DEMONSTRATIO MATHEMATICA
Vol. XI No 4 1978

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ON FREE ALGEBRAS IN SOME EQUATIONAL CLASSES DEFINED BY REGULAR EQUATIONS

Introduction

Let K be an equational class of algebras of some fixed type τ without nullary operations. Let $x \cdot y = x$ be an identity in K, where $x \cdot y$ is a term of τ, in which the variable y occurs. Following J. Płonka [3] an identity $f = g$ in K is called regular if the set of variables occuring in the term f is the same as that in g. Let R(K) denote the equational class of algebras defined by all regular identities holding in K. It is known (see [5]) that any algebra in R(K) is the Płonka sum of a semilattice-ordered system of algebras from K (see § 1 for definitions). J. Płonka [6] proved that a free algebra in R(K) is the sum of a semilattice-ordered system of finitely generated free algebras from K. In this paper we give a necessary and sufficient condition for the algebra $\mathfrak{A}$ to be free in the class R(K). Our theorem gives a characterization of free algebras in the classes of distributive quasilattices, Padmanabhan quasilattices and sums of Boolean algebras (see [2], [4], [7] for definitions).

1. Preliminaries

Let K denote an equational class of algebras of some fixed type τ without nullary operations. (For these and other standard algebraic notions see [1]).

By a semilattice-ordered system of algebras in K we mean a triple

$$<\mathcal{J}, <\mathfrak{A}_i>_{i\in I}, <\varphi_{i,j}>_{i,i,j\in I}>,$$

where $\mathcal{J} = (I, \vee)$ is a join semilattice, $<\mathfrak{A}_i>_{i\in I}$ is a family of algebras in K indexed by the set I, and if $i \leqslant j$, $i,j \in I$, then $\varphi_{i,j}$ is a homomorphism from $\mathfrak{A}_i$ into $\mathfrak{A}_j$ satisfying the following two conditions.

(i) $\varphi_{i,i}$ is the identity mapping on A_i.

(ii) If $i \leqslant j \leqslant k$, then $\varphi_{j,k} \cdot \varphi_{ij} = \varphi_{i,k}$.

Given such a family of algebras in K, J. Płonka (see [3]) constructed an algebra of type τ in the following manner. Let $A = \dot{\bigcup}_{i\in I} A_i$, the disjoint sum of the carriers of the carriers of the algebras $\mathfrak{A}_i$. For an n-ary operation symbol f of τ we define its realization on A by setting

$$f(x_1,\ldots,x_n) = f(\varphi_{i_1,j}(x_1),\ldots,\varphi_{i_n,j}(x_n)),$$

where $j = i_1 \vee \ldots \vee i_n$, $x_r \in A_{i_r}$, $r = 1,\ldots,n$. We call the resulting algebra $\mathfrak{A} = (A,F)$ the Płonka sum of the semilattice-ordered system $<\mathcal{J}, <\mathfrak{A}_i>_{i\in I}, <\varphi_{i,j}>_{i\leqslant j, i,j\in I}>$.

T h e o r e m 1. [5] If there is a term $x \cdot y$ of τ in which the variable y occurs such that $x.y = x$ is an identity in K, then R(K) consists of all isomorphic copies of Płonka sums of semilattice-ordered systems of algebras in K.

From now, we will assume that $x \cdot y = x$ is an identity in K. Now, let $\{a_j, j \in J\}$ be free generators of a free algebra $\mathcal{F}_{R(K)}$. Let $J_0 = \{k_1,\ldots,k_n\}$ be a finite subset of J. Let $\mathcal{F}_{J_0}$ be the subalgebra of $\mathcal{F}_{R(K)}$ generated by all elements of the form

$$g_i = a_{k_i} \cdot a_{k_2} \cdot \ldots \cdot a_{k_{i-1}} \cdot a_{k_1} \cdot a_{k_{i+1}} \cdot \ldots \cdot a_{k_n},$$

where $i = 1,2,\ldots,n$. If $n = 1$, then $g_1 = a_{k_1}$.

L e m m a 2. [6] $\mathcal{F}_{J_0}$ is a free algebra in K with n free generators g_i.

L e m m a 3 [6] $\mathcal{F}_{R(K)}$ is the Płonka sum of the semilattice-ordered system of algebras $\mathcal{F}_{J_0}$, where J_0 ranges over all non-void finite subsets of J.

