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ON THE CAUCHY PROBLEM FOR THE n-CALORIC EQUATION AND FOR THE TIME-PLANE

1. Let $x = (x_1, x_2)$, $X = (x_1, x_2, t)$. In this paper we shall construct a continuous function $u = U(X)$ defined for $t \geqslant 0$ and all x_1, x_2 which is the solution $u(X)$ for the n-caloric equation

$$P^n u(X) = 0, \tag{1}$$

where n is a positive integer, $n \geqslant 2$

$$P = D^2_{x_1} + D^2_{x_2} - D_t, \quad P^n = P(P^{n-1}), \quad P^0 = Id$$

in the domain

$$W = \left\{ X: \; |x_i| < \infty, \quad i = 1,2; \; t > 0 \right\} \tag{2}$$

and satisfies the initial conditions

$$D^i_t u(X) = f_i(x) \text{ for } X \in S = \left\{ X; \; |x_j| < \infty \;, \; j = 1,2, \; t = 0 \right\}, \tag{3}$$
$$(i = 0, 1, \ldots, n, 1).$$

2. Let $Y = (y_1, y_2, s)$, $y = (y_1, y_2)$, and let

$$r = |x,y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Let us consider the function

$$(4) \qquad G_n(X,Y) = (t-s)^{n-1}\, U(X,Y),$$

where

$$(5) \quad U(X,Y) = \begin{cases} (t-s)^{-1} \exp((4(t-s))^{-1}(-r)^2 & \text{for } s < t \\ 0 & \text{for } s \geq t,\ X \neq Y \end{cases}$$

is the fundamental solution of the equation $Pu(X) = 0$.

Let P_X^n denote the operator P^n acting on the variables x_1, x_2, t.

L e m m a 1. The function $G_n(X,Y)$ given by formulas (4), (5) satisfies the n-caloric equation $P_X^n G(X,Y) = 0$.

P r o o f . We shall prove this lemma by induction

1^0 For $n = 2$ we have $G_2(X,Y) = (t-s)U(X,Y)$, then $P_X(t-s)U(X,Y) = \Delta_x(t-s)U(X,Y) - D_t(t-s)U(X,Y) =$ $= (t-s)P_X U(X,Y) - U(X,Y)$ and consequently

$$P_X^2 G_2(X,Y) = -P_X U(X,Y) = 0.$$

2^0 Let us assume that the function $G_{k-1}(X,Y) = (t-s)^{k-2}U(X,Y)$ satisfies the equation $P_X^{k-1}\, G_{k-1} = 0$. We shall prove that the function $G_k = (t-s)^{k-1}\, U(X,Y)$ satisfies the equation $P_X^k G_k(X,Y) = 0$. Namely

$$P_X^k G_k(X,Y) = P_X^{k-1}(P_X(t-s)^{k-1}\, U(X,Y)) = P_X^{k-1}((t-s)^{k-1}\, \Delta_x U(X,Y) -$$

$$- (t-s)^{k-1}\, D_t U(X,Y) - (k-1)(t-s)^{k-2}\, U(X,Y)) =$$

$$= P_X^{k-1}((t-s)^{k-1}\, P_X U(X,Y) - (k-1)(t-s)^{k-2}\, U(X,Y)) = 0,$$

because

$$P_X U(X,Y) = 0 \quad \text{and} \quad P_X^{k-1}(t-s)^{k-2}\, U(X,Y) = 0.$$

L e m m a 2. The function $U_1(X,y) = U(X,Y)\big|_{s=0} = t^{-1}\exp((-4t)^{-1}r^2)$ satisfies the equation

$$D_t^i\, U_1(X,y) = \Delta_x^i U_1(X,y) \qquad (i = 1,2,\dots,n-1). \tag{6}$$

P r o o f . Since $D_tU_1(X;y) = \Delta_x U_1(X;y)$ thus

$$D_t^2U_1(X;y) = D_tD_tU_1(X;y) = D_t\Delta_xU_1(X;y) = \Delta_xD_tU_1(X;y) =$$

$$= \Delta_x^2U_1(X;y).$$

We shall assume that the function $U_1(X,y)$ satisfies the equation (6) for $i = k < n-1$, then for $i = k+1$ we have

$$D_t^{k+1}U_1(X,y) = D_t^kD_tU_1(X,y) = D_t^k\Delta_xU_1(X,y) =$$

$$= \Delta_xD_t^kU_1(X,y) = \Delta_x^{k+1}U_1(X,y).$$

We shall prove the following

L e m m a 3. Let the function $f(y)$ be of the class C^{2n} in the set $E_2 = \{x: |x_i| < \infty,\ i = 1,2\}$ and bounded with its derivatives up to the order $2n$ on E_2. Then

