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Stanisław Wroński

# ON FIELDS OF SETS WITH A NOWHERE DENSE BOUNDARY

It is well-known that every Boolean algebra is isomorphic to a field of all closed-open subsets of a topological space (see [3]). It seems to be interesting to ask under what conditions a Boolean algebra is isomorphic to a field of all subsets with a nowhere dense boundary. Let us say that a Boolean algebra is representable iff this is a case. In this paper we shall give a necessary and sufficient condition for a Boolean algebra to be representable in the above sense. We shall prove the following theorem.

T h e o r e m . A Boolean algebra $\mathfrak{A}$ is representable iff it is atomic and there exists a mapping $D : \mathfrak{A} \to \mathfrak{A}$ satisfying the following conditions:

(1) $D(\mathbf{1}) = \mathbf{1}$ $D(\mathbf{0}) = \mathbf{0}$,

(2) $D(D(a)) = D(a)$,

(3) $D(-D(a)) = D(-a)$,

(4) $D(a \wedge b) = D(a) \wedge D(b)$.

(5)*) If $X \subseteq \mathfrak{A}$ has an upper bound $a$ such that $D(a) = \mathbf{0}$ then $X$ has the supremum in $\mathfrak{A}$.

(6) If $X \subseteq \mathfrak{A}$ is such that for every $a \in X$, $a \leq D(a)$ then $X$ has the supremum in $\mathfrak{A}$.

P r o o f . To prove the sufficiency let us assume that $\mathfrak{A}$ is an atomic Boolean algebra and $D : \mathfrak{A} \to \mathfrak{A}$ is a mapping

*) After completing this paper the author has learnt that the condition (5) can be dispensed with because it can be derived from the remaining conditions.

such that the conditions (1),...,(6) are satisfied. It is known (see [3]) that the mapping $a \mapsto \{x \leqslant a \mid x$ is an atom$\}$ is a complete embedding of the atomic Boolean algebra into the field of all sets of its atoms. Thus, without loss of generality we can assume that $\mathcal{A}$ is a field of sets, every atom of $\mathcal{A}$ is a singleton and all existing suprema in $\mathcal{A}$ are set-theoretical. We shall prove succesively the following:

(i) The family of sets $\mathfrak{a} = \{a \in \mathcal{A} \mid a \subseteq D(a)\}$ can be used as a family of open subsets of the unit element $\mathbf{1}$ of the field $\mathcal{A}$, i.e. $\mathfrak{a}$ is closed under finite intersections and arbitrary unions.

(ii) The family of sets $\Phi = \{a \in \mathcal{A} \mid D(a) = \mathbf{0}\}$ is the family of all nowhere dense subsets of the topological space $\langle \mathbf{1}, \mathfrak{a} \rangle$.

(iii) The field $\mathcal{A}$ is the family of all subsets with a nowhere dense boundary of the topological space $\langle \mathbf{1}, \mathfrak{a} \rangle$.

To prove (i) observe first that $\mathbf{1}, \mathbf{0} \in \mathfrak{a}$ by (1). Assume that $a, b \in \mathcal{A}$, $a \subseteq D(a)$, $b \subseteq D(b)$, then $a \cap b \subseteq D(a) \cap D(b) = D(a \cap b)$ by (4). Now let us assume that $a \subseteq D(a)$ for every $a \in X \subseteq \mathcal{A}$. Since by (4) it follows that the mapping $D$ is monotone and by (6) it follows that $\bigcup X \in \mathcal{A}$ thus $\bigcup X \subseteq \bigcup D(X) \subseteq D(\bigcup X)$.

Now it will be convenient to prove two lemmas, the first characterising the interior operation of the topological space $\langle \mathbf{1}, \mathfrak{a} \rangle$ for sets belonging to the field $\mathcal{A}$ and the second stating an useful property of the family $\Phi$.

(*) For every $a \in \mathcal{A}$, $Ia = a \cap D(a)$ $(Ca = a \cup -D(-a))$

(**) If $a \in \Phi$ and $b \subseteq a$ then $b \in \Phi$.

To prove (*) assume that $b \in \mathcal{A}$, $b \subseteq D(b)$ and $b \subseteq a$. Then $D(b) \subseteq D(a)$ which gives that $b \subseteq a \cap D(a)$ and therefore we have only to check that $a \cap D(a) \in \mathfrak{a}$. Indeed, $a \cap D(a) \subseteq D(a) = D(a) \cap D(D(a)) = D(a \cap D(a))$ by virtue of (2) and (4).

To prove (**) assume that $a \in \Phi$ and $b \subseteq a$. Let $At(a)$, $At(b)$ be the sets of all atoms of $\mathcal{A}$ contained in $a, b$ respectively. Since $a \in \Phi$ and $At(b) \subseteq At(a)$ then $\bigcup At(b) \in \mathcal{A}$ by (5). Clearly $\bigcup At(b) \subseteq b$. To prove the converse inclusion suppose that $x \in b$. Since $a$ is an element of the atomic field $\mathcal{A}$ then $\bigcup At(a) = a$ and it follows that $x \in \bigcup At(a)$. Since every atom of $\mathcal{A}$ is a singleton thus $\{x\}$ must be an atom of $\mathcal{A}$ and consequently $\{x\} \in At(b)$. This means that $x \in \bigcup At(b)$ and thus we get that $b = \bigcup At(b) \in \mathcal{A}$.

