DEMONSTRATIO MATHEMATICA
Vol. XII No 2 1979

K. D. Singh, Rakeshwar Singh

ON ALMOST PARA-CONTACT METRIC MANIFOLDS WITH SEMI-SYMMETRIC METRIC CONNECTIONS

Introduction

Semi-symmetric connections have been studied by various mathematicians including Yano [6], Mishra [1], Imai [2] and S.I.Hussain [4]. Recently I.Sato defined and studied almost paracontact manifolds [3] and has shown that it is similar to almost product manifolds.

In the present paper we study semi-symmetric metric connections on an almost para-contact manifold, in relation to a Riemannian connection. The later part of the paper is devoted to the study of curvature tensor and Nijenhuis tensor.

1. Preliminaries

Let M^n be an n-dimensional real differentiable manifold equipped with a C^∞-(1,1) tensor field f, a C^∞-vector field T, and C^∞-1 form A, satisfying

$$(1.1)\qquad \text{a)}\ \bar{\bar{X}} = X - A(X)T, \quad \text{where } \bar{X} \stackrel{\text{def}}{=\!=} f(X), \qquad \text{b)}\ A(T) = 1.$$

Then the structure (f, T, A) on M^n is called an almost paracontact structure [3] and M^n is said to be an almost para-contact manifold. It can be verified that on M^n the following holds

$$(1.2)\qquad \text{a)}\ \bar{T} = 0 \quad \text{b)}\ A(\bar{X}) = 0 \quad \text{c)}\ \operatorname{rank}(f) = n-1.$$

An almost para-contact manifold M^n with structure (f,T,A) always admits a positive definite Riemannian metric g [3] which satisfies

$$(1.3)\qquad \begin{aligned} &a)\ g(\overline{X},\overline{Y}) = g(X,Y) - A(X)A(Y),\\ &b)\ g(X,T) = A(X). \end{aligned}$$

M^n endowed with such a metric g is called almost para-contact metric manifold with structure (f,T,A,g).
From (1.3) a) it follows that

$$(1.4)\qquad g(\overline{\overline{X}},\overline{\overline{Y}}) = g(\overline{X},\overline{Y}).$$

If we put

$$(1.5)\qquad F(X,Y) \overset{def}{=\!=} g(\overline{X},Y),$$

then we have the following

$$(1.6)\qquad \begin{aligned} &a)\ F(X,Y) - F(Y,X) = 0,\\ &b)\ F(X,\overline{Y}) - F(\overline{X},Y) = 0,\\ &c)\ F(T,Y) = 0. \end{aligned}$$

A linear connection ∇ is said to be semi-symmetric connection on the almost para-contact manifold M^n if its torsion tensor

$$S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

satisfies the formula

$$(1.7)\qquad S(X,Y) = A(Y)X - A(X)Y.$$

∇ is said to be semi-symmetric metric connection with respect to the associated Riemannian metric g if

$$(1.8)\qquad \nabla_X g = 0.$$

We define ∇ to be a semi-symmetric metric f-connection iff in addition to (1.7) and (1.8) ∇ satisfies

(1.9) $$(\nabla_X f) = 0.$$

Suppose D is a Riemannian connection on M^n, then we can always put [6]

(2.0) $$\nabla_X Y = D_X Y + u(X,Y),$$

u being a tensor of (1.2) type satisfying

(2.1) $$g(u(X,Y),Z) = - g(u(X,Z),Y).$$

Obviously we have

(2.2) $$S(X,Y) = u(X,Y) - u(Y,X).$$

Yano [6] has expressed the value of $u(X,Y)$ in terms of S and S', both being tensors of (1.2) type as follows

(2.3) $$u(X,Y) = \frac{1}{2}\left\{S(X,Y) + S'(X,Y) + S'(Y,X)\right\},$$

where

(2.4) $$g(S(Z,X),Y) \overset{\text{def}}{=\!=} g(S'(X,Y),Z).$$

Thus (2.0) takes the form of

(2.5) $$\nabla_X Y = D_X Y + \frac{1}{2}\left\{S(X,Y) + S'(X,Y) + S'(Y,X)\right\}.$$

It can be verified that [6]

(2.6) $$S'(X,Y) = A(X)Y - g(X,Y)T$$

and thus we get

(2.7). $$\nabla_X Y = D_X Y + A(Y)X - g(X,Y)T.$$

It is easy to verify that

$$
(2.8)\qquad
\begin{aligned}
&a)\ u(X,Y) = S'(Y,X),\\
&b)\ g(S(X,Y),T) = 0,\\
&c)\ u(X,T) = S(X,T) = S'(T,X) = \bar{\bar{X}},\\
&d)\ S'(X,Y) - S'(Y,X) = S(Y,X).
\end{aligned}
$$

