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A NON-LINEAR HILBERT-RIEMANN PROBLEM WITH BOUNDARY CONDITIONS OF THE CLASS L_p^α

1. Introduction

The composite mixed Hilbert-Riemann problem in the class of analytic functions has been considered by I.S.Rogozhina [4] and Lu Chien Ke [2], and in the class of pseudoanalytic function by J.Wolska-Bochenek [7]. In the above-mentioned papers the authors were seeking a solution to the Hilbert-Riemann problem in the class of Hölder functions in connected and multiconnected domains. The present paper provides a generalization of these results with respect to the class in which the solution is sought. Namely, we solve a non-linear Hilbert-Riemann problem with boundary conditions of the class L_p^α.

2. The formulation of the problem

Assume that in the complex variable plane E we are given a doubly connected domain D^+ bounded by two non-intersecting closed Lapunov curves L and L_0. The curve L_0 comprises the curve L. The orientation of L and L_0 is selected in such a way that the right-angle rotation from the positive direction of the tangent to the interior normal with respect to the domain D^+ is consistent with the rightangle rotation from the real axis OX to the imaginary axis OY. Let D^- denote the domain lying inside the curve L, where we assume that the origin of the coordinate system lies within

D^-. Without loss of generality we may assume that $L_0 = \{z: |z| = 1\}$, is a circle comprising L (see [3], p.37).

We introduce the following notation: $\bar{S} = D^- \cup L \cup D^+ \cup L_0$; $S^- = E \setminus \bar{S}$.

The non-linear Hilbert-Riemann problem to be investigated here consists in finding a function $\phi(z)$ sectionally holomorphic in the domains D^+ and D^-, whose boundary values $\phi^+(t)$ and $\phi^-(t)$ satisfy, almost everywhere on L and L_0 respectively, the boundary conditions:

$$(1)\quad \begin{cases} \text{(a)} \quad \phi^+(t) = G(t)\phi^-(t) + g\left[t, \phi^+(t), \phi^-(t)\right], & t \in L, \\ \text{(b)} \quad \mathrm{Re}\left[c(t_0)\phi^+(t_0)\right] = H(t_0), & t_0 \in L_0, \end{cases}$$

where the functions G, g, c, H are given.

To solve this problem we make the following assumptions:

I. The complex function $G(t)$ defined for every $t \in L$ satisfies Hölder's condition

$$(2)\qquad G(t) \in H^{\mu}(L, M_G, k_G) \quad \text{and} \quad \bigwedge_{t \in L} G(t) \neq 0, \qquad \mu \in (0,1\rangle .$$

II. The complex function $g(t,u_1,u_2)$ is defined in the domain $t \in L$; $u_1, u_2 \in E$ and belongs to the class L_p^{α}, i.e. it satisfies the inequalities

$$(3)\quad \begin{cases} \left\| g(t,u_1,u_2) \right\|_{L_p} \leqslant M_g \\ \left\| g(t,u_1,u_2) - g(t',u_1',u_2') \right\|_{L_p} \leqslant k_g |t-t'|^{\alpha} + k_g' \left[\left\| u_1 - u_1' \right\|_{L_p} + \left\| u_2 - u_2' \right\|_{L_p} \right], \end{cases}$$

where $p \in (1, +\infty)$, $\alpha \in (0,1\rangle$, M_g, k_g, $k_g' > 0$ are constants.

III. The complex function $c(t_0)$ is defined for every $t_0 \in L_0$ and satisfies Hölder's condition

$$(4)\qquad c(t_0) \in H^{\mu}(L_0, M_c, k_c), \qquad \mu \in (0,1\rangle .$$

IV. The real function $H(t_0)$ is defined for every $t_0 \in L_0$ and belongs to L_p^{α}

(5) $H(t_0) \in L_p^\alpha(L_0, M_H, k_H)$, $p \in (1,+\infty)$, $\alpha \in (0,1\rangle$, M_H, $k_H > 0$.

