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ON PARACONTACT MANIFOLDS

In [1] C.S.Hsu has obtained a necessary and sufficient condition for a (2n+1)-dimensional manifold to admit a (ϕ,ξ,η)-structure which is closely related to almost contact structure. An almost paracontact structure is defined and studied by I.Sato [2].

In the present paper we obtain certain properties of almost paracontact structure manifold and finally in the last part of the paper we obtain a necessary and sufficient condition for a manifold to admit an almost paracontact structure.

Let us consider an r-dimensional real manifold M_r of differentiability class C^∞. Let there exist in M_r:

(i) a (1,1) tensor ϕ,

(ii) a non-zero vector field ξ,

(iii) a non-zero 1-form η

which satisfy the following postulates:

(P_1) $$\phi(\xi) = 0,$$

(P_2) $$\phi^2(X) = X - \eta(X)\xi$$

for an arbitrary vector field X. Then we say that M_r in consideration has an almost paracontact structure and M_r is an almost paracontact manifold.

It can be easily shown that (P_1) and (P_2) imply:

(P_3) $$\eta(\xi) = 1,$$

(P_4) $$\eta\phi(X) = 0,$$

(P_5) $$\phi^3(X) = \phi(X),$$

(P_6) $$\phi^4(X) = \phi^2(X).$$

Firstly, we consider the solution of $\phi(X) = 0$. This equation implies $\phi^2(X) = 0$ or $\eta(X)\xi = X$, in consequence of (P_2). Thus the only solution of $\phi(X) = 0$ is $X = \xi$ up to a factor of proportionality. Hence the rank of ϕ is $r-1$.

We will now obtain the eigen values and eigen vectors of ϕ. Let μ be an eigen value of ϕ, the corresponding eigen vector being P. Then $\phi(P) = \mu P$ or $\phi^2(P) = \mu\phi(P) = = \mu^2 P$. Consequently using (P_2), we get $(1-\mu^2)P = \eta(P)\xi$.

Thus there are two cases:

Case I: $P = \xi$ up to a factor of proportionality. Then using (P_3), $\mu^2 = 0$. Consequently, there is a single eigen value 0, the corresponding eigen vector being ξ.

Case II: P and ξ are linearly independent. Then $\mu = \pm 1$ and $\eta(P) = 0$. Since the rank of ϕ is $r-1$, there are, say h, eigen values 1 and $r-h-1$ eigen values -1.

So, over the differentiable manifold M_r we have three distributions L, M and N of dimensions h, $r-h-1$ and 1 respectively corresponding to the eigen values 1, -1 and 0 respectively.

Let us now agree that if X, Y, Z, U occur in any equation, the equation stands for arbitrary vector fields X, Y, Z, U.

L e m m a : The distributions L, M and N are complementary distributions generated by complementary projection operators l, m and n respectively defined by

(1) $$2l \overset{\text{def}}{=} I + \phi - \eta \otimes \xi$$

(2) $$2m \overset{\text{def}}{=} I - \phi - \eta \otimes \xi$$

(3) $$n \overset{\text{def}}{=} \eta \otimes \xi$$

or equivalently

(4) $$2l \overset{\text{def}}{=\!=} \phi^2 + \phi,$$

(5) $$2m \overset{\text{def}}{=\!=} \phi^2 - \phi,$$

(6) $$n \overset{\text{def}}{=\!=} -\phi^2 + I,$$

where I is the identity (1.1) tensor.

P r o o f . First we show that l, m, n are complementary projection operators. We have

$$2l + 2m + 2n = \phi^2 + \phi + \phi^2 - \phi + 2I - 2\phi^2 = 2I$$

i.e. $l + m + n = I$. From (P_5) and (P_6) we have

$$4l^2 = 2l \cdot 2l = (\phi^2+\phi)(\phi^2+\phi) = \phi^4 + 2\phi^3 + \phi^2 = 2(\phi^2 + \phi) = 4l$$

i.e. $l^2 = l$.

Similarly,

$$m^2 = m$$

and

$$n^2 = n.$$

Further, $4lm = 2l(2m) = (\phi^2+\phi)(\phi^2-\phi) = \phi^4 - \phi^3 + \phi^3 - \phi^2 = 0$ i.e. $lm = 0$. From (P_5) and (P_6) we have

$$4ml = 2m \cdot 2l = (\phi^2-\phi)(\phi^2+\phi) = \phi^4 + \phi^3 - \phi^3 - \phi^2 = 0$$

i.e. $ml = 0$. Thus $lm = 0 = ml$.

Similarly

$$ln = 0 = nl$$

and

$$mn = 0 = nm.$$

Moreover, using (P_5) and (1) we get

$$2\phi l = \phi(2l) = \phi(\phi^2 + \phi) = \phi^3 + \phi^2 = 2l,$$

i.e. $\phi 1 = 1$. From (P_5) and (1) we have

$$2l\phi = (\phi^2+\phi)\phi = \phi^3 + \phi^2 = 2l,$$

i.e. $l\phi = l$. Thus

(7) $$l\phi = l = \phi l \quad \text{and} \quad \phi^2 l = l.$$

Similarly

(8) $$m\phi = -m = \phi m, \quad \phi^2 m = m$$

(9) $$n\phi = 0 = \phi n, \quad \phi^2 n = 0.$$

Thus l, m, n are complementary projection operators on M_r.

To complete the proof of the lemma we now show that L, M, N are the complementary distributions corresponding to the complementary projection operators l, m, n,: i.e.