T h e o r e m 1.1. In an almost para-contact manifold M^n, the torsion tensor of the semi-symmetric metric connection satisfies the following identities

$$
(2.9)\qquad
\begin{aligned}
&a)\ S(X,T) = \bar{X},\\
&b)\ S(\bar{\bar{X}},T) - S(X,T) = 0,\\
&c)\ S(\bar{\bar{X}},Y) = A(Y)X - A(X)A(Y)T,\\
&d)\ S(\bar{\bar{X}},Y) + S(X,\bar{\bar{Y}}) = S(X,Y),\\
&e)\ A(S(X,Y)) = 0,\\
&f)\ \overline{\overline{S(X,Y)}} = S(X,Y).
\end{aligned}
$$

Now we wish to establish certain identities among the (0,3) type tensors defined by [4]

$$
(3.0)\qquad
\begin{aligned}
S'(X,Y,Z) &\overset{\text{def}}{=} g(S(X,Y),Z),\\
u'(X,Y,Z) &\overset{\text{def}}{=} g(u(X,Y),Z)
\end{aligned}
$$

or equivalently

$$S'(X,Y,Z) = \begin{vmatrix} g(Y,T) & g(X,T) \\ g(Y,Z) & g(X,Z) \end{vmatrix}$$

and

$$u'(X,Y,Z) = \begin{vmatrix} g(Y,T) & g(Z,T) \\ g(X,Y) & g(X,Z) \end{vmatrix} .$$

T h e o r e m 1.2. The following relations hold in an almost para-contact metric manifolds

$$(3.1)\quad \begin{aligned} &a)\ u'(X,\overline{Y},\overline{Z}) = s'(\overline{X},\overline{Y},Z) = 0,\\ &b)\ u'(X,Y,Z) = S'(Z,Y,X),\\ &c)\ u'(X,Y,Z) = -u'(X,Z,Y),\\ &d)\ s'(X,Y,Z) = -S'(Y,X,Z),\\ &e)\ S'(X,Y,Z) - S'(X,Z,Y) = u'(X,Y,Z),\\ &f)\ u'(\overline{X},Y,Z) - u'(X,\overline{Y},Z) - u'(X,Y,\overline{Z}) = 0,\\ &g)\ u'(\overline{\overline{X}},Y,Z) - u'(X,Y,Z) = 0. \end{aligned}$$

T h e o r e m 1.3. The connections ∇, D and the (0,3) type tensor u' of the almost para-contact metric manifold (f,T,A,g) are related by the following

$$(3.2)\quad \begin{aligned} &a)\ (\nabla_X F)(Y,Z) = (D_X F)(Y,Z) + u'(X,\overline{Y},Z) - u'(X,Y,\overline{Z}),\\ &b)\ (\nabla_X F)(Y,Z) = (D_X F)(\overline{Y},\overline{Z}). \end{aligned}$$

The proof is an easy consequence of (1.5), (1.6)a) and (2.0).

C o r o l l a r y . It follows from (3.2)a) that

$$(\nabla_X F)(Y,Z) = (D_X F)(Y,Z)$$

iff

$$(3.3)\qquad u'(X,\overline{Y},Z) = u'(X,Y,\overline{Z}).$$

T h e o r e m 1.4. We have

$$(3.4)\qquad \nabla_X\overline{Y} = D_X\overline{Y} - F(X,Y)T$$

and

$$(3.5)\qquad \overline{\nabla_X\overline{Y}} = \overline{(D_X\overline{Y})}.$$

2. The curvature tensor

We denote by R and K the curvature tensors of the semi-symmetric metric connection ∇ and the Riemannian connection D respectively i.e.

$$(3.6)\quad \begin{aligned} &a)\ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ &b)\ K(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \end{aligned}$$

Then we state the following theorem.

Theorem 2.1. The two curvature tensors are related by the following equation

$$\begin{aligned} (3.7)\quad R(X,Y)Z = {} & K(X,Y)Z + A(D_Y Z)X - A(D_X Z)Y + g(Y,D_X Z) - \\ & - g(X,D_Y Z) + X(A(Z))Y - Y(A(Z))X + \\ & + A(Z)S(X,Y) + \{Yg(X,Z) - Xg(Y,Z)\} T + \\ & + g(X,Z)D_Y T - g(Y,Z)D_X T + g(X,Z)Y + \\ & - g(Y,Z)X + g(Y,Z)A(X)T - g(X,Z)A(Y)T + \\ & + g([X,Y],Z)T. \end{aligned}$$