V. The index of the problem satisfies the condition

$$\operatorname{ind}[G(T)]_L = \frac{1}{2\pi i} \ln[G(t)]_L = \frac{1}{2\pi}[\arg G(t)]_L = \varkappa_H \geqslant 0,$$

$$\operatorname{ind}\left[\frac{\overline{c(t_0)}}{c(t_0)}\right]_{L_0} = \frac{1}{2\pi i} \ln\left[\frac{\overline{c(t_0)}}{c(t_0)}\right]_{L_0} = \frac{1}{\pi}\left[\arg \overline{c(t_0)}\right] = \varkappa_R \geqslant 0.$$

3. The solution of the problem

Making use of the theory developed in [2], [7], we can state that if the problem (1) has a solution $\phi(z)$, then

$$\phi(z) = \phi_1(z) + X_1(z)\phi_0(Z); \tag{6}$$

the function $\phi_1(z)$ is a solution of the Hilbert problem and is defined by the formula

$$\phi_1(z) = \frac{X_1(z)}{2 i} \int_L \frac{g[\tau, \phi_1^+(\tau), \phi_1^-(\tau)]}{X_1^+(\tau)(\tau - z)} d\tau + X_1(z)P_1(z), \tag{7}$$

where $P_1(z)$ is an arbitrary polynomial and $X_1(z)$ is a canonical solution of the homogeneous Hilbert problem [see [3]]

$$X_1^+(t) = G(t)X_1^-(t), \quad t \in L, \tag{8}$$

and is defined by the formula

$$X_1(z) = \begin{cases} z^{-\varkappa_H} \exp\Gamma_1(z), & z \in D^+ \cup L_0 \cup S^-, \\ \exp\Gamma_1(z), & z \in D^-, \end{cases} \tag{9}$$

where

$$\Gamma_1(z) = \frac{1}{2\pi i} \int_L \frac{\ln\left[\tau^{H^{\varkappa}} G(\tau)\right]}{\tau - z} d\tau:, \tag{10}$$

the function $\phi_0(z)$ from (6) is a solution of the Riemann problem (cf. (16))

$$(11)\quad \mathrm{Re}\left[c(t_0)X_1^+(t_0)\phi_0^+(t_0)\right] = H(t_0) - \mathrm{Re}\left[c(t_0)\phi_1^+(t_0)\right], \qquad t_0 \in L_0,$$

and can be expressed by the formula

$$(12)\qquad \phi_0(z) = \frac{1}{2}\left[\phi_3(z) + \phi_3^*(z)\right] + X_0(z)P_0(z),$$

where $P_0(z)$ is an arbitrary polynomial and

$$(13)\qquad \phi_3(z) = \frac{X_0(z)}{\pi i}\int_{L_0}\frac{\phi_4(\tau_0)}{X_0^+(\tau_0)(\tau_0 - z)}\,d\tau_0, \qquad \tau_0 \in L_0,$$

$$(14)\qquad \phi_4(\tau_0) = H(\tau_0) - \mathrm{Re}\ c(\tau_0)\phi_1^+(\tau_0)\ ,$$

$$(15)\qquad \phi_3^*(z) = \overline{\phi_3\left(\frac{1}{\bar{z}}\right)}\ ,$$

$$(16)\qquad X_0(z) = \begin{cases} \mathrm{const}\ \exp\Gamma_0(z), & z \in \bar{S}\setminus L_0, \\ \mathrm{const}\ z^{-\varkappa_R}\exp\Gamma_0(z), & z \in S^-, \end{cases}$$

$$(17)\qquad \Gamma_0(z) = \frac{1}{2\pi i}\int_{L_0}\frac{\ln\left[\tau_0^{-\varkappa_R}\dfrac{\overline{c(\tau_0)}}{c(\tau_0)}\right]}{\tau_0 - t}\,d\tau_0.$$

The canonical solutions $X_1^{\pm}(t)$ for $t \in L$, $X_0^{\pm}(t_0)$ for $t_0 \in L_0$ satisfy Hölder's condition

$$(18)\quad X_1^{\pm}(t) \in H^{\mu}(L, k_{X_1}, M_{X_1}),\ X_0^{\pm}(t_0) \in H^{\mu}(L_0, k_{X_0}, M_{X_0}), \qquad \mu \in (0,1\rangle$$

and the inequalities

$$(19) \qquad 0 < m_{X_1} \leqslant \left| X_1^{\pm}(t) \right| \leqslant M_{X_1}, \qquad 0 < m_{X_0} \leqslant \left| X_0^{\pm}(t_0) \right| \leqslant M_{X_0},$$

where m_{X_1}, m_{X_0}, M_{X_1}, M_{X_0} are lower and upper bounds of Hölder's coefficients of the canonical solution of the respective Hilbert problems.

