DEMONSTRATIO MATHEMATICA
Vol. XII No 2 1979

Tadeusz Pruszko

OPERATIONAL EQUATIONS IN SPACE B_0^* AND BOUNDARY VALUE PROBLEMS

It is proved in [5] that any linear value problem $x' = f(t,x)$, $lx=r$ has exactly one solution (under some assumptions on f) in the Banach space of continuous functions on a compact interval. Making use of Bittner's operational calculus $CO(L^1, L^0, S, T, s)$ where

(i) L^0 is a B_0 space, L^1 is a B_0^* space;

(ii) $S: L^1 \to L^0$, $T: L^0 \to L^1$ are algebraically linear operations such that T is continuous and $S \circ T = id_{L^0}$;

(iii) the projection $s: L^1 \to \operatorname{Ker} S$, called a limit condition, given by $sx = x - T \circ Sx$ is a continuous operation,

one can formulate a type of "boundary value problem" as follows

$$(1) \qquad \begin{aligned} Sx &= f(x) \\ lx &= c \qquad (c \in \operatorname{Ker} S). \end{aligned}$$

Since the theorem about open mappings is true also for locally convex topological spaces we are able to prove that the problem (1) posseses exactly one solution in the space B_0^*. This is certainly a generalization of the A. Lasota's and Z. Opial's result ([5]) that permits to use the space of all continuous functions on the interval $[0,\infty)$, with a sequence of semi-norms $q_k(x) = \sup\{|x(t)| : t \in [0,k]\}$ $(k = 1,2,\ldots)$. As a result we obtain the existence and uniqueness for the system of differential equations

$$x' = f(t,x)$$

with the boundary value condition $x_\infty + \lambda \cdot x(o) = r$ where $\lambda \in R$, $r \in R^n$, $x_\infty = \lim_{t \to \infty} x(t)$.

1. The theorem on the existence and uniqueness of the solution of problem (1)

If E is a B_0 space then $nv(E)$ denotes the family of all nonempty subsets of E. A mapping $H : E \to nv(E)$ will be called upper semicontinuous if its graph $\{(x,y) : y \in H(x)\}$ is closed in $E \times E$, and compact if, for any bounded subset X of E the closure of the set $\bigcup_{x \in X} H(x)$ is compact in E. An upper semicontinuous and compact mapping $H : E \to nv(E)$ will be called completely continuous.

Let $\{q_n\}$ be a family of semi-norms in L^1 such that if $q_n(x) = 0$ for $n = 1,2,\ldots,$ then $x = 0$. We shall denote by $\bar{L}^1$ the complementation of L^1 in the paranorm

$$|x| = \sum_{n=1} 2^{-n} \frac{q_n(x)}{1 + q_n(x)} .$$

In the proof of Theorem 1 we shall make use of the following Lemma which is an immediate corollary from Theorem 10 of [7].

L e m m a 1. Let E be a B_0 space and let $g : E \to E$ be a mapping of the form $g = I - h$ where I is the identity on E, whereas $h : E \to E$ is a compact map. Then if $g : E \to g(E)$ is one-to-one mapping, then $g : E \to E$ is open.

T h e o r e m 1. Assume that $\dim(\mathrm{Ker}S) < \infty$ and some mappings $l : \bar{L}^1 \to \mathrm{Ker}S$, $f\ \bar{L}^1 \to L^o$, $F : \bar{L}^1 \to nv(L^o)$ satisfy the following conditions:

(a) $T \circ F : \bar{L}^1 \to nv(\bar{L}^1)$, where $T \circ F(x) = T(F(x))$ for each $x \in \bar{L}^1$, is completely continuous mapping

(b) f is a continuous mapping and $f(x_1) - f(x_2) \in F(x_1 - x_2)$ for any $x_1, x_2 \in \bar{L}^1$

(c) l is a linear continuous operation.

Then, if $x = 0$ is the unique solution of the problem

$$(2) \qquad S(x) \in F(x), \quad lx = 0$$

then in the space L^1 there exists exactly one solution of (1).

P r o o f . As the solution of problem (1) is equivalent to the solution of the following equation:

$$(x,c) = (T\circ f(x)+c, c+c_0-lx)$$

therefore the theorem will be proved if we demonstrate that the mapping $g : \bar{L}^1 x \operatorname{Ker}S \to \bar{L}^1 x \operatorname{Ker}S$ given by the formula

$$g(x,c) = (x,c) - (T\circ f(x) + c,\ c + c_0 - lx)$$

is a homeomorphism.

