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SOME PROPERTIES OF WRONSKIAN IN D-R SPACES OF THE TYPE Q-L, I

In this paper we shall consider D-R spaces in the sense of [4], which satisfy an additional condition, namely the so-called Quasi-Leibniz condition (shortly: Q-L condition, cf. [3]).

Consider a D-R space (X,D,R), where X is a linear ring over a field $\mathcal{F}$.

D e f i n i t i o n 1. The D-R space (X,D,R) is called a D-R space of the type Q-L if

1^o X is a commutative linear ring over the field $\mathcal{F}$.

2^o The operator D satisfies the Q-L condition:

$$(1)\quad D(x\cdot y) = x\cdot Dy+Dx\cdot y+d\cdot Dx\cdot Dy \text{ for all } x,y\in \mathcal{D}_D, \text{ where } d\in\mathcal{F}.$$

If $d = 0$ then we get the D-R space of the Leibniz type (see [3]). In the sequel we shall assume that $d \neq 0$, and we shall write X or (X,D) instead of (X,D,R). We may do so, because all facts we are going to show do not depend upon R but only upon X and D.

P r o p o s i t i o n 1. If X is a D-R space of the type Q-L then

$$1^o\quad (2)\quad D^n(x\cdot y) = \sum_{k=0}^{n}\binom{n}{k} d^k \sum_{j=0}^{n-k}\binom{n-k}{j}\left(D^{k+j}x\right)\cdot\left(D^{n-j}y\right),$$

for all $x,y\in\mathcal{D}_D n, n\in N$,

2° (3) $$D\left(\prod_{j=1}^{n} x_j\right) = \sum_{l=1}^{n} d^{l-1} \sum_{\sigma_l \in C_n^l} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j \cdot x_k,$$

for all $x_j \in \mathcal{D}_D$, $j = 1,\ldots,n$

where we write: $C_n^l = 2^{\{p_1,\ldots,p_l\}}$, where $p_j \in \{1,\ldots,n\}$, $j = 1,\ldots,l \leqslant n$,

$$\sigma'_l = \{1,2,\ldots,n\} \setminus \sigma_l \quad \text{for } \sigma_l \in C_n^l.$$

The proof is by induction.

1°. Let $n = 1$ and let $x,y \in \mathcal{D}_D$ be arbitrarily fixed. Then

$$D(x\cdot y) = x\cdot Dy + Dx\cdot y + d\cdot Dx\cdot Dy$$

by our assumptions (1).

Suppose that formula (2) holds for an arbitrarily fixed $n \geqslant 1$. Then for $n+1$ we get

$$D^{n+1}(xy) = D\left(\sum_{j=0}^{n}\binom{n}{j} d^j \sum_{k=0}^{n-j}\binom{n-j}{k} D^{k+j}x\ D^{n-k}y\right) =$$

$$= \sum_{j=0}^{n}\binom{n}{j} d^{j+1} \sum_{k=0}^{n-j}\binom{n-j}{k} D^{k+1+j}x\cdot D^{n+1-k}y +$$

$$+ \sum_{j=0}^{n}\binom{n}{j} d^j \sum_{k=0}^{n-j}\binom{n-j}{k} D^{k+1+j}x\ D^{n-k}y + D^{k+j}x\ D^{n+1-k}y =$$

$$= \sum_{j=0}^{n}\binom{n}{j} d^j \left[\sum_{k=1}^{n+1-j}\binom{n-j}{k-1} D^{k+j}x\ D^{n+1-k}y + \sum_{k=0}^{n-j}\binom{n-j}{k} D^{k+j}x D^{n+1-k}y\right] +$$

$$+ \sum_{j=1}^{n+1}\binom{n}{j-1} d^j \sum_{k=0}^{n+1-j}\binom{n+1-j}{k} D^{k+j}x\ D^{n+1-k}y =$$