Applying Sochocki-Plemelj's formulas [3] to the function $\varnothing_1(z)$ given by the formula (7), we obtain the system of integral equations

$$(20) \begin{cases} \varnothing_1^+(t) = \frac{1}{2}\, g\left[t, \varnothing_1^+(t), \varnothing_1^-(t)\right] + \frac{X_1^+(t)}{2\pi i} \int\limits_L \frac{g\left[\tau, \varnothing_1^+(\tau), \varnothing_1^-(\tau)\right]}{X_1^+(\tau)(\tau - t)}\, d\tau + \\ \qquad + X_1^+(t) P_1(t), \\ \varnothing_1^-(t) = -\frac{1}{2} \frac{X^-(t)}{X^+(t)}\, g\left[t, \varnothing_1^+(t), \varnothing_1^-(t)\right] + \\ \qquad + \frac{X_1^-(t)}{2\pi i} \int\limits_L \frac{g\left[\tau, \varnothing_1^+(\tau), \varnothing_1^-(\tau)\right]}{X_1^+(\tau)(\tau - t)}\, d\tau + X_1^-(t) P_1(t). \end{cases}$$

We introduce the following notations

$$(21) \begin{cases} \varphi_j(t) = \begin{cases} \varnothing_1^+(t) & \text{for } j = 1, \\ \varnothing_1^-(t) & \text{for } j = 2, \end{cases} \\ Y_j(t) = \begin{cases} X_1^+(t) P_1(t) & \text{for } j = 1, \\ X_1^-(t) P_1(t) & \text{for } j = 2, \end{cases} \\ g_j\left[t, \varphi_1(t), \varphi_2(t)\right] = \begin{cases} \frac{1}{2} g\left[t, \varphi_1(t), \varphi_2(t)\right] & \text{for } j=1, \\ -\frac{1}{2} \frac{X_1^-(t)}{X_1^+(t)}\, g\left[t, \varphi_1(t), \varphi_2(t)\right] & \text{for } j=2, \end{cases} \\ f_j(t) = \begin{cases} \frac{X_1^+(t)}{2\pi i} \int\limits_L \frac{g\left[\tau, \varphi_1(\tau), \varphi_2(\tau)\right]}{X_1^+(\tau)(\tau - t)}\, d\tau & \text{for } j = 1, \\ \frac{X_1^-(t)}{2\pi i} \int\limits_L \frac{g\left[\tau, \varphi_1(\tau), \varphi_2(\tau)\right]}{X_1^+(\tau)(\tau - t)}\, d\tau & \text{for } j = 2, \end{cases} \end{cases}$$

With the above notations the system (20) can be written in the following form

(22) $$\varphi_j(t) = g_j\left[t,\varphi_1(t),\varphi_2(t)\right] + f_j(t) + Y_j(t) \text{ for } j=1,2.$$

Putting

(23) $$\begin{aligned}&\varphi^0(t) = \left[\varphi_1(t),\varphi_2(t)\right],\ \varphi^1(t) = \phi(t),\ \phi(t) = \left[\phi^+(t),\phi^-(t)\right],\\ &\varphi^2(z) = \phi_0(z),\ \varphi^3(z) = \phi_3(z),\ \varphi^4(t_0) = \phi_4(t_0)\end{aligned}$$

and making use of the equalities (21) and (22), in view of the formulas (6), (12), (13) and (14), we obtain the following system of strongly singular integral equations

(24) $$\begin{cases}\varphi^1(t) = \varphi^0(t) + X_1(t)\varphi^2(t),\\ \varphi^2(z) = \frac{1}{2}\left[\varphi^3(z) + \overline{\varphi^3(z)}\right] + X_0(z)P_0(z),\\ \varphi^3(z) = \frac{X_0(z)}{2\pi i}\int\limits_{L_0}\frac{\varphi^4(\tau_0)}{X^+(\tau_0)(\tau_0-z)}\,d\tau_0,\\ \varphi^4(t_0) = H(t_0) - \mathrm{Re}\left[c(t_0)\varphi^c(t_0)\right].\end{cases}$$

If the problem (1) has a solution of the form (6), then the system (24) has a solution and conversely.