(10) $$L = \left\{ lX : X \in \mathcal{F}_0^1(M) \right\}$$

(11) $$M = \left\{ mX : X \in \mathcal{F}_0^1(M) \right\}$$

(12) $$N = \left\{ nX : X \in \mathcal{F}_0^1(M) \right\}.$$

To prove (10), let

$$Z' \in \left\{ lX : X \in \mathcal{F}_0^1(M) \right\}.$$

Then we have

$$Z' = lX,$$

$$\phi Z' = \phi lX = lX = Z',$$

$$lX \in L.$$

Conversely, let $Z \in L$, then

$$\phi Z = Z.$$

Also

(13) $$Z = lZ + mZ + nZ = lZ + mZ + n\phi Z = lZ + mZ.$$

Again from (13)

(14) $$Z = l\phi Z + m\phi Z = lZ - mZ.$$

From (13) and (14), if $Z = lZ$ then $Z \in \left\{lX : X \in \mathcal{F}^1_0(M)\right\}$.
Hence $L = \left\{lX : X \in \mathcal{F}^1_0(m)\right\}$.
Similarly

$$M = \left\{mX : X \in \mathcal{F}^1_0(M)\right\}$$

and

$$N = \left\{nX : X \in \mathcal{F}^1_0(M)\right\}.$$

Thus, over the differentiable manifold M_r we have three complementary distributions L, M and N of dimensions h, r-h-1 and 1 generated by the complementary projection operators l, m and n respectively.

We will now prove our main theorem.

T h e o r e m . A necessary and sufficient condition for M_r to admit an almost paracontact structure is that there exist three complementary distributions L, M and N of dimensions h, r-h-1 and 1 respectively which together span a linear manifold of dimension r.

P r o o f . The necessary condition immediately follows from the lemma proved above.

The condition is sufficient: Suppose there are three complementary distributions L, M and N of dimensions h, r-h-1 and 1 respectively which together span M_r.

Let $\underset{x}{P}$; $x = 1,2,\dots,h$ and $\underset{a}{Q}$; $a = h+1,\dots,n-1$ be linearly independent basis vectors in L and M respectively and ξ be a vector in N. Then $\left\{\underset{x}{P}, \underset{a}{Q}, \xi\right\}$ is linearly independent.

Consequently, there exists an inverse set $\left\{\overset{x}{p}, \overset{a}{q}, \eta\right\}$ satisfying

$$\overset{y}{p}(\underset{x}{P}) = \delta^y_x, \quad \overset{y}{p}(\underset{a}{Q}) = 0, \quad \overset{y}{p}(\xi) = 0,$$

$$\overset{b}{q}(\underset{x}{P}) = 0, \quad \overset{b}{q}(\underset{a}{Q}) = \delta^b_a, \quad \overset{b}{q}(\xi) = 0,$$

$$\eta(\underset{x}{P}) = 0, \quad \eta(\underset{a}{Q}) = 0, \quad \eta(\xi) = 1$$

and

$$\overset{x}{p}(X)\underset{x}{P} + \overset{a}{q}(X)\underset{a}{Q} + \eta(X)\xi = X.$$

Let us put

$$\phi(X) = \overset{x}{p}(X)\underset{x}{P} + \epsilon \overset{a}{q}(X)\underset{a}{Q}, \quad \text{where} \quad \epsilon = \pm 1,$$

then by virtue of the above equations, we have

$$\eta\phi(X) = 0, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1$$

and

$$\phi^2(X) = \overset{x}{p}\, \phi(X)\underset{x}{P} + \epsilon\, \overset{a}{q}\phi(X)\underset{a}{Q} =$$

$$= \overset{x}{p}\left\{\overset{y}{p}(X)\underset{y}{P} + \epsilon\, \overset{b}{q}(X)\underset{b}{Q}\right\}\underset{x}{P} + \epsilon\, \overset{a}{q}\left\{\overset{y}{p}(X)\underset{y}{P} + \epsilon\, \overset{b}{q}(X)\underset{b}{Q}\right\}\underset{a}{Q} =$$

$$= \overset{x}{p}(X)\underset{x}{P} + \overset{a}{q}(X)\underset{a}{Q} = X - \eta(X)\xi\,.$$

Thus M_r admits an almost paracontact structure.
This proves the theorem.

R e m a r k s :

(i) Let

$$\phi_1(X) = \overset{x}{p}(X)\underset{x}{P} - \overset{a}{q}(X)\underset{a}{Q},$$

$$\phi_2(X) = \overset{x}{p}(X)\underset{x}{P} + \overset{a}{q}(X)\underset{a}{Q}.$$

In the latter case, we have

$$\phi^2(X) = \phi(X).$$

(ii) By virtue of (4), (5) and (6) we have

$$l + m = \phi^2,$$

$$l - m = \phi,$$

$$l + m + n = I.$$

(iii) Again from (7), (8) and (9) we obtain

$$\phi^{2r} l = l, \quad \phi^{2r+1} l = l,$$

$$\phi^{2r} m = m, \quad \phi^{2r+1} m = - m,$$

$\phi^r n = 0$ for every positive integer r.

REFERENCES

[1] C.J. H s u : Note on (ϕ, ξ, η)-structure. Tôhoku Math. J., 13 (1961) 434-442.

[2] I. S a t o : On structure similar to the almost contact structure. Tensor, NS., 30 (1976) 219-224.

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