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} d^j \sum_{k=0}^{n+1-j} \binom{n+1-j}{k} D^{k+j}x \; D^{n+1-k}y \; +$$

$$+ \sum_{j=0}^{n} \binom{n}{j} d^j \left[\sum_{k=1}^{n-j} \binom{n+1-k}{k} D^{k+j}x \; D^{n+1-k}y + D^{n+1}x \cdot D^j y + D^j x \cdot D^{n+1} y \right] =$$

$$= \sum_{j=0}^{n} \binom{n}{j} d^j \sum_{k=0}^{n+1-j} \binom{n+1-j}{k} D^{k+j}x \; D^{n+1-k}y \; +$$

$$+ \sum_{j=1}^{n+1} \binom{n}{j-1} d^j \sum_{k=0}^{n+1-j} \binom{n+1-j}{k} D^{k+j}x \; D^{n+1-k}y =$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} d^j \sum_{k=0}^{n+1-j} \binom{n+1-j}{k} D^{k+j}x \; D^{n+1-k}y$$

for all $x, y \in {}_D{}^{n+1}$, which was to be proved.

2^o. Let $n = 1$. Then we have $Dx_1 = Dx_1$ for all $x_1 \in \mathcal{D}_D$. For $n = 2$, by our assumption (1) we get

$$D(x_1 \cdot x_2) = Dx_1 \cdot x_2 + x_1 \cdot Dx_2 + d \cdot Dx_1 \cdot Dx_2 \quad \text{for all} \quad x_1, x_2 \in \mathcal{D}_D.$$

Suppose that formula (3) holds for an arbitrarily fixed $n \geqslant 2$. Then we get for $n+1$:

$$D(x_1 \ldots x_n \cdot x_{n+1}) =$$

$$= D(x_1 \ldots x_n) \cdot x_{n+1} + x_1 \ldots x_n \cdot Dx_{n+1} + d \cdot D(x_1 \ldots x_n) \cdot Dx_{n+1} =$$

$$= \sum_{l=1}^{n} d^{l-1} \sum_{\sigma_l \in C_n^l} \prod_{\substack{j \in \sigma_l \\ l \in \sigma_l'}} Dx_j x_k x_{n+1} + x_1 \ldots x_n Dx_{n+1} \; +$$

$$+\sum_{l=1}^{n} d^{l} \sum_{\sigma_l \in C_n^l} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j x_k Dx_{n+1} = \sum_{j=1}^{n} \prod_{\substack{k=1 \\ k\neq j}}^{n} Dx_j x_k \, x_{n+1} + x_1 \ldots$$

$$\ldots x_n Dx_{n+1} + \sum_{l=2}^{n} d^{l-1} \sum_{\sigma_l \in C_n^l} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j \, x_k \, x_{n+1} +$$

$$+ \sum_{\sigma_{l-1} \in C_n^{l-1}} \prod_{\substack{j\in\sigma_{l-1} \\ k\in\sigma'_{l-1}}} Dx_j Dx_{n+1} x_k + d^n \, Dx_1 \cdot Dx_2 \ldots Dx_n \cdot Dx_{n+1} =$$

$$= \sum_{j=1}^{n+1} \prod_{\substack{k=1 \\ k\neq j}}^{n+1} Dx_j \cdot x_k + \sum_{l=2}^{n} d^{l-1} \sum_{\substack{\sigma_l \in C_{n+1}^l \\ n+1 \notin \sigma_l}} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j \cdot x_k +$$

$$+ \sum_{\substack{\sigma_l \in C_{n+1}^l \\ n+1 \in \sigma_l}} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j \cdot x_k + d^n \prod_{j=1}^{n+1} Dx_j =$$

$$= d^0 \sum_{\sigma_1 \in C_{n+1}^1} \prod_{\substack{j\in\sigma_1 \\ k\in\sigma'_1}} Dx_j \cdot x_k + \sum_{l=2}^{n} d^{l-1} \sum_{\sigma_l \in C_{n+1}^l} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j \cdot x_k +$$

$$+ d^n \sum_{\sigma_{n+1} \in C_{n+1}^{n+1}} \prod_{\substack{k\in\sigma_{n+1} \\ j\in\sigma'_{n+1}}} Dx_j \cdot x_k = \sum_{l=1}^{n+1} d^{l-1} \sum_{\sigma_l \in C_{n+1}^l} \prod_{\substack{j\in\sigma_l \\ k\in\sigma'_l}} Dx_j \cdot x_k$$

which was to be proved.