3. The Nijenhuis tensor

In this section we study the Nijenhuis tensor in relation to the semi-symmetric metric connection and establish various identities involving it. The Nijenhuis tensor is defined by

$$(3.8)\quad \begin{aligned} &a)\ N(X,Y) = [\bar{X},\bar{Y}] + \overline{\overline{[X,Y]}} - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]}, \\ &\text{or} \\ &b)\ N(X,Y) = [\bar{X},\bar{Y}] + [X,Y] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]} - A([X,Y])T. \end{aligned}$$

If we put

$$(3.9)\quad \begin{aligned} &a)\ B(X,Y) \overset{\text{def}}{=\!=} \overline{\overline{[X,Y]}} + [X,Y], \\ &b)\ W(X,Y) \overset{\text{def}}{=\!=} \overline{[\bar{X},Y]} + \overline{[X,\bar{Y}]}. \end{aligned}$$

Then (3.8)a) reduces to

$$N(X,Y) = B(X,Y) - W(X,Y). \tag{4.0}$$

Further if we put

$$(4.1)\qquad \begin{aligned} &a)\ B(X,Y,Z) \overset{def}{=\!=} g(B(X,Y),Z),\\ &b)\ W(X,Y,Z) \overset{def}{=\!=} g(W(X,Y),Z),\\ &c)\ N(X,Y,Z) \overset{def}{=\!=} g(W(X,Y),Z) \end{aligned}$$

then it is evident from the definitions that

$$N(X,Y,Z) = B(X,Y,Z) - W(X,Y,Z). \tag{4.2}$$

Theorem 3.1. The Nijenhuis tensor N defined on M^n with the Riemannian connection D satisfies the following identity

$$N(X,Y) = (D_{\bar X} f)(Y) - (D_{\bar Y} f)(X) - \overline{(D_X f)(Y)} + \overline{(D_Y f)(X)}. \tag{4.3}$$

Theorem 3.2. $B(X,Y)$ defined by (3.9)a) satisfies the following equation

$$B(X,Y) = \nabla_{\bar X}\bar Y - \nabla_{\bar Y}\bar X + \nabla_X Y - \nabla_Y X - A([X,Y])T - S(X,Y). \tag{4.4}$$

Remark 3.1. Let ∇ be semi-symmetric metric f-connection over M^n. Then from (4.4) it follows that $B(X,Y)$ takes the form of

$$B(X,Y) = [X,Y] + A([X,Y])T + \overline{(\nabla_{\bar X} Y - \nabla_{\bar Y} X)}. \tag{4.5}$$

Theorem 3.3. The following relations always hold on an almost para-contact metric manifold

$$
(4.6)\quad
\begin{aligned}
&a)\ B(\bar{X},Y) = [\bar{X},Y] + [X,\bar{Y}] - A([\bar{X},Y])T - \\
&\qquad - A(X)\ [T,\bar{Y}] + \bar{Y}(A(X))T, \\
&b)\ B(X,\bar{Y}) = [X,\bar{Y}] + [\bar{X},Y] - A([X,\bar{Y}])T - \\
&\qquad - A(Y)[\bar{X},Y] + \bar{X}(A(Y))T, \\
&c)\ B(\bar{X},\bar{Y}) = [X,Y] + [\bar{X},\bar{Y}] - A([\bar{X},\bar{Y}])T - A(X)[T,Y] - \\
&\qquad - A(Y)[X,T] + A(X)(T(A(Y)))T - \\
&\qquad - A(Y)(T(A(X)))T - X(A(Y))T + Y(A(X))T.
\end{aligned}
$$

The proof of above follows from (3.8)a) and (1.1)a). As a consequence of Theorem 3.3 and (3.8)a) we can state the following theorem.

T h e o r e m 3.4a). The following identities are satisfied on M^n with the structure (f,T,A,g)

$$
(4.7)\quad
\begin{aligned}
&a)\ B(\bar{X},Y) - B(X,\bar{Y}) = \{A([X,\bar{Y}]) - A([\bar{X},Y])\}T + \\
&\qquad + A(Y)[\bar{X},T] - A(X)[T,\bar{Y}] + \bar{Y}(A(X))T - \bar{X}(A(Y))T, \\
&b)\ B(\bar{X},\bar{Y}) - B(X,Y) = A([X,Y]) - A[\bar{X},\bar{Y}]T - \\
&\qquad - A(X)[T,Y] - A(Y)[X,T] + A(X)T(A(Y))T - \\
&\qquad - A(Y)T(A(X))T - X(A(Y))T + Y(A(X))T, \\
&c)\ N(X,Y) = B(X,Y) - \overline{B(\bar{X},Y)} - A(X)\overline{[T,\bar{Y}]}.
\end{aligned}
$$

From (4.0) we find that if in an almost para-contact structure $B(X,Y) = 0$ then Nijenhuis tensor is of the form $N(X,Y) = -W(X,Y)$ on the other hand, from (4.7)c) we have

$$N(X,Y) = -A(X)[T,\bar{Y}]$$

and therefore, in view of these we state the following theorem.