T h e o r e m 4 [6] A free algebra in R(K) is the Płonka sum of a semilattice ordered system of finitely generated free algebras from K.

The following example shows that the convers of Theorem 4 is not true. Consider distributive quasilattices, i.e. algebras with two binary operations + and . satisfying the following axioms

$$x+x = x \qquad x\cdot x = x,$$

$$x+y = y+x, \qquad x\cdot y = y\cdot x,$$

$$(x+y)+z = x+(y+z), \qquad (x\cdot y)\cdot z = x\cdot(y\cdot z),$$

$$x+(y\cdot z) = (x+y)\cdot(x+z), \qquad x\cdot(y+z) = (x\cdot y)+(x\cdot z).$$

It is known (see [4]), that every distributive quasilattice is the Płonka sum of the semilattice-ordered system of distributive lattices. Let $\mathfrak{A} = (\{a_1,a_2,a_3\}+,\cdot)$ be an algebra satisfying all mentioned axioms and $x+y = x\cdot y$. $\mathfrak{A}$ is the Płonka sum of the semilattice-ordered system $\langle \mathcal{I}, \langle \mathfrak{A}_i \rangle_{i\in I}, \langle \varphi_{i,j} \rangle_{i\leqslant j, i,j\in I} \rangle$, where $I = \{1,2,3\}$, $1,2<3$, $A_i = \{a_i\}$. $\mathfrak{A}$ is the sum of one element lattices, it is generated by a_1 and a_2, but it is not a free distributive quasilattice with two generators, because of the following lemma.

L e m m a 5. [8] A free distributive quasilattice with two generators has six elements.

2. <u>Main result</u>

The following theorem gives a characterization of free algebras in the class R(K).

T h e o r e m 6. An algebra $\mathfrak{A}$ is a free algebra in the class R(K) with α free generators $\{g_j, j\in J\}$ iff it is the

Płonka sum of the semilattice-ordered system $\langle \mathcal{J}, \langle \mathfrak{A}_i \rangle_{i \in I}, \langle \varphi_{i,j} \rangle_{i \leq j, i, i \in I} \rangle$ such that

1) $\mathcal{J}$ is the free semilattice with the set J of free generators,
2) every algebra $\mathfrak{A}_i$ is a finitely generated free algebra from the class K, moreover $\mathfrak{A}_i$ has exactly n free generators g_j^i $(j = 1,\dots,n)$ iff $i = i_1 \vee \dots \vee i_n$, where $i_1,\dots,i_n \in J$, $i_j \neq i_k$ for $i \neq k$ and if $i \in J$, then $\mathfrak{A}_i$ is freely generated by $g_1^i = g_{i_1} = g_i$,
3) for $i \leq k$, $\varphi_{i,k} : \mathfrak{A}_i \to \mathfrak{A}_k$ is a monomorphic extension of the mapping $\varphi^o_{i,k}$ defined by $\varphi^o_{i,k}(g_j^i) = \varphi_{i_j,i}(g_{i_j}) = g_j^k$, where $j = 1,\dots,n$.

P r o o f. $(\Longrightarrow)$ Let $\mathfrak{A}$ be a free algebra in the class $R(K)$ with free generators $g_j, j \in J$. By Lemmas 2,3 and Theorem 4, $\mathfrak{A}$ is the Płonka sum of the semilattice-ordered system $\langle \mathcal{J}, \langle \mathfrak{A}_i \rangle_{i \in I}, \langle \varphi_{i,j} \rangle_{i \leq j, i,j \in I} \rangle$, where every $\mathfrak{A}_i$ a finitely generated free algebra from the class K and $\mathcal{J}$ is the join semilattice of all finite subsets of J. We will prove now, that $\mathcal{J}$ is a free semilattice generated by one element subsets. Let $\mathfrak{S}$ be a semilattice. Let f_o be a mapping of one element subsets of J into S. Let us define $f(\{i_1,\dots,i_k\}) = f_o(\{i_1\}) \vee \dots \vee f_o(\{i_k\})$, $i_1,\dots,i_k \in J$. It is easy to see that f is a homomorphism from $\mathcal{J}$ into $\mathfrak{S}$ extending f_o. Hence $\mathcal{J}$ is a free semilattice. If $i \leq j$, then $\varphi^o_{i,j}(g_k^i) = \varphi^o_{i,j}(g_{i_k} \cdot g_{i_2} \cdot \dots \cdot g_{i_{k-1}} \cdot g_{i_1} \cdot g_{i_{k+1}} \cdot \dots \cdot g_{i_r}) =$
$= g_{i_k} \cdot g_{i_2} \cdot \dots \cdot g_{i_{k-1}} \cdot g_{i_1} \cdot g_{i_{k+1}} \cdot \dots \cdot g_{i_r} \cdot \dots \cdot g_{i_n} = g_k^j$,
where $k = 1,\dots,r$, $g_1^i,\dots,g_r^i$ are all free generators of $\mathfrak{A}_i$ and $g_1^j,\dots,g_n^j$ are all free generators of $\mathfrak{A}_j$.