In the sequel we shall assume that X has a unit e and we will consider a polynomial $Q(D)$ of some special form. These conditions will be denoted by A1 and A2. So we have:

(A1) (X,D) is a D-R space of the type Q-L and X has a unit e.

(A2) The polynomial $Q(D)$ is of the form

$$Q(D) = \sum_{k=0}^{N} a_k D^k, \quad \text{where} \quad a_k \in X,\ a_N = I. \tag{4}$$

Our aim is to investigate the two following equations:

$$Q(D)x = 0,\ x \in X \qquad (I) \tag{5}$$

$$Q(D)x = y,\ x,y \in X \qquad (II) \tag{6}$$

We shall try to solve them using the method of Wronskian, which was applied in [2] for a D-R space of Leibniz type.

If the condition (A1) is fulfilled we get at once

$$D(e) = D(e\cdot e) = e\cdot De + De\cdot e + d\cdot De\cdot De \tag{7}$$

so we have

$$De(e+dDe) = 0. \tag{8}$$

This implies that either

$$De = 0 \qquad (E1) \tag{9}$$

or

$$De = \frac{-1}{d}e \qquad (E2) \tag{10}$$

or

$$De(e+dDe) = 0 \text{ and } De \neq 0 \text{ and } De \neq \frac{-e}{d} \qquad (E3). \tag{11}$$

In the case (E2) $\ker D = \{0\}$ and the theory becomes trivial, so in this paper we consider the case (E2), leaving the case (E3) to be considered in another paper.

Let $A \in L(X)$ and $\dim \ker A \neq 0$.

Definition 2. The determinant

$$W(x_1,\dots,x_m) = \begin{vmatrix} x_1, & x_2,\dots\dots, & x_m \\ Ax_1, & Ax_2,\dots\dots, & Ax_m \\ \vdots & \vdots & \vdots \\ A^{m-1}x_1 & A^{m-1}x_2,\dots, & A^{m-1}x_m \end{vmatrix}, \tag{12}$$

where $x_1,x_2,\dots,x_m \in \mathcal{D}_{A^{m-1}}$

is called the Wronskian of elements $x_1,x_2,\dots,x_m \in \mathcal{D}_{A^{m-1}}$.

Definition 3. A subset $X_0 \subset X$ is called an A-modul if it is a kerA-modul in the sense of Johnson [5].

Definition 4. The rank of an A-modul X_0 is called an A-dimension of X_0, briefly A-dimX_0 or $\dim_A X_0$.

Definition 5. Elements $x_1,x_2,\dots,x_m \in X$ are said to be A-linearly dependent if there exist $z_1,z_2,\dots,z_m \in \ker A$, non vanishing simultaneously such that the A-linear combination of elements $x_1,x_2,\dots,x_m$ i.e. the element

$$z_1x_1+z_2x_2+\dots+z_mx_m \tag{13}$$

is equal to zero.

The elements $x_1,x_2,\dots,x_m$ are said to be A-linearly independent if the condition $z_1x_1+z_2x_2+\dots+z_mx_m = 0$ implies $z_1=z_2=z_3=\dots=z_m = 0$.

Definition 6. A system of elements $x_1,x_2,\dots,x_m$ belonging to $\mathcal{D}_{A^{m-1}}$ is said to be fundamental if $W(x_1,x_2,\dots,x_m)$ is invertible.