Before we show the existence of at least one solution of the system (24) with the help of J.Schauder's fixed point theorem [5], we state some lemmas.

L e m m a 1. If the function $g(t,u_1,u_2)$ satisfies the assumption II, then the function $\varphi^0(t)$ defined by (22), (23) belongs to the class L_p^β, where $\beta = \min(\alpha,\mu)$.

P r o o f . We make use of the results obtained by B.W.Chwedelidze [1] and R.Słysz [6]. According to the assumption II the function $g(t,u_1,u_2)$ satisfies the inequality (3), hence we have

$$g[t,\varphi_1(t),\varphi_2(t)] \in L_p^{\alpha}(L,M_g,k_g + 2kk_g'),$$

where k is an arbitrary fixed positive number. In view of (21), we have

$$g_1[t,\varphi_1(t),\varphi_2(t)] \in L_p^{\alpha}\left[L, \frac{1}{2} M_g, \frac{1}{2}(k_g + 2kk_g')\right],$$

$$g_2[t,\varphi_1(t),\varphi_2(t)] \in L_p^{\alpha}\left[L, \frac{1}{2}\frac{M_{X_1}}{m_{X_1}} M_g, \frac{1}{2}\frac{M_{X_1}}{m_{X_1}}(k_g+2kk_g')+k_X M_g \frac{M_{X_1}}{m_{X_1}^2}\right].$$

Let $M = \min\left(\frac{1}{2}, \frac{1}{2}\frac{M_{X_1}}{m_{X_1}}\right)$, then we have

$$g_j\left[t,\varphi_1(t),\varphi_2(t)\right] \in L_p^{\alpha}\left[L, MM_g, M(k_g+2kk_g')+k_X \frac{M}{m_{X_1}} M_g\right], \quad j=1,2.$$

The functions $f_j(t)$ defined by (21) belong to the class

$$L_p^{\alpha}\left[L, a_1 \frac{M_{X_1}}{m_{X_1}} M_g, a_2 \frac{M_{X_1}}{m_{X_1}} M_g + a_3 \frac{M_{X_1}}{m_{X_1}}(k_g+2kk_g')+\frac{M_{X_1}}{m_{X_1}^2} k_{X_1} M_g\right].$$

The functions $Y_j(t)$ defined by the formula (21) satisfy conditions (18), (19) and we have

$$Y_j(t) \in H^{\mu}(L, M_{X_1} P_{X_1}, M_{P_1} k_{X_1} + M_{X_1} k_{P_1}), \quad j = 1,2.$$

We also have

$$(25)\quad \varphi^{0}(t) \in L_p^{\beta}\Bigg[L, MM_g + a_1 \frac{M_{X_1}}{m_{X_1}} M_g + M_{X_1} P_{X_1}, M(k_g+2kk_g') + k_{X_1} \frac{M}{m_{X_1}} M_g + a_2 \frac{M_{X_1}}{m_{X_1}} M_g + a_3 \frac{M_{X_1}}{m_{X_1}}(k_g+2kk_g') + \frac{M_{X_1}}{m_{X_1}^2} k_{X_1} M_g + M_{P_1} k_{X_1} + M_{X_1} k_{P_1}\Bigg],$$

where $\beta = \min(\alpha, \mu)$. Let us denote

$$M+a_1\frac{M_{X_1}}{m_{X_1}} = A_1, \quad M+a_3\frac{M_{X_1}}{m_{X_1}} = A_2, \quad \frac{M}{m_{X_1}} + \frac{M_{X_1}}{m_{X_1}^2} = A_3, \quad a_2\frac{M_{X_1}}{m_{X_1}} = A_4.$$

We then have

$$(26)\qquad \varphi^o(t) \in L_p^\beta\Big[L, A_1M_g+M_{X_1}P_{X_1},\ A_2(k_g+2kk_g') + A_3k_{X_1}M_g + A_4M_g+M_{P_1}k_{X_1}M_{X_1}k_{P_1}\Big],$$

where $A_1,\dots,A_4$ are constants dependent of M_{X_1}, m_{X_1}, M.