If the mapping $h : \bar{L}^1 x \operatorname{Ker}S \to \bar{L}^1 x \operatorname{Ker}S$ is defined by the formula

$$h(x,c) = (T\circ f(x) + c,\ c + c_0 - lx),$$

then the mapping g has the form $g = I - h$, where I is identity on $\bar{L}^1 x \operatorname{Ker}S$.

1^o. Let us notice in the first place that $g : \bar{L}^1 x \operatorname{Ker}S \to g(\bar{L}^1 x \operatorname{Ker}S)$ is a one-to-one mapping. Indeed, let $g(x_1,c_1) = g(x_2,c_2)$. Then it follows from the definition of g that

$$(x_1-x_2, c_1-c_2) = (T\left[f(x_1)-f(x_2)\right]+c_1-c_2,\ c_1-c_2 - l(x_1-x_2))$$

and this equality is equivalent to the following system of two equations

$$(1.1) \qquad x_1-x_2 = T\left[f(x_1) - f(x_2)\right] + c_1-c_2$$

$$(1.2) \qquad l(x_1-x_2) = 0.$$

As, according to our assumption (b) $f(x_1) - f(x_2) \in F(x_1-x_2)$, therefore it follows from (1.1) that

$$(1.3) \qquad x_1-x_2 \in T\circ F(x_1-x_2) + c_1-c_2.$$

Hence we get from (1.2) and (1.3) $S(x_1-x_2) \in F(x_1-x_2)$, $l(x_1-x_2) = 0$. By our assumptions problem (2) has only a zero solution and hence we obtain from condition (1.1) the equality

$$c_2-c_1 = s(c_2-c_1) = s\circ T\left[f(x_1)-f(x_2)\right] = 0$$

then we have $(x_1,c_1) = (x_2,c_2)$. Consequently $g: \bar{L}^1 x \operatorname{KerS} \longrightarrow g(\bar{L}^1 x \operatorname{KerS})$ is a one-to-one mapping.

2^0 Next we shall show that h is a compact mapping, i.e. that for any arbitrary bounded set $X \subset \bar{L}^1 x \operatorname{KerS}$ the closure of the set $h(X)$ is compact in $\bar{L}^1 x \operatorname{KerS}$.

Let $(y_n,\bar{c}_n) \in h(X)$ $(n=1,2,\ldots)$. We construct a sequence

$$(1.4) \qquad \left\{h(x_n,c_n) - h(0,0)\right\} = \left\{\left(T\left[f(x_n)-f(0)\right] + c_n,\ c_n-lx_n\right)\right\},$$

where $(x_n,c_n) \in X$ and $h(x_n,c_n) = (y_n,\bar{c}_n)$ $(n=1,2,\ldots)$. As the sequence $\left\{(x_n,c_n)\right\}$ is bounded, therefore the sequences $\{x_n\}$ and $\{c_n\}$ are bounded, too, in $\bar{L}^1$ and KerS respectively. It follows from assumption (b) that $T\left[f(x_n)-f(0)\right] \in \bigcup_{n=1}^{\infty} T\circ F(x_n)$. By assumption (a) the mapping $T\circ F$ is compact, and therefore the sequence $\left\{T\left[f(x_n)-f(0)\right]\right\}$ contains a convergent subsequence. As $\dim(\operatorname{KerS}) < \infty$, therefore the bounded sequence $\left\{c_n\right\}$ contains a convergent subsequence. The linear and continuous operation l maps the bounded sequence $\left\{x_n\right\}$ into the bounded sequence $\left\{lx_n\right\}$ where $lx_n \in \operatorname{KerS}$ $(n=1,2,\ldots)$. We may assume now, without any loss of generality, that $\left\{T\left[f(x_n)\right]\right\}$, $\left\{c_n\right\}$, and $\left\{lx_n\right\}$ are convergent. It follows therefore from (1.4) that the sequence $\left\{h(x_n,c_n)\right\}$ is convergent. Therefore the closure of $h(X)$ is compact an consequently the mapping h is compact.

Now in view of Lemma 1 g is open.

3^o. Let $V(0,0)$ be such a neighbourhood of zero in $\bar{L}^1 \times \mathrm{KerS}$ that

$$(1.5)\qquad p_n(x,c) < n \quad \text{for} \quad (x,c) \in V(0,0) \ (n=1,2,\dots),$$

where p_n is a sequence of seminorms in $\bar{L}^1 \times \mathrm{KerS}$ such that $p_n < p_{n-1}$ $(n=1,2,\dots)$. Let $\partial V(0,0)$ be the boundary of the neighborhood $V(0,0)$. We shall show the existence of such a neighborhood $W(0,0)$ that condition

$$(1.6)\qquad (y,\bar{c}) \in \partial V(x,c) \Longrightarrow g(y,\bar{c}) \notin W(g(x,c))$$

holds for $(x,c) \in \bar{L}^1 \times \mathrm{KerS}$.