If $y_1,y_2,\dots,y_n$ are generators for an A-modul X_0 then we write

(14) $$\langle y_1,\dots,y_n\rangle_A = X_0.$$

By $\langle y_1,y_2,\dots,y_n\rangle_{\mathcal{F}}$ we denote the linear span of elements $y_1,\dots,y_n$. Taking into account these definitions we get

P r o p o s i t i o n 2. If the condition A1 is satisfied then we have

(15) $$D^k(z\cdot x) = z\cdot D^k x, \quad \text{for all} \quad z \in \ker D, \quad k=0,1,2,\dots$$

P r o o f. Suppose that $z \in \ker D$ and $z \in \mathcal{D}_D$ are arbitrarily fixed. Then

$$D(zx) = Dz\cdot x+z\cdot Dx+d\cdot Dz\cdot Dx = z\cdot Dx$$

$$D^{k+1}(zx) = D(D^k zx) = D(z\cdot D^k x) = zD^{k+1}x+Dz\cdot D^{k+1}x+Dz\cdot D^{k+1}x =$$

$$= z\cdot D^{k+1}x.$$

P r o p o s i t i o n 3. For all $n \in N$ $\ker D^n$ is a D-modul.

P r o o f . It suffices to check that

(16) $$zw \in \ker D^n, \quad \text{for all} \quad z \in \ker D,\ w \in \ker D^n.$$

But it is an immediate consequence of (15), because we have

$$D^n(zw) = z\, D^n w = 0.$$

P r o p o s i t i o n 4. Let the conditions A1 and A2 be satisfied. Then we have: The set of all solutions of equation (I), i.e. the set

(17) $$\ker Q(D) = \left\{x \in X:\ Q(D)x = 0\right\}$$

is a D-modul.

P r o o f : It suffices to show that

(18) $zw \in \ker Q(D)$ for all $z \in \ker D$, $w \in \ker Q(D)$.

But we have

$$Q(D)(zw) = \sum_{k=0}^{N} a_k D^k(zw) = z \cdot Q(D)x = 0.$$

It is well known that we have

T h e o r e m 1. Let $A \in L(X)$, $\dim \ker A \neq 0$, $x_1, x_2, \ldots, x_m \in X$ and X be a commutative linear ring over $\mathcal{F}$.

If the Wronskian $W(x_1, \ldots, x_m)$ is invertible, then the elements $x_1, x_2, \ldots, x_m$ are A-linearly independent.

C o r o l l a r y 1. If (X,D) is a D-R space of the type Q-L and $x_1, \ldots, x_m \in X$ are D-linearly independent then the elements $x_1, \ldots, x_m$ are linearly independent.

P r o o f : It follows from (E1) that $De = 0$, so we have $e \in \ker D$.

C o r o l l a r y 2.

(19) $$\dim \ker D \geqslant 1.$$

P r o o f : This is an immediate consequence of the condition (E1), for $De = 0$ and $\langle e \rangle = \langle e \rangle_{\mathcal{F}} \subset \ker D$.

C o r o l l a r y 3. Let X_0 be a D-modul in X. Then we have

(20) $$\dim_D X_0 \leqslant \dim_{\mathcal{F}} X_0 = \dim X_0 .$$

P r o o f : If the elements $x_1, \ldots, x_m$ are generators for a D-modul X_0 then they are D-linearly independent, hence linearly independent. Thus they belong to the basis of X_0.

C o r o l l a r y 4. Let the Wronskian $W(x_1, \ldots, x_m)$ be invertible. Then

(21) $$< x_1,\dots,x_m>_D \supset < x_1,\dots,x_m>_{\mathcal{F}} .$$

P r o o f : From (E1) we get

$$c_1x_1+\dots+c_mx_m = (c_1e)x_1+\dots+(c_me)x_m \in <x_1,\dots,x_n>_{\mathcal{F}} .$$

P r o p o s i t i o n 5. Let $x_1,\dots,x_r \in X$ be D-linearly independent. Then

(22) $< x_1,\dots,x_r>_D = <x_1,\dots,x_r>_{\mathcal{F}}$ if and only if $\ker D = <e>_{\mathcal{F}}$.

P r o o f : Sufficiency. Let $z \in \ker D$ and $z \notin <e>_{\mathcal{F}}$. Then we have

$$zx_1 \in < x_1,\dots,x_r>_D \quad \text{and} \quad zx_1 \notin <x_1,\dots,x_r>_{\mathcal{F}}$$

because $zx_1 = c_1x_1+\dots+c_rx_r$ implies $(z-c_1e)x_1+\dots+c_rx_r = 0$. Thus $z = c_1e$ which contradicts our assumption. Necessity is obvious.