L e m m a 2. If the functions $c(t_o)$ and $H(t_o)$ satisfy the assumptions III and IV, respectively, and $\varphi^o(t_o) \in L_p^\beta$, then the function $\varphi^4(t_o)$ defined by (24) belongs to the class L_p^β.

P r o o f . Making use of the assumptions III, IV and of the Lemma 1 we obtain

$$(27)\qquad \left\|\varphi^4(t_o)\right\|_{L_p} = \left(\int_{L_o}\left|\varphi^4(t_o)\right|^p dt_o\right)^{\frac{1}{p}} =$$

$$= \left(\int_{L_o}\left|H(t_o) - \mathrm{Re}\left[c(t_o)\varphi^o(t_o)\right]\right|^p dt_o\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant \left(\int_{L_o}\left|H(t_o)\right|^p dt_o\right)^{\frac{1}{p}} + \left(\int_{L_o}\left|\mathrm{Re}\left[c(t_o)\varphi^o(t_o)\right]\right|^p dt_o\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant b_1M_H + b_2M_c(A_1M_g+M_{X_1}P_{X_1}) = B_1 + B_2M_g + B_3M_{X_1}P_{X_1},$$

$$(28)\quad \left\| \varphi^4(t_o+\Delta t_o)-\varphi^4(t_o)\right\| = \left(\int\limits_{L_o} \left|\varphi^4(t_o+\Delta t_o)-\varphi^4(t_o)\right|^p dt_o\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant \left(\int\limits_{L_o} \left|H(t_o+\Delta t_o) - H(t_o)\right|^p dt_o\right)^{\frac{1}{p}} +$$

$$+ \left(\int\limits_{L_o} \left|\mathrm{Re}\left[c(t_o+\Delta t_o)\varphi^o(t_o+\Delta t_o)-c(t_o)\varphi_o(t_o)\right]\right|^p dt_o\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant b_3k_H\left|\Delta t_o\right|^{\alpha} + b_4M_c\left[A_2(kg+2kk_g')+A_3k_{X_1}M_g+A_4M_g+M_{P_1}k_{X_1} +\right.$$

$$\left.+ M_{X_1}k_{P_1}\right]\left|\Delta t_o\right|^{\alpha} + b_5k_c(A_1M_g+M_{X_1}P_{X_1})\left|\Delta t_o\right|^{\mu} \leqslant$$

$$\leqslant \left[B_4+B_5(k_g+2kk_g')+B_6k_{X_1}M_g+B_7M_g+B_8M_{P_1}k_{X_1} +\right.$$

$$\left.+ B_9M_{X_1}k_{P_1}+B_{10}M_{X_1}P_{X_1}\right]\left|\Delta t_o\right|^{\beta} ,$$

where $\beta = \min(\alpha,\mu)$, B_i $(i = 1,\dots,10)$ are positive constants depending upon M_H, k_H, M_c, k_c.

L e m m a 3. If the function $\varphi^A(\tau_o)$ belongs to L_p^{β} in agreement with Lemma 2, then the function $\varphi^3(z)$ defined by the ineality (24) belongs to the class L_p^{β}.

P r o o f . The proof of this lemma follows from the papers of B.W.Chwedelidze [1], R.Słysz [6] and from Lemma 2. Hence we have

$$(29)\qquad \varphi^3(z) \in L_p^{\beta}\left[L_o,\tilde{B}_1+\tilde{B}_2M_g+\tilde{B}_3M_{X_o}P_{X_o},\tilde{B}_4+\tilde{B}_5(k_g+2kk_g') +\right.$$

$$\left.+ \tilde{B}_6k_{X_o}M_g+\tilde{B}_7M_g+\tilde{B}_8M_{P_o}k_{X_o}+\tilde{B}_9M_{X_o}k_{P_o}+\tilde{B}_{10}M_{X_o}P_{X_o}\right] ,$$

where $\tilde{B}_i$ $(i = 1,\dots,10)$ are positive constants.

L e m m a 4. If $\varphi^3(z) \in L_p^\beta$, then $\varphi^2(z)$ given by (24) also belongs to L_p^β.