Assume that condition (1.6) is false. Then for any neighborhood $W_n(0,0) = \left\{(x,c) : p_n(x,c) < \frac{1}{n}\right\}$ $(n=1,2,\dots)$ there exist such elements $(x_n,c_n) \in \bar{L}^1 \times \mathrm{KerS}$ and $(y_n,\bar{c}_n) \in \partial V(x_n,c_n)$ that

$$(1.7)\qquad g(y_n,\bar{c}_n) \in W_n(g(x_n,c_n)),$$

where

$$g(y_n,\bar{c}_n) = (y_n,c_n) - h(y_n,\bar{c}_n), \quad g(x_n,c_n) = (x_n,c_n) - h(x_n,c_n).$$

Therefore

$$(1.8)\qquad (y_n-x_n,\bar{c}_n-c_n) = h(y_n,\bar{c}_n) - h(x_n,c_n) + g(y_n,\bar{c}_n) - g(x_n,c_n).$$

Condition (1.7) is equivalent to the following condition

$$g(y_n,c_n) - g(x_n,c_n) \in W_n(0,0)$$

and therefore

$$p_n\left(g(y_n,\bar{c}_n) - g(x_n,c_n)\right) < \frac{1}{n} \qquad (n=1,2,\dots).$$

Hence

$$(1.9)\qquad \lim_{n\to\infty} \left[g(y_n,\bar{c}_n) - g(x_n,c_n)\right] = 0.$$

We get from the definition of h that

(1.10) $h(y_n,\bar{c}_n)-h(x_n,c_n) = \Big(T\big[f(y_n)-f(x_n)\big]+\bar{c}_n-c_n,\bar{c}_n-c_n-l(y_n-x_n)\Big)$.

As $(y_n,\bar{c}_n)-(x_n,c_n) \in \partial V(0,0)$ $(n=1,2,\ldots)$, therefore by (1.5) the sequence $\{(y_n-x_n,\bar{c}_n-c_n)\}$ is bounded in $\bar{L}^1 \times \operatorname{Ker} S$. Now, just as in part 2° we notice that the sequences $\{l(x_n-y_n)\}$, $\{\bar{c}_n-c_n\}$, $\{T[f(y_n)-f(x_n)]\}$ include convergent subsequences. Therefore without any loss of generality we can assume that the sequence $(u_n,\tilde{c}_n) = \{h(y_n,\bar{c}_n)-h(x_n,c_n)\}$ is convergent. Let

(1.12) $$\lim_{n\to\infty}(u_n,\tilde{c}_n) = (u,\tilde{c}).$$

It follows from conditions (1.8), (1.9), (1.12) that

(1.13) $$\lim_{n\to\infty}(y_n-x_n,\bar{c}_n-c_n) = (u,\bar{c})$$

and that

$$(u,\tilde{c}) = \lim_{n\to\infty}\Big(T\big[f(y_n)-f(x_n)\big]+\bar{c},\ \bar{c}-lu\Big).$$

Put

(1.14) $$z = \lim_{n\to\infty} T\big[f(y_n) - f(x_n)\big].$$

By assumption (b) we have

(1.15) $$T\big[f(y_n) - f(x_n)\big] \in T\circ F(y_n-x_n) \quad (n=1,2,\ldots)$$

and as the mapping $T\circ F$ is upper semicontinuous, we get from conditions (1.13), (1.14), (1.15)

(1.16) $$z \in T\circ F(u).$$

Next, taking the limits in (1.11), we get

(1.17) $$(u,\tilde{c}) = (z+\bar{c},\bar{c}-lu).$$

Therefore from (1.16) and (1.17) we get

$$Su \in F(u), \quad lu = 0.$$

As by our assumptions problem (2) has only a zero solution, therefore $u = 0$. It follows from (1.16) that $sz = 0$. By (1.17) $sz+\tilde{c} = 0$ therefore $\tilde{c} = 0$. Hence $(u,\tilde{c}) = (0,0)$.