$(\Longleftarrow)$ Let $\mathfrak{A}$ be the Płonka sum of the semilattice-ordered system described in the Theorem. By Theorem 1, $\mathfrak{A}$ belongs to the class $R(K)$. Let $a \in A_i$, $b \in A_j$. Since $x \cdot y$ is a term,

(1) $$a \cdot b = \varphi_{i,i \vee j}(a) \cdot \varphi_{j,i \vee j}(b) = \varphi_{i,i \vee j}(a)$$

Using (1), it is easy to check that $x \cdot y$ define a P-function (for the definition see [3]). Moreover, for $a, b \in A_i$, we have $a \cdot b = a$, $b \cdot a = b$, furthermore if $a \in A_i$, then $\varphi_{i,j}(a) = a \cdot b$ for $i \leq j$, $b \in A_j$ and finally $i \leq j$ iff $g_k^j \cdot g_k^i = g_k^j$. Hence, by Theorem 2 in [3], the decomposition $\mathfrak{A}$ into the described sum can be obtained by means of $x \cdot y$. Therefore, $\varphi_{i_k,i}(g_{i_k}) = g_{i_k} \cdot x$, where x is an arbitrary element of $\mathfrak{A}_i$ and $i = i_1 \vee \ldots \vee i_k \vee \ldots \vee i_n$, $i_1, \ldots, i_n \in J$. Let $g_1^i, \ldots, g_n^i$ be all free generators of $\mathfrak{A}_i$. Since $g_{i_1} \cdot \ldots \cdot g_{i_n} =$

$= \varphi_{i_1,i}(g_{i_1}) \cdot \ldots \cdot \varphi_{i_n,i}(g_{i_n}) = g_1^i \cdot \ldots \cdot g_n^i \in \mathfrak{A}_i$, we can

assume that $x = g_{i_1} \cdot \ldots \cdot g_{i_n}$. Hence $g_k^i = \varphi_{i_k,i}(g_{i_k}) =$
$= g_{i_k} \cdot (g_{i_1} \cdot \ldots \cdot g_{i_n}) = g_{i_k} \cdot g_{i_2} \cdot \ldots \cdot g_{i_{k-1}} \cdot g_{i_1} \cdot g_{i_{k+1}} \cdot \ldots \cdot g_{i_n}$.
Therefore, for $i \in I$, all generators of $\mathfrak{A}_i$ can be expressed by generators g_j with help of . . Hence $\mathfrak{A}$ is generated by the set $\{g_j, j \in J\}$.

Let $\mathfrak{B}$ be an algebra from R(K). By Theorem 1, $\mathfrak{B}$ is the Płonka sum of a semilattice-ordered system $\langle \mathcal{L}, \langle \mathfrak{B}_i \rangle_{i \in L}, \langle \psi_{i,j} \rangle_{i \leq j, i,j \in L} \rangle$. Let $f_0 : J \to L$. Since $\mathcal{J}$ is a free semilattice, f_0 can be extended to a homomorphism $f : \mathcal{J} \to \mathcal{L}$. We prove now that any mapping $h_0 : \{g_j, j \in J\} \to B$ can be extended to a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$. We define a mapping h as follows:

$$h(a) = h^i(a) \quad \text{for} \quad a \in A_i,$$

where $h^i : \mathfrak{A}_i \to \mathfrak{B}_{f(i)}$ is a homomorphic extension of the mapping

$$h_0^i(g_j^i) = \psi_{f(i_j), f(i)}\left[h_0(g_{i_j})\right], \quad i = i_1 \vee \ldots \vee i_n.$$

h_0^i can be extended to a homomorphism because $\mathfrak{A}_i$ is free in K. Let $F_t \in F$ be a fundamental n-ary operation of the algebra $\mathfrak{A}$. Let $x_r \in \mathfrak{A}_{i_r}$ for $r = 1, \ldots, n$, $i = i_1 \vee \ldots \vee i_n$,