Similarly, as in [2] we obtain

T h e o r e m 2. Let the condition (A1) be satisfied and let $x_1,\dots,x_m$ be a fundamental system. Then there exists an operator $Q(D)$ of order N, such that

(23) $$\ker Q(D) \supset <x_1,\dots,x_N>_D .$$

P r o o f : Assume that $x = z_1x_1+\dots+z_Nx_N$. Then we infer that the Wronskian $W(x_1,\dots,x_N,x)$ is equal zero, for we have

(24) $$W(x_1,\dots,x_N,x) = \begin{vmatrix} x_1,\dots, & x_N, & z_1x_1+\dots+z_Nx_N \\ Dx_1,\dots, & Dx_N, & z_1Dx_1+\dots+z_NDx_N \\ \vdots & \vdots & \vdots \\ D^Nx_1,\dots, & D^Nx_N, & z_1D^Nx_1+\dots+z_ND^Nx_N \end{vmatrix} =$$

$$= \begin{vmatrix} x_1, & x_2,\dots, & x_N, & 0 \\ Dx_1, & Dx_2,\dots, & Dx_N, & 0 \\ \vdots & \vdots & \vdots & \vdots \\ D^N x_1, & D^N x_2,\dots, & D^N x_N, & 0 \end{vmatrix} = 0.$$

On the other hand we have

$$(25)\quad W(x_1,\dots,x_N,x) = \sum_{i=0}^{N} (-1)^{i+1+N+1} D^i x\, W_i(x_1,\dots,x_N) D^k x = W \sum_{k=0}^{N} a_k D^k x,$$

where

$$(26)\quad W_k(x_1,\dots,x_N) = \begin{vmatrix} Dx_1 & ,\dots\dots, & Dx_N \\ \vdots & & \vdots \\ D^{k-1}x_1, & \dots\dots, & D^{k-1}x_N \\ D^{k+1}x_1, & \dots\dots, & D^{k+1}x_N \\ \vdots & & \vdots \\ D^N x_1 & ,\dots\dots, & D^N x_N \end{vmatrix}, \quad \begin{matrix} k=0,1,\dots,N \\ x_1,x_2,\dots,x_N \in \mathcal{D}_D^N \end{matrix}$$

$$(27)\qquad a_k = \frac{(-1)^{n+k}\, W_k(x_1,\dots,x_N)}{W(x_1,\dots,x_N)}.$$

Hence we obtain

$$(28)\qquad Q(D)x = \sum_{k=0}^{N} a_k \cdot D^k = 0$$

and

$$(29)\qquad a_N = \frac{(-1)^{2N}\, W_N(x_1,\dots,x_N)}{W(x_1,\dots,x_N)} = \frac{W(x_1,\dots,x_N)}{W(x_1,\dots,x_N)} = I.$$

We therefore conclude that

$\langle x_1,\dots,x_N \rangle_D \subset \ker Q(D)$ and $Q(D)$ satisfies (A2).

C o r o l l a r y 1.

(30) $$\dim_D \ker Q(D) \leqslant N.$$

This follows from the fact that $x_1,\dots,x_N$ are generators for $\ker Q(D)$. It is well known that in the case where the elements of X are functions of one variable $t \in \mathcal{F}$ the condition $W(x_1,\dots,x_N) \neq 0$ is sufficient for the elements $x_1,\dots,x_N$ to be linearly independent.

T h e o r e m 3. Let the condition (A1) be satisfied. Then we have

(31) $$D\,W + \sum_{l=1}^{N} (-d)^{l-1}\, a_{N-l}\cdot W = 0,$$

where a_k are (as in (26)) obtained from $W(x_1,\dots,x_N,x)$ in the following way:

$$W = W(x_1,\dots,x_N),$$

$$a_k = W^{-1}\cdot W_k(x_1,\dots,x_N).$$

P r o o f : We can rewrite the formula (31) in the form:

(32) $$DW + \sum_{k=0}^{N-1} (-d)^{N-k-1}\, a_k\, W = 0$$

or

(33) $$DW = -\sum_{k=0}^{N-1} (-d)^{N-k-1}\, a_k\, W.$$

To prove the first equality we need some lemmas.