From the other side we have $(y_n-x_n,\bar{c}_n-c_n) \in \partial V(0,0)$ $(n=1,2,\dots)$ and therefore $P_V(u,c) = 1$ where P_V is the Minkowski functional for the neighborhood $V(0,0)$. This contradicts the fact that $(u,c) = (0,0)$. Thus condition (1.16) has been proved.

4°. Finally we shall that $\operatorname{Im} g = \bar{L}^1 x \operatorname{Ker} S$. It follows from (1.6) and from the fact that g is an open mapping we get

$$W(g(x,c)) \subset g(V(x,c)). \tag{1.18}$$

Assume that $\bar{L}^1 x \operatorname{Ker} S \setminus \operatorname{Im} g \neq \emptyset$. As g is open, therefore there exists an element $(y,\bar{c}) \in \bar{L}^1 x \operatorname{Ker} S$ such that $(y,\bar{c}) \notin \operatorname{Im} g$. and $(y,\bar{c}) \in \overline{\operatorname{Im} g}$. Since $(y,\bar{c}) \in \overline{\operatorname{Im} g}$ there exists an element $(y_0,\bar{c}_0) \in \operatorname{Im} g$ such that $(y,\bar{c}) \in W(y_0,\bar{c}_0)$. Assume $g(x_0,c_0) = (y_0,\bar{c}_0)$. From condition (1.18) we obtain the relation $W(y_0,\bar{c}_0) \subset g(V(x_0,c_0))$. As $(y,\bar{c}) \in W(y_0,\bar{c}_0)$ we have $(y,\bar{c}) \in g(V(x_0,c_0))$.

Thus we obtained a contradiction with the condition $(y,c) \notin \operatorname{Im} g$. Therefore g is a surjection. q.e.d.

2. Certain boundary value problem for the equation $x' = f(t,x)$ in the unlimited interval

Let L^1 be the space of all absolutely continuous functions $x : [0,\infty) \to R^m$ with the sequence of seminorms

$$q_k(x) = \sup\{|x(t)| : t \in [0,k]\} \qquad (k=1,2,\dots).$$

It can be easily seen that in this case $\bar{L}^1$ is the space of all continuous functions $C[0,\infty)$.

Let L^0 be the space of such functions $y : (0, \infty) \to R^m$ that if $y = (y_1, y_2, \dots, y_m)$ then the functions $y_i : (0, \infty) \to R^1$ are locally integrable. In L^0 we define a sequence of semi-norms

$$p_k(y) = \max \left\{ \int_0^k |y_i(t)| dt \ : \ i=1,2,\dots,m \right\} \qquad (k=1,2,\dots).$$

The derivative operation and the integral we define as follows

$$S = \frac{d}{dt}, \quad T = \int_0^t .$$

Then $sx = (x(0))$ and the space KerS is isomorphic with R^m. We can easily verify that the operational calculus defined in this way satisfies conditions (i)-(iii).

Let $M \subset C[0, \infty)$ be a subspace of functions possessing a finite limit $\lim_{t \to \infty} x(t) = x_\infty$ and let $\lambda \in R^1$. On the space M we define a continuous linear operation $\tilde{l} : M \to R^m$, $\tilde{l}x = x_\infty + \lambda x(0)$. We infer easily from the Han-Banach's Theorem that there exists a continuous linear operation $l : C[0, \infty) \to R^m$ such that $lx = \tilde{l}x$ for $x \in M$.

In the defined model of the operational calculus we shall prove the following

L e m m a 2. If $\varphi : (0, \infty) \to R^1$ is a non-negative integrable function and if the mapping $F : C[0, \infty) \to nv(L^0)$ is defined by the formula

$$F(x) = \left\{ y \in L^0 \ : \ |y(t)| \leqslant \varphi(t) |x(t)| \ \text{ a.e. on } \ (0, \infty) \right\}$$

then the mapping $T \circ F : C[0, \infty) \to nv(C[0, \infty))$, $T \circ F(x) = T(F(x))$ is completely continuous.