T h e o r e m 3.4b). A necessary condition for the almost para-contact structure to satisfy $B(X,Y) = 0$ is that

$$W(X,Y) = A(X)\left[T,\bar{Y}\right].$$

It is easy to verify that

$$(4.8)\quad \begin{aligned} &a)\ A(B(X,Y)) = A(\left[\bar{X},\bar{Y}\right]),\\ &b)\ A(B(X,Y)) = 0,\\ &c)\ A(N(X,Y)) = A(\left[\bar{X},\bar{Y}\right]) = A(B(X,Y)). \end{aligned}$$

Moreover we have

$$(4.9)\quad \begin{aligned} &a)\ A(W(X,Y)) = 0,\\ &b)\ W(X,Y) - \overline{B(\bar{X},Y)} = A(X)\overline{\left[T,\bar{Y}\right]}. \end{aligned}$$

T h e o r e m 3.5. We have

$$(5.0)\quad \begin{aligned} N(X;Y) = A(X)\left\{N(T,Y) + \overline{\left[T,\bar{Y}\right]}\right\} + \\ + A(\left[\bar{X},\bar{Y}\right])T - W(X,Y) + \overline{W(\bar{X},Y)}. \end{aligned}$$

T h e o r e m 3.6. In an almost para-contact metric structure (f,T,A,g) $W(X,Y) = 0$ if

$$(5.1)\quad B(X,Y) = A(X)\left\{N(T,Y) + \overline{\left[T,\bar{Y}\right]}\right\} + A(\left[\bar{X},\bar{Y}\right])T.$$

From (4.9)(b), (5.0) reduces to

$$(5.2)\quad N(X,Y) = A(X)N(T,Y) - \overline{B(\bar{X},Y)} + \overline{W(\bar{X},Y)} + A(\left[\bar{X},\bar{Y}\right])T.$$

Now it is clear that Nijenhuis tensor must vanish in order that $B(X,Y) = 0$ as well as $W(X,Y) = 0$.

But from (5.2) it follows that if $B(X,Y) = 0 = W(X,Y)$, then

$$N(X,Y) = A(X)N(T,Y) + A(\left[\bar{X},\bar{Y}\right])T.$$

Therefore, we have

$$A([\bar{X},\bar{Y}])T = 0$$

or

$$A([\bar{X},\bar{Y}]) = 0.$$

Thus we have the following theorem.

Theorem 3.7. In order that in an almost para-contact metric structure (f,T,A,g), $B(X,Y) = 0 = W(X,Y)$ it is necessary for the 1-form A to satisfy

$$A([\bar{X},\bar{Y}]) = 0 \quad \text{for all} \quad X,Y.$$

Theorem 3.8. An almost para-contact metric structure with semi symmetric metric f-connection has vanishing Nijenhuis tensor.

Proof. From (4.3) and (2.7) we have

$$N(X,Y) = \nabla_{\bar{X}}fY - f\nabla_{\bar{X}}Y - \nabla_{\bar{Y}}fX + f\nabla_{\bar{Y}}X - \overline{\nabla_X fY} +$$

$$+ \overline{f\nabla_X Y} + \overline{\nabla_Y fX} - \overline{f\nabla_Y X} =$$

$$= (\nabla_{\bar{X}}f)(Y) - (\nabla_{\bar{Y}}f)(X) - \overline{(\nabla_X f)(Y)} + \overline{(\nabla_Y f)(X)} = 0,$$

in view of ∇ being semi-symmetric metric f-connection. Thus the Nijenhuis tensor vanishes.

REFERENCES

[1] R.S. Mishra : Affine connections in an almost Grayan manifolds. Tensor, NS, 23 (1972) 317-322.

[2] T. Imai : Notes on semi-symmetric metric connection, Tensor, NS, 24 (1972) 293-296.

[3] I. Sato : On a structure similar to almost contact structure. Tensor, NS, 30 (1976) 219-224.

[4] A. Sharfuddin, S.I. Husain: Semi-symmetric metric connections on almost contact manifolds. Tensor, NS, 30 (1976) 133-139.
[5] N.J. Hicks: Notes on differential geometry. Princeton 1965.
[6] K. Yano: On semi-symmetric metric connections, Rev. Roumaine Math. Pures Appl. 15 (1970) 1579-1586.

DEPARTMENT OF MATHEMATICS AND ASTRONOMY, LUCKNOW UNIVERSITY, LUCKNOW (INDIA)
Received October 5, 1977.