Now we are able to characterize the family of all nowhere dense subsets of the topological space $\langle \mathbf{1}, \mathcal{Q} \rangle$ by proving (ii). Suppose first that $a \subseteq \mathbf{1}$ and $ICa = \mathbf{0}$. Since $Ca \in \mathcal{A}$ then $\mathbf{0} = ICa = Ca \cap D(Ca)$ by (*) and next $\mathbf{0} = D(\mathbf{0}) = D(Ca \cap D(Ca)) = D(Ca) \cap D(D(Ca)) = D(Ca) \cap D(Ca) = D(Ca)$ which means that $Ca \in \Phi$ and thus one gets that $a \in \Phi$ applying (**). We have proved that every nowhere dense subset of the topological space $\langle \mathbf{1}, \mathcal{Q} \rangle$ belongs to the family $\Phi$. Now, let us suppose that $a \in \Phi$. Then $D(a) = \mathbf{0}$ and consequently $D(-a) = D(-D(a)) = D(-\mathbf{0}) = \mathbf{1}$ by virtue of (3) and (1). Next applying (*) one gets that $ICa = (a \cup -D(-a)) \cap D(a \cup -D(-a)) = (a \cup -\mathbf{1}) \cap D(a \cup -\mathbf{1}) = a \cap D(a) = a \cap \mathbf{0} = \mathbf{0}$ which means that $a$ is a nowhere dense subset of the topological space $\langle \mathbf{1}, \mathcal{Q} \rangle$

Now we are in the position to prove the condition (iii). Having proved that $\Phi$ is the family of all nowhere dense subsets of the topological space $\langle \mathbf{1}, \mathcal{Q} \rangle$ we need only to show that for every $a \subseteq \mathbf{1}$, $Fr(a) \in \Phi$ iff $a \in \mathcal{A}$. Let us suppose first that $a \in \mathcal{A}$. Then

$$Fr(a) = Ca - Ia = (a \cup -D(-a)) - (a \cap D(a)) =$$

$$= (-a - D(-a)) \cup (a - D(a)) \cup -(D(-a) \cup D(a)).$$

Now applying the conditions (1), (3) and (4) one can show that each the component of the sum above belongs to $\Phi$. Indeed,

$$D(-a - D(-a)) = D(-a) \cap D(-D(-a)) = D(-a) \cap D(a) =$$

$$= D(-a \cap a) = D(\mathbf{0}) = \mathbf{0}.$$

$$D(a - D(a)) = D(a) \cap D(-D(a)) = D(a) \cap D(-a) = \\ = D(a \cap -a) = D(\mathbf{0}) = \mathbf{0}.$$

$$D((-D(-a) \cup D(a))) = D(-D(-a) \cap -D(a)) = \\ = \mathbf{0}(-D(-a)) \cap D(-D(a)) = D(a) \cap D(-a) = D.$$

We know that $\Phi$ is an ideal of $\mathcal{A}$ by virtue of (ii) and thus it follows that the whole sum (i.e. Fr(a)) belongs to $\Phi$ which was to be shown. Now let us suppose that $Fr(a) \in \Phi$. Then $a \cap Fr(a) \in \Phi \subseteq \mathcal{A}$ by virtue of (**). Next observing that $Ia \in \mathcal{A}$ and $a = Ia \cup (a \cap Fr(a))$ we get that $a \in \mathcal{A}$ which finishes the proof of sufficiency.

To prove the necessity let us suppose that we are given a topological space $\langle \mathbf{1}, \mathfrak{a} \rangle$ with a family $\mathfrak{a}$ of open subsets and with a field $\mathcal{A}$ of all subsets having a nowhere dense boundary. First we shall prove that the field $\mathcal{A}$ is atomic. Suppose the converse, then there exists $a \in \mathcal{A}$ such that $a \neq \mathbf{0}$ and there is no atom of $\mathcal{A}$ contained in a. We claim that for every $b \subseteq a$ if $b \in \mathcal{A}$ then $b \in \mathfrak{a}$. Indeed, since Fr(b) is a nowhere dense set and $b = Ib \cup (b \cap Fr(b))$ then $b \cap Fr(b) = \mathbf{0}$ because in the opposite case there exists a nowhere dense atom $\{x\} \subseteq a$ such that $x \in b \cap Fr(b))$. Now, let us consider the ideal $\mathcal{B} = \{b \in \mathcal{A} | b \subseteq a\}$. It is clear that $\mathcal{B}$ is a nontrivial field of sets and $\mathcal{B} \subseteq \mathfrak{a}$. Thus for every $X \subseteq \mathcal{B}$, $\bigcup X \in \mathcal{B}$ and consequently the field $\mathcal{B}$ as a nontrivial complete and completely distributive Boolean algebra would be atomic, a contradiction. Now, if we define a mapping $D : \mathcal{A} \longmapsto \mathcal{A}$ by putting $D(a) = ICa$ for every $a \in \mathcal{A}$ then it is easy to check that the conditions (1),...,(4) are satisfied. To prove (5) observe that the sum of a family of sets having a nowhere dense upper bound must be a nowhere dense set and thus it must have a nowhere dense boundary. To prove (6) assume that $a \subseteq ICa$ for every $a \in X \subseteq \mathcal{A}$. Observing that set x has a nowhere dense boundary iff $ICx \subseteq CJx$ we get succesively $ICa \subseteq CIa$, $ICa \subseteq ICIa$ and $a \subseteq ICIa$ for every $a \in X$. This gives that $\bigcup X \subseteq ICI\bigcup X$ and consequently $IC\bigcup X \subseteq ICICI\bigcup X = ICI\bigcup X \subseteq CI\bigcup X$ which was to be shown. Q.E.D.

It is worth noticing that by a theorem of Von Neumann (see [2]) it follows that every $\sigma$-field of sets with a complete and $\sigma$-finite measure is representable and thus it is atomic. On the other hand one can construct an example of atomic field of sets which is not representable.

## REFERENCES

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INSTITUTE OF MATHEMATICS, UNIVERSITY, ŁÓDŹ
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