P r o o f . In the first place we shall show that $T \circ F$ is a compact. Let $X \subset C[0, \infty)$ be a bounded set. We shall show that the closure of the set $\bigcup_{x \in X} T \circ F(x)$ is compact in $C[0, \infty)$. As X is bounded, therefore for any positive integer k there exists such a positive number N_k that $\sup\{q_k(x) : x \in X\} \leq N_k$. Thus in every integral $[0,k]$ $(k=1,2,\dots)$ the functions belonging to the set $Y = \bigcup_{x\in X} T\circ F(x)$ are all bounded by the constant $N_k \int_0^k \varphi(t)dt$ and are equicontinuous, because for $t_1, t_2 \in [0,k]$ we have

$$\left| \int_0^{t_1} y(z)dz - \int_0^{t_2} y(z)dz \right| \leq N_k \int_{t_2}^{t_1} \varphi(z)dz \quad \text{for} \quad y \in Y.$$

Let $\{y_n\}$ be some sequence of functions belonging to Y. By Ascoli's theorem sequence $\{y_n\}$ contains for any $k=1,2,\dots$ a subsequence $\{y_{n_i}^k\}_{i=1,2,\dots}$ convergent in the seminorm q_k where $\{y_{n_i}^{k+1}\}$ is a subsequence of $\{y_{n_i}^k\}$ for any k. Now we can select by means of the diagonal method the subsequence $\{y_{n_i}^i\}_{i=1,2,\dots}$ of $\{y_n\}$ convergent in any seminorm q_k $(k=1,2,\dots)$. Thus we have shown that the closure of Y is compact. Hence $T\circ F$ is compact, too.

Next we shall show that the mapping $T\circ F$ is upper semicontinuous, i.e. that the conditions

(2.1) $\qquad x_n \to x, \quad u_n \to u, \quad u_n \in T\circ F(x_n) \quad (n=1,2,\dots)$

imply

$$u \in T\circ F(x).$$

Let

(2.2) $$u_n = \int_0^t y_n(z)dz,$$

where

(2.3) $\qquad y_n \in F(x_n)$, i.e. $|y_n(t)| \leq \varphi(t)|x_n(t)|$ a.e. on $(0,\infty)$.

As the sequence $\{x_n\}$ is bounded in $C[0,\infty)$ therefore the function $h(t) = \sup\{|x_n(t)| : n=1,2,\dots\}$ $t \in [0,\infty)$ is locally integrable.

Now from (2.3) we have

$$(2.4) \qquad |y_n(t)| \leqslant \varphi(t)h(t) \quad \text{a.e. on } (0,\infty).$$

Let $L_1(0,k)$ denote the space of integrable functions $v:(0,k) \to R^m$ and let $P_k: L^0 \to L_1(0,k)$ be a continuous linear operation defined by the formula $P_k(y)(t) = y(t)$ for $t \in (0,k)$. Since for the sequence $\{y_n\}$ condition (2.4) holds then in the space L^0 there exists a function y such that the sequence $\{P_k y_n\}$ contains a subsequence $\{P_k y^k_{n_i}\}_{i=1}^{\infty}$ weakly convergent to the $P_k y$ in the space $L_1(0,k)$ for $k=1,2,\dots$ (cf. [3] th.IV.8.9 and [8] th.4.2). Moreover, let the sequence $\{P_{k+1} y^{k+1}_{n_i}\}_{i=1}^{\infty}$ be a subsequence of $\{P_k y^k_{n_k}\}_{i=1}^{\infty}$ for $k=1,2,\dots$.

Now we can select by means of the diagonal method the sequence $\{y^i_{n_i}\}_{i=1}^{\infty}$ of $\{y_n\}$ weakly convergent to the function y in the space L^0. Hence the sequence $\{Ty^i_{n_i}\}_{i=1}^{\infty}$ is weakly convergent to the function Ty, in the space $C[0,\infty)$ by the continuity of T. Since $Ty^i_{n_i} \xrightarrow[w]{} u$ as $Ty_n \to u$ by (2.1), therefore

$$(2.5) \qquad Ty = u.$$

On the other hand, by (2.4) it follows that there is a sequence $\{v^k_n\}_{n=1}^{\infty}$ of convex combinations of $P_k y_{n+1}$, $P_k y_{n+2}$, $P_k y_{n+3}, \dots$, such that $p_k(v^k_n - P_k y) \to 0$ if $n \to \infty$. Therefore there is a subsequence $\{v^k_{n_j}\}_{j=1}^{\infty}$ of the sequence $\{v^k_n\}_{n=1}^{\infty}$ such that

$$v_{n_j}(t) \to y(t) \quad \text{if } j \to \infty, \quad \text{a.e. on } (0,k), (k=1,2,\dots).$$

From the convexity of the set $F(x)(t) = \{z \in R^m : |z| \leq \varphi(t)|x(t)|\}$ for every $t \in (0,\infty)$ and the convergence of $\{x_n\}$ to x it follows that $y(t) \in F(x)(t)$ a.e. on $(0,k)$, $(k=1,2,\dots)$. Consequently by (2.5) we have $y \in F(x)$ and therefore $u \in T\circ F(x)$. q.e.d.