P r o o f . Lemma 4 follows directly from Lemma 3, if we apply to (24) the formulas given in [3] (p.42). Namely we have

$$(30)\qquad \varphi^2(z) \in \Big[L_p^\beta\ L_0, B_1'+B_2'M_g+B_3'M_{X_0}P_{X_0}, B_4'+B_5'(k_g+2kk_g') +$$

$$+ B_6'k_{X_0}M_g+B_7'M_g+B_8'M_{P_0}k_{X_0}+B_9'M_{X_0}k_{P_0}+B_{10}'M_{X_0}P_{X_0}\Big],$$

where B_i' $(i = 1,\dots,10)$ are positive constants.

L e m m a 5. If $\varphi^0(t) \in L_p^\beta$, $\varphi^2(t) \in L_p^\beta$, $X_1(t) \in H^\mu$, then $\varphi^1(t) \in L_p^\beta$.

P r o o f . Lemma 5 follows from Lemma 1 and Lemma 4. Hence we have

$$(31)\qquad \varphi^1(t) \in L_p^\beta\Big\{L, A_1M_g+M_{X_1}P_{X_1}+k_{X_1}(B_1'+B_2'M_g+B_3'M_{X_0}P_{X_0}),$$

$$A_2(k_g+2kk_g')+A_3k_{X_1}M_g+A_4M_g+M_{P_1}k_{X_1}+M_{X_1}k_{P_1} +$$

$$+ M_{X_1}\Big[B_4'+B_5'(k_g+2kk_g')+B_6'k_{X_0}M_g+B_7'M_g +$$

$$+ B_8'M_{P_0}k_{X_0}+B_9'M_{X_0}k_{P_0}+B_{10}'M_{X_0}P_{X_0}\Big]\Big\}.$$

From the lemmas given above it follows that the functions $\varphi^1(t), \varphi^2(z), \varphi^3(z), \varphi^4(z)$ belong to the class $L_p^\beta(L)$. Hence it suffices to make use of Schauder's theorem to show that there exists a solution of the equation which is the superposition of equations (24). To this aim we consider the functional space Λ consisting of all pairs of functions $\varphi^1(t) = [\varphi_1^1(t), \varphi_2^1(t)]$ integrable on the curve L with the power p. In this space the norm is defined by the formula

$$(32)\qquad \left\|\varphi^1(t)\right\| = \max_{j=1,2}\left(\int_L \left|\varphi_j^1(t)\right|^p dt\right)^{\frac{1}{p}}, \qquad p \in (1,+\infty).$$

The operations of addition and multiplication by a scalar are defined in the usual way. With this definition, Λ is a Banach space. In this space we consider the set $Z(\varrho,\varkappa)$ of points such that

$$(33)\qquad \varphi^1(t) \in L_p^{\beta}(L,\varrho,\varkappa), \quad \beta = \min(\alpha,\mu),$$

where ϱ and $\varkappa$ are arbitrary fixed positive constants. The set Z is convex and closed. Let us transform it by means of the formula

$$(34)\qquad \psi^1(t) = \varphi^0(t) + X_1(t)\tilde{\varphi}^2(t),$$

where $\tilde{\varphi}^2(t)$ is formed by substituting to the first of the equations (24) in place of $\varphi^2(t)$ the respective functions from the remaining three equations in (24). The formula (34) associates to every point $\varphi^1(t) \in Z$ a point $\psi^1(t) \in Z'$. We shall find conditions assuring that $Z' \subset Z$.

L e m m a 6. If the constant k_g' is sufficiently small, then Z' is a subset of Z.

P r o o f . From Lemma 5 we infer that the operation defined by (34) transforms the set Z into its subset Z', whenever the following inequalities hold

$$(35)\qquad \begin{cases} A_1M_g+M_{X_1}P_{X_1}+k_{X_1}(B_1'+B_2'M_g+B_3'M_{X_0}P_{X_0}) \leq \varrho, \\ A_2(k_g+2\varkappa k_g')+A_3k_{X_1}M_g+A_4M_g+M_{P_1}k_{X_1}+M_{X_1}k_{P_1} + \\ + M_{X_1}\Big[B_4'+B_5'(k_g+2\varkappa k_g')+B_6'k_{X_0}M_g+B_7'M_g + \\ + B_8'M_{P_0}k_{X_0}+B_9'M_{X_0}k_{P_0}+B_{10}'M_{X_0}P_{X_0}\Big] \leq \varkappa . \end{cases}$$