$i_r = r_1 \vee \ldots \vee r_{k_r}$, where $r_1,\ldots,r_{k_r} \in J$, let $g_1^{i_r},\ldots,g_{k_r}^{i_r}$ be all free generators of $\mathfrak{A}_{i_r}$. Then $x_r = f_r(g_1^{i_r},\ldots,g_{k_r}^{i_r})$ for some term f_r.

$$h\left[F_t(x_1,\ldots,x_n)\right] = h^i\left[F_t(\varphi_{i_1,i}(x_1),\ldots,\varphi_{i_n,i}(x_n))\right] =$$

$$= F_t\left[h^i\circ\varphi_{i_1,i}(x_1),\ldots,h^i\circ\varphi_{i_n,i}(x_n)\right] =$$

$$= F_t\left\{h^i\circ\varphi_{i_1,i}\left[f_1(g_1^{i_1},\ldots,g_{k_1}^{i_1})\right],\ldots,h^i\circ\varphi_{i_n,i}\left[f_n(g_1^{i_n},\ldots,g_{k_n}^{i_n})\right]\right\} =$$

$$= F_t\left\{f_1\left[h^i\circ\varphi_{1_1,i}(g_{1_1}),\ldots,h^i\circ\varphi_{1_{k_1},i}(g_{1_{k_1}})\right],\ldots\right.$$

$$\left.\ldots,f_n\left[h^i\circ\varphi_{n_1,i}(g_{n_1}),\ldots,h^i\circ\varphi_{n_{k_n},i}(g_{n_{k_n}})\right]\right\} = F_t\left\{f_1\left[h^i(g_1^i),\ldots,\right.\right.$$

$$\left.h^i(g_{k_1}^i)\right],\ldots,f_n\left[h^i(g_1^i),\ldots,h^i(g_{k_n}^i)\right]\right\} = F_t\left\{f_1\left[\psi_{f(1_1),f(i)}\circ h_0(g_{1_1}),\right.\right.$$

$$\left.\ldots,\psi_{f(1_{k_1}),f(i)}\circ h_0(g_{1_{k_1}})\right],\ldots,f_n\left[\psi_{f(n_1),f(i)}\circ h_0(g_{n_1}),\ldots,\right.$$

$$\left.\left.\psi_{f(n_{k_n}),f(i)}\circ h_0(g_{n_{k_n}})\right]\right\} = F_t\left\{f_1\left[\psi_{f(i_1),f(i)}\circ h^{i_1}(g_1^{i_1}),\ldots,\right.\right.$$

$$\left.\psi_{f(i_1),f(i)}\circ h^{i_1}(g_{k_1}^{i_1})\right],\ldots,f_n\left[\psi_{f(i_n),f(i)}\circ h^{i_n}(g_1^{i_n}),\ldots,\right.$$

$$\left.\left.\psi_{f(i_n),f(i)}\circ h^{i_n}(g_{k_n}^{i_n})\right]\right\} = F_t\left\{\psi_{f(i_1),f(i)}\circ h^{i_1}\left[f_1(g_1^{i_1},\ldots,g_{k_1}^{i_1})\right],\ldots,\right.$$

$$\left.\psi_{f(i_n),f(i)}\circ h^{i_n}\left[f_n(g_1^{i_n},\ldots,g_{k_n}^{i_n})\right]\right\} = F_t\left[h(x_1),\ldots,h(x_n)\right].$$

This result applies, of course, to equational classes of distributive quasilattices, Padmanabhan quasilattices and sums of Boolean algebras (see [2], [4], [7] for definitions). Added in print. After the submission of this paper the author have learned that a similar result was proved independently by A. Mitschke in her doctoral thesis.

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Received February 3, 1978.