Now we shall formulate lemma which has been proved in [6] for bounded intervals.

L e m m a 3. If $\varphi:(0,\infty) \to R^1$ is a non-negative integrable function satisfying the condition

$$\int_0^\infty \varphi(t)dt < \ln|\lambda|$$

then the problem

$$(*) \qquad x' \in F(x), \quad \tilde{l}x = 0,$$

where F is the mapping defined in Lemma 2 has exclusively a zero solution.

P r o o f . If $x \in F(x)$, then we have from the definition of F

$$(2.6) \qquad |x'(t)| \leq \varphi(t)|x(t)| \quad \text{a.e. on } (0,\infty).$$

We shall show that if the function x satisfies the inequality (2.6) then $x \in M$ i.e. there exists a finite limit $\lim_{t\to\infty} x(t) = x_\infty$. In fact, from (2.6) follows the inequality

$$|x(t)| \leq |x(0)| + \int_0^t \varphi(z)|x(z)|dz.$$

From Gronwall's inequality we obtain

$$(2.7) \qquad |x(t)| \leq |x(0)| \exp\left[\int_0^\infty \varphi(z)dz\right]$$

what together with inequalities (2.6) gives

$$(2.8) \qquad |x'(t)| \leq |x(0)| \exp\left[\int_0^\infty \varphi(z)dz\right] \varphi(t).$$

Now we obtain from (2.8) that the integral $\int_0^\infty |x'(t)|dt$ is convergent. Therefore let $x_\infty = \int_0^\infty x\,(t)dt$.

Then

$$(2.9)\qquad \left| x(t) - \int_0^\infty x'(t)dt \right| \leqslant \left| \int_z^\infty x(z)dz \right|$$

and as $\int_0^\infty |x'(z)|dz$ is convergent, therefore we obtain from the estimation (2.9) that $\lim_{t\to\infty} x(t) = x_\infty$. Thus $x \in M$.

Next we show that problem (*) has only a zero solution. If $x(0) = 0$ then from (2.7) we obtain $x(t) \equiv 0$. If, on the contrary, $x(0) \neq 0$, then (2.7) implies

$$(2.10)\qquad \ln \frac{|x_\infty|}{|x(0)|} \leqslant \int_0^\infty \varphi(z)dz.$$

As $x \in M$, therefore $\tilde{l}x = x_\infty + \lambda \cdot x(0)$. Therefore from (*) follows the equality

$$\frac{|x_\infty|}{|x(0)|} = |\lambda|$$

what together with condition (2.10) gives

$$\ln |\lambda| < \int_0^\infty \varphi(t)dt.$$

This contradicts our assumption. Therefore problem (*) has exclusively a zero solution.

From Theorem 1 and Lemmas 2 and 3 it follows that

T h e o r e m 2. Assume that:

(α) the integrable and non-negative function $\varphi : (0,\infty) \to R^1$ satisfies the assumption of Lemma 3;

(β) the continuous function $f : [0,\infty) \times R^m \to R^m$ satisfies the conditions

$$|f(t,s) - f(t,\bar{s})| \leqslant \varphi(t)|s-\bar{s}| \quad \text{and} \quad f(t,0) = 0 \quad \text{for} \quad t \in [0,\infty).$$

Then there exists one and only one solution of the problem

$$x'(t) = f(t,x(t))$$

$$x_{\infty} + \lambda \cdot x(0) = r, \qquad r \in R^m.$$

REFERENCES

[1] R. Bittner: Operational calculus in linear spaces, Studia Math. 20(1961) 1-18.

[2] R. Bittner: Algebraic and analytic properties of solutions of abstract differential equations, Rozprawy Matematyczne, Warszawa 1964.

[3] N. Dunford, J.T. Schwartz: Linear operators, Part I, New York 1958.

[4] A. Lasota, Z. Opial: On the existence and uniqueness of solution of nonlinear functional equations, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astronom. Phys., 15(1967) 797-800.

[5] A. Lasota, Z. Opial: On the existence of solutions of linear problems for ordinary differential equations, ibidem, 14(1966) 371-376.

[6] J. Myjak: A boundary value problem for nonlinear differential eouations with a retarded argument, Ann. Polon. Math. 27(1973) 133-142.

[7] M. Nagumo: Degree of mapping in locally convex linear topological spaces, Amer. J. Math. 73(1951) 497-511

[8] S. Rolewicz: Metric linear spaces, Warszawa 1973.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, GDAŃSK
Received February 18, 1977.