$$\lim D_t^i(4\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y)U_1(X;y)dy = \Delta^if(x_0) \qquad (i = 1,\dots,n)$$

when

$$X \to (x_0,0) \in S, \quad x_0 = (x_1^0,x_2^0).$$

P r o o f . At first we shall prove that the integrals

$$I(X) = \iint_{E_2} f(y)D_t^iU_1(X,y)dy \qquad (i = 0,1,2,\dots,n-1)$$

are uniformly convergent in every set

$$W_1 = \left\{X: \ |x_i| \le a_i \quad i = 1,2; \ t_1 \le t \le t_2\right\},$$

a_i, t_i $(i = 1,2)$ being arbitrary positive numbers. The derivatives $D_t^i U_1(X;y)$ are linear combinations of the functions

$$t^{\alpha}(r^2)^{\beta} \exp((-4t)^{-1} r^2),$$

where β is nonnegative number, and α is negative number. Consequently it is sufficient to prove that the integrals

$$J(X) = \iint_{E_2} f(y)t^{\alpha}(r_1^2)^{\beta} \exp((-4t)^{-1}r^2)dy$$

are uniformly convergent in the set W_1. For the integral $J(X)$ we have the following estimation

$$|J(X)| \le M \iint_{E_2} t^{\alpha}(r^2)^{\beta} \exp((-4t)^{-1}r^2)dy,$$

where $M = \sup |f(y)|$.

Upon the change of variables

$$x_1-y_1 = 2\sqrt{t}\, u \cos\varphi, \quad x_2-y_2 = 2\sqrt{t}\, u \sin\varphi$$

we get

$$|J(X)| \le M \int_0^{2\pi} d\varphi \int_0^{\infty} 4(4tu^2)^{\beta}\, t^{\alpha+1}\, u \exp(-u^2)du =$$

$$= 2M\pi 4^{\beta+1}\, t^{\alpha+\beta+1} \int_0^{\infty} u^{2\beta+1} \exp(-u^2)du \le M_1 t^{\alpha+\beta+1} \text{ for } X \in W_1,$$

where M_1 is a convenient positive constant.

It follows from the above inequality that the integral $J(X)$ is uniformly convergent in W_1. Let $K(x_0,R)$ denote the circle with center at x_0 and radius $R > 0$. By uniform convergence of the integrals $I(X)$ and by Lemma 2 for $X \in W_2$ we have

$$D_t^i \frac{1}{4\pi}\iint_{E_2} f(y)U_1(X,y)dy = B_1^i(X) + B_2^i(X) \qquad (i = 1,2,\dots,n-1),$$

where

$$B_1^i(X) = \frac{1}{4\pi} \iint_{E_2\setminus K(x_0,R)} f(Y)D_t^i U_1(X,y)dy,$$

$$B_2^i(X) = \frac{1}{4\pi} \iint_{K(x_0,R)} f(Y)\Delta_Y^i U_1(X,y)dy, \quad i = 1,2,\dots,n-1.$$

Since $\lim B_k^i(X) = B_k^i(x_0,t)$ as $x \to x_0$, $0 < t$ $(k=1,2;\ i=1,2,\dots,n-1)$, it is sufficient to prove, that

(a) $\lim B_1^i(x_0,t) = 0$ as $t \to 0_+$,

(b) $\lim B_2^i(x_0,t) = \Delta^i f(x_0)$ as $t \to 0_+$. $\qquad (i=1,2,\dots,n-1)$

Ad (b). Since the functions $f(y)$ and $\Delta_Y^i U_1(X,y)\big|_{x=x_0}$ are of class C^{2i-2} $(i=1,2,\dots,n-1)$ in the circle $K(x_0,R)$, thus [1]

$$\iint_{K(x_0,R)} \left(f(y)\Delta_y^i U_1(X;y)\Big|_{x=x_0} - \Delta^i f(y)U_1(X,y)\Big|_{x=x_0}\right) dy =$$

$$= -\sum_{k=0}^{i-1} \int_{\partial K(x_0,R)} (\Delta^k f(y) D_{n_y} (\Delta_y^{i-k-1} U_1(X;y))\Big|_{x=x_0} -$$

$$- \Delta_y^i U_1(X;y)\Big|_{x=x_0} D_{n_y} \Delta^{i-k-1} f(y))\, d\sigma_y,$$

where n is the inward normal to the boundary $\partial K(x_0,R)$, of the domain $K(x_0,R)$.