L e m m a 1. Let

$\sigma_1 \in C_n^1$, $\tau_{n-1} \in C_n^{n-1}$ be such that $\sigma_1 \cap \tau_{n-1} \neq \emptyset$.

Then

$$(34) \qquad W(x_{k_1},x_{k_2},\ldots,x_{k_l},x_{r_1},\ldots,x_{r_{n-1}}) = 0,$$

where $k_1,\ldots,k_l \in \sigma_l$, $r_1,\ldots,r_{n-l} \in \tau_{n-l}, x_1,\ldots,x_n \in \mathcal{D}_D n-1$.

P r o o f : Let $p_o \in \sigma_l \cap \tau_{n-l}$. This means that $p_o = k_{i_o} = r_{j_o}$ for some i_o, j_o, such that $1 \leqslant i_o \leqslant l$ and $1 \leqslant j_o \leqslant n-l$. Then we have

$$(35) \qquad W(x_{k_1},\ldots,x_{k_l},\ldots,x_{r_{n-l}}) =$$

$$= W(x_{k_1},\ldots,x_{p_o},\ldots,x_{k_l},x_{r_1},\ldots,x_{p_o},\ldots,x_{r_{n-l}}) = 0$$

L e m m a 2. Let $\sigma_l \in C_n^l$, $1 \leqslant l \leqslant n$. Then we have

$$(36) \quad T_1\sigma_l \cap \sigma'_l = \emptyset \text{ if and only if } \sigma_l = \{n-l+1, n-l+2, \ldots, n-1, n\},$$

where

$$(37) \qquad T_1\sigma_l = \sigma_l + 1 = \{n-l+2, n-l+3, \ldots, n, n+1\} .$$

P r o o f : Necessity is obvious.

Sufficiency. Let $T_1\sigma_l \cap \sigma'_l = \emptyset$ and $\sigma_l \neq \{n-l+1,\ldots,n\}$. Then there exists k_o such that $k_o \in \sigma'_l$ and $n-l+1 \leqslant k_o \leqslant n$. Let $k_1 = k_o-1$. If $k_1 \notin \sigma_l$ then we take $k_2 = k_1-1$ and so on. In this way after a finite number of steps we get the element $k_{j_o} \in \sigma_l$ such that $k_{j_o} + 1 \in \sigma'_l$. Thus $T_1\sigma_l \cap \sigma'_l \neq \emptyset$, which contradicts with the assumption that $T_1\sigma_l \cap \sigma'_l = \emptyset$.

P r o o f of Theorem 3: Let S_N denote the set of all permutations of the set $\{1,2,\ldots,N\}$, Proposition 1 and Lemmas 1 and 2 imply together that

$$(38)\quad DW=D\begin{vmatrix} x_1,\dots\dots,x_N \\ \vdots \qquad \vdots \\ D^{N-1}x_1,\dots,D^{N-1}x_N \end{vmatrix} = \sum_{\alpha\in S_N} \operatorname{sgn}\alpha\cdot D\left(x_{\alpha(1)}\cdot Dx_{\alpha(2)}\cdots D^{N-1}x_{\alpha(N)}\right) =$$

$$=\sum_{\alpha\in S_N}\operatorname{sgn}\alpha \sum_{l=1}^{N} d^{l-1} \sum_{\sigma_l\in C_N^l} \prod_{\substack{j\in\sigma_l\\ k\in\sigma_l'}} D^j x_{\alpha(j)}\, D^{k-1}x_{\alpha(k)} =$$

$$=\sum_{l=1}^{N} d^{l-1}\sum_{\sigma_l\in C_N^l}\sum_{\alpha\in S_N}\operatorname{sgn}\alpha \prod_{\substack{j\in\sigma_l\\ k\in\sigma_l'}} D^j x_{\alpha(j)}\, D^{k-1}x_{\alpha(k)} =$$

