution, we attain, as a special case, the recurrence relations for the same moments of the inflated geometric distribution.

C o r o l l a r y 2. The recurrence relations for the simple and the central moments of the inflated geometric distribution have the following forms

$$\bar{m}_r = \frac{1}{q}\left\{\beta 1^r - \beta p(1+1)^r + p\sum_{j=0}^{r-1}\binom{r}{j+1}\bar{m}_{r-1-j}\right\}, \tag{3.5}$$

$$\bar{m}_{r+1} = \beta 1^r\left(1 - \frac{p}{q}\right) + p\left(\frac{\bar{m}_r}{q} + \frac{d\bar{m}_r}{dp}\right), \tag{3.6}$$

$$\bar{\mu}_{r+1} = \alpha^r\beta\left(1 - \frac{p}{q}\right)^{r+1} - \beta\left(1 - \frac{p}{q}\right)\bar{\mu}_r + p\left(\frac{\alpha r}{q^2}\bar{\mu}_{r-1} + \frac{d\bar{\mu}_r}{dp}\right), \tag{3.7}$$

$$\begin{aligned}\bar{\mu}_r = {} & \beta\frac{1}{q}\left[\alpha\left(1 - \frac{p}{q}\right)\right]^{r-1} - \beta\frac{p}{q}(1+1)\left[1+\alpha\left(1 - \frac{p}{q}\right)\right]^{r-1} - \\ & - \left(\alpha\frac{p}{q} + \beta 1\right)\bar{\mu}_{r-1} + \frac{p}{q}\sum_{j=0}^{r-1}\binom{r-1}{j}\bar{\mu}_j + \frac{p}{q}\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_{j+1} + \\ & + \frac{p}{q}\left(\alpha\frac{p}{q} + \beta 1\right)\sum_{j=0}^{r-2}\binom{r-1}{j}\bar{\mu}_j.\end{aligned} \tag{3.8}$$

To the relations given here the same remarks refer as those made above for the negative binomial distribution on the possibility of obtaining formulae for the distribution without inflation. Simpler relations of that kind were already known before. And so, the equivalent of formula (3.1) (i.e., under the condition that $\beta = 0$) was offered by R.Risser and C.E. Traynard [14] (p.320, 2nd ed., p.91) in 1933, next, by W.Krysicki [11] (p.23, (5-I)) in 1957 and lastly, by the present writer [7] (p.20, (2.5)) in 1971. The discussion of the methods used in the derivation of this formula, which are different from the one mentioned above, can be found in [7] (pp.9,15,20,21).

In 1934 A.T.Craig [1] (p.264) obtained formulae (3.2) and (3.3) for the Poisson distribution without inflation (cf. [7], p.10), and in 1965 A.R.Kamat [10] (p.47, (17)) obtained formula (3.4) also for the very distribution (the formula of a different but equivalent form was obtained by K.Pearson [13] (p.161, XVII) as early as 1924 (cf. [7], p.7). Formula (3.5) for the case of $\beta = 0$ can be found in [7] (p.21, (2.8)).

It is also worth mentioning that the relations obtained here for distribution (2.1) may as well be given for the case of the so-called Pólya-Eggenberger distribution which is obtained from (2.1) by substituting $p = \eta/1+\eta$ and $n = \lambda/\eta$, where $\lambda > 0$ and $\eta > 0$ (cf. [7], pp.22,31,37,40 and 44). We omit an effective presentation which can easily be carried out.

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INSTITUTE OF MATHEMATICS, UNIVERSITY, ŁÓDŹ
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