Therefore we have

$$(7)\quad B_2^i(x_0,t) = B_3^i(x_0,t) - \sum_{k=0}^{i-1} C_k^i(x_0,t) + \sum_{k=0}^{i-1} L_k^i(x_0,t),$$

where

$$B_3^i(x_0,t) = \iint_{K(x_0,R)} \Delta^i f(y)U_1(X,y)\Big|_{x=x_0} dy \quad (i=1,2,\dots,n-1),$$

$$C_k^i(x_0,t) = \int_{\partial K(x_0,R)} \Delta^k f(y)D_n\Delta_y^{i-k-1} U_1(X,Y)\Big|_{x=x_0} d\sigma_y$$

$$(k=0,1,\dots,i-1;\ i=1,2,\dots,n-1),$$

$$L_k^i(x_0,t) = \int_{\partial K(x_0,R)} D_n\Delta^{i-k-1} f(y)\Delta_y^i U_1(X,y)\Big|_{x=x_0} d\sigma_y$$

$$(k=0,1,\dots,i-1;\ i=1,2,\dots,n-1).$$

Expanding the function $\Delta^i f(y)$ by formula

$$\Delta^i \bar{f}(y) = \begin{cases} \Delta^i f(y) & \text{for} \quad y \in K(x_0,R) \\ 0 & \text{for} \quad y \in E_2 \setminus K(x_0,R) \end{cases}$$

and by Weierstrass theorem we have

$$\lim B_3^i(x_0,t) = \Delta^i f(x_0) \quad \text{as} \quad t \to 0_+.$$

We must still prove that the other integrals in formula (7) tend to zero when $t \to 0_+$. Since $D_{n_y} U(X,Y) = - D_r U$ for $y \in \partial K(x_0,R)$ and by Lemma 2 we get

$$C_k^i(x_0,t) = -\int_{\partial K(x_0,R)} \Delta^k f(y) D_r(\Delta^{i-k-1} U_1(X;y))\Big|_{x=x_0} d\sigma_y =$$

$$= \int_{\partial K(x_0,R)} \Delta^k f(y) D_t^{i-k-1} D_r(U_1(X;y))\Big|_{x=x_0} d\sigma_y =$$

$$= \int_{\partial K(x_0,R)} \Delta^k f(y) D_t^{i-k-1}(R(-2t^2)^{-1} \exp((-4t)^{-1} R^2))\, d\sigma_y.$$

Since the derivatives $D_t^{i-k-1}((-2t^2)^{-1} R \exp((-4t)^{-1} R^2))$ are linear combinations of the functions

$$t^{-\alpha} R^{2\beta} \exp((-4t)^{-1} R^2) \quad \left(\beta \geq 0,\ \ \alpha > 0\right)$$

we must show that the functions

$$S(x_0,t) = \int_{\partial K(x_0,R)} \Delta^k f(y) t^{-\alpha} R^{2\beta} \exp((-4t)^{-1} R^2) d\sigma_y$$

tend to zero when $t \to 0_+$.

For that purpose we present $S(x_0,t)$ in the form

$$S(x_0,t) = M_2\sqrt{t} \int_{\partial K(x_0,R)} f(y)((4t)^{-1} R^2)^{\alpha+\frac{1}{2}} \exp((-4t)^{-1} R^2) d\sigma_y,$$

where M_2 is the convenient constant. By the assumptions of Lemma 3 and by the inequality

$$(8) \qquad A^a e^A \leq a^a e^{-a} \quad \text{for} \quad A \geq 0, \quad a > 0$$

we obtain

$$|S(x_0,t)| \leq M_3\sqrt{t} \to 0 \quad \text{when} \quad t \to 0_+;$$

M_3 being the convenient positive constant.

Arguing similarly as for the integrals $C_k^i(x_0,t)$ we may prove that the integrals $L_k^i(x_0,t)$ $(k=0,1,\dots,i-1;\ i=1,2,\dots,n-1)$ tend to zero as $t\to 0_+$, which implies (b).

In order to prove (a) we observe that the functions $D_t^iU_1(X,y)$ are the linear combinations of the functions

$$t^{1-\beta}((t)^{-1}r^2)^{\alpha}\exp((-4t)^{-1}r^2)\qquad (\alpha\geqq 0,\ \beta\geqq 2).$$

Then the functions $B_1^i(X)$ are also linear combinations of the integrals

$$g(x_0,t) =$$

$$= M_4 t\iint\limits_{E_2\setminus K(x_0,R)} f(y)((4t)^{-1}r^2)^{\alpha+\beta}(r^{2\beta})^{-1}\exp((-4t)^{-1}r^2)\Big|_{x=x_0}dy,$$

M_4 being the convenient positive constant.

Now we have by (8)

$$|g(x_0,t)|\leqq M_5 t\left|\iint\limits_{E_2\setminus K(x_0,R)} r^{-2}\Big|_{x=x_0}dy\right|\leqq M_6 t\to 0\ \text{ as }\ t\to 0_+,$$

where M_5, M_6 are the convenient positive constants. Therefore the condition (a) is satisfied.

Let

$$Q(X)=\sum_{i=0}^{n-1} t^i\,u_i(X), \tag{9}$$

where

$$u_i(X)=(4\pi)^{-1}\iint\limits_{E_2} h_i(y)U_1(X,y)dy,\quad X\in W,\ (i=0,1,\dots,n-1). \tag{9a}$$

Now we shall prove the following

L e m m a 4. Let the functions $f_i(y)$, $h_i(y)$ be of the class $C^{2n-2i-2}$ $(i=0,1,\dots,n-1)$ in E_2 and bounded with their derivatives up to the order $2n-2i-2$ in the set E_2. Let the function $Q(X)$ given by the formulas (9), (9a) satisfy the initial conditions (3). Then the functions f_k, h_k $(k=0,1,\dots,n-1)$ satisfy the system of the equations

$$(10)\quad \frac{1}{k!} f_k(x) = \frac{1}{k!}\sum_{i=0}^{k} \Delta^{k-i} h_i(x)\binom{k}{i} i! \quad (k=0,1,\dots,n-1) \text{ for } x \in E_2.$$

P r o o f . Making use of the initial conditions (3) and of Lemma 3 we can prove the formula (10). In virtue of Weierstrass' theorem and by the initial conditions (3) for $i = 0$, we have

$$f_0(x_0) = \lim Q(X) = h_0(x_0) \quad \text{when} \quad X \to (x_0, 0_+).$$

From the initial conditions (3), for $i = 1$, we get

$$\lim D_t Q(X) = f_1(x_0) \quad \text{as} \quad X \to (x_0, 0_+).$$

On the other hand we have

$$D_t Q(X) = u_1(X) + D_t u_0(X) + tD_t u_1(X) + \sum_{i=2}^{n-1} \left(it^{i-1} u_i(X) + t^i D_t u_i(X)\right),$$

where $u_i(X)$ $(i=0,1,\dots,n-1)$ are given by formula (9a). Now it follows from Weierstrass theorem and lemma 3, that

$$\lim D_t Q(X) = \lim D_t u_0(X) + \lim u_1(X) = \Delta h_0(x_0) + h_1(x_0)$$

when $X \to (x_0, 0_+)$. Hence

$$f_1(x_0) = \Delta h_0(x_0) + h_1(x_0).$$

In view of the initial conditions (3), for $i = k \leq n-1$, $2 \leq i$, we have

$$\lim D_t^k Q(X) = f_k(x_o) \quad \text{when} \quad X \to (x_o, 0_+).$$

On the other hand we have

$$D_t^k Q = \sum_{i=0}^{n-1} D_t^k(u_i t^i) = D_t^k u_o + \ldots + D_t^k(t^k u_k) +$$

$$+ \sum_{i=k+1}^{n-1} D_t^k(t^{k+1} u_{k+1}).$$

Moreover we obtain

$$D_t^k(tu_1) = tD_t^k u_1 + \binom{k}{1} D_t^{k-1} u_1$$

$$D_t^k(t^2 u_2) = \binom{k}{0} t^2 D_t^k u_2 + \binom{k}{1} 2t D_t^{k-1} u_2 + \binom{k}{2} 2! D_t^{k-2} u_2$$

....

$$D_t^k(t^k u_k) = \sum_{i=0}^{k} \binom{k}{i} D_t^i(t^k) D_t^{k-i} u_k = \sum_{i=0}^{k} \binom{k}{i} k(k-1)\ldots(k-i+1) t^{k-i} D_t^{k-i} u_k)$$

Thus we have

$$\lim D_t^k Q = \lim \sum_{i=0}^{k} D_t^k(t^i U_i) = \lim(D_t^k u_o + \binom{k}{1} D_t^{k-1} u_1 +$$

$$+ 2!\binom{k}{2} D_t^{k-2} u_2 + \ldots + \binom{k}{k} k! u_k).$$

Now by Lemma 3 and Weierstrass' theorem we obtain

$$\lim_{X \to (x_o, 0_+)} D_t^k Q = \Delta^k f_o(x_o) + \binom{k}{1} \Delta^{k-