$$=\sum_{l=1}^{N} d^{l-1}\sum_{\sigma l\in C_N^l}\sum_{\alpha\in S_N}\operatorname{sgn}\alpha \prod_{\substack{j\in\sigma_l\\ k\in\sigma' l}} D^j x_{\alpha(j)}\, D^{k-1}x_{\alpha(k)} =$$

$$=\sum_{l=1}^{N} d^{l-1}\sum_{S_N}\operatorname{sgn}\alpha\cdot x_{\alpha(1)}\, Dx_{\alpha(2)}\cdots$$

$$\cdots D^{N-l-1}x_{\alpha(N-1)}\, D^{N-l+1}x_{\alpha(N-l+1)}\, D^N x_{\alpha(N)} =$$

$$=\sum_{l=1}^{N} d^{l-1}\begin{vmatrix} x_1,\dots\dots\dots\dots,x_N \\ Dx_1,\dots\dots\dots\dots,Dx_N \\ \vdots \qquad\qquad \vdots \\ D^{N-l-1}x_1,\dots,D^{N-l-1}x_N \\ D^{N-l+1}x_1,\dots,D^{N-l+1}x_N \\ \vdots \qquad\qquad \vdots \\ D^N x_1,\dots\dots\dots,D^N x_N \end{vmatrix} = \sum_{l=1}^{N} d^{l-1}\, W_{N-l} =$$

$$=\sum_{l=1}^{N} d^{l-1}(-1)^{2N-l}\, a_{N-l}\cdot W = -\sum_{l=1}^{N}(-d)^{l-1}\, a_{N-l}\cdot W.$$

Thus

(39) $$D\,W = -\sum_{l=1}^{N} (-d)^{l-1}\, a_{N-l}\, W$$

and

$$D\,W = -\sum_{k=0}^{N-1} (-d)^{N-k-1}\, a_k\, W$$

which implies that

(40) $$D\,W + \sum_{k=0}^{N-1} (-d)^{N-k-1}\, a_k\, W = 0.$$

Let $\mathcal{R}_D$ denote the set of all right inverses of D i.e.

(41) $$\mathcal{R}_D = \left\{ R \in L(X) : \mathcal{D}_R = X \text{ and } RX \subset \mathcal{D}_D \text{ and } DR = I \right\}.$$

Similarly as in [2] we obtain

Theorem 4. Let

1° (X,D,R) be a D-R space of the type Q-L with unit e,

2° F be an initial operator for D corresponding to an operator $R \in \mathcal{R}_D$ (see [1]),

3° the operator $I + R\sum_{k=0}^{N-1} (-d)^{N-k-1} a_k$ be invertible.

Then we have

(42) $$W = W(x_1,\ldots,x_N) = 0 \text{ if and only if } F\,W = 0.$$

Proof: From (40) we obtain

(43) $$D\,W + D\,R\sum_{k=0}^{N-1} (-d)^{N-k-1} a_k \cdot W = D\left(I + R\sum_{k=0}^{N-1} (-d)^{N-k-1}\, a_k\right) W = 0.$$

Thus we infer that

(44) $$W + R\sum_{k=0}^{N-1}(-d)^{N-k-1}a_k \cdot W = z \in \ker D.$$

Taking into account that F is a projection onto kerD, we obtain

(45) $$FW + FR\sum_{k=0}^{N-1}(-d)^{N-k-1}a_k \cdot W = Fz = z =$$

$$= W + R\sum_{k=0}^{N-1}(-d)^{N-k-1}a_k \cdot W.$$

This means that

(46) $$FW = I + R\sum_{k=0}^{N-1}(-d)^{N-k-1}a_k \cdot W.$$

Hence (46) is equivalent to the following statement:

(47) $$FW = 0 \text{ if and only if } W = 0,$$

because $I + R\sum_{k=0}^{N-1}(-d)^{N-k-1}a_k \cdot W$ is invertible.

This finishes our proof.

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