The inequalities (35) hold for any selection of the constants ϱ and $\varkappa$ if we have

(36) $$2k'_g(A_2+B'_5M_{X_1}) < 1,$$

i.e. if

(37) $$k'_g < \frac{1}{2(A_2+B_5M_{X_1})},$$

where A_2 and B'_5 are positive constants depending upon m_{X_1}, M_{X_1} and $M = \min\left(\frac{1}{2}, \frac{M_{X_1}}{2m_{X_1}}\right)$.

If the consition (37) holds, i.e. if the constant k'_g is sufficiently small, then the set Z' is a subset of Z.

L e m m a 7. The transformation (34) of the set Z is continuous with respect to the norm of the space Λ.

P r o o f . It suffices to show that if a sequence $\left\{\overset{(n)}{\varphi}{}^1(t)\right\} \in Z$ is convergent to a function $\varphi^1(t) \in Z$, then the sequence $\left\{\overset{(n)}{\psi}{}^1(t)\right\} \in Z$ is convergent in norm to the function $\psi^1(t) \in Z$ according to the formula (34), i.e.

(38) $$\overset{(n)}{\psi}{}^1(t) = \overset{(n)}{\varphi}{}^0(t) + X_1(t)\,\overset{(n)}{\tilde{\varphi}}{}^2(t).$$

Making use of the notation (21), the equation (22) and the assumption (3), we obtain

$$\left\|\overset{(n)}{\psi}{}^1(t)-\psi(t)\right\| = \left\|\overset{(n)}{\varphi}{}^0(t)-\varphi^0(t)\right\| + \left\|X_1(t)\right\|\cdot\left\|\overset{(n)}{\tilde{\varphi}}{}^2(t)-\tilde{\varphi}^2(t)\right\| =$$

$$= \left\| g_j\left[t, \overset{(n)}{\varphi}_1(t), \overset{(n)}{\varphi}_2(t)\right] - g_j\left[t,\varphi_1(t),\varphi_2(t)\right]\right\| +$$

$$+ \left\|\overset{(n)}{f}_j(t)-f_j(t)\right\| + \left\|X_1(t)\right\| \left\|\overset{(n)}{\tilde{\varphi}}{}^2(t)-\tilde{\varphi}^2(t)\right\| \leq$$

$$\leq \underset{(n>N_1)}{2k'_g\frac{\varepsilon}{6k'_g}} + \underset{(n>N_2)}{2k'_gM_1\frac{\varepsilon}{6k'_gM_1}} + \underset{(n>N_3)}{2k'_gM_2M_{X_1}\frac{\varepsilon}{6k'_gM_2M_{X_1}}} \leq \varepsilon$$

for $n = \max(N_1, N_2, N_3)$. M_1 and M_2 are positive constants depending upon m_{X_1}, M_{X_1}. Hence the transformation (38) is continuous in the norm of the space Λ.

L e m m a 8. The transformed set Z is compact.

P r o o f . From the inequalities (35) it follows that the functions $\psi^1(t)$ defined by (34) are jointly bounded and continuous, hence Z' is compact ([8], p.27).

We see that all the assumptions of Schauder's theorem are satisfied. Hence there exists, on the curve L, at least one solution $\overset{*}{\varphi}{}^1(t) \in L_p^{\beta}(L,\varrho,\varkappa)$ of the equation being the superposition of the equations (24). Substituting the solution $\overset{*}{\varphi}{}^1(t)$ into the equation (6) we obtain a function $\overset{*}{\phi}(z)$ which is sectionally analytic in D^+ and D^-, and its boundary values satisfy the conditions (1).

We can now summarize our results in the following theorem.

T h e o r e m . If the given functions $G(t)$, $g(t,u_1,u_2)$, $H(t_0), c(t_0)$ and the index of the problem (1) satisfy the assumptions I-V, and the constant k_g' of the problem satisfies the inequality (37), then there exists at least one solution $\phi(z)$, partly holomorphic in the domains D^+ and D^-, whose boundary values satisfy almost everywhere the boundary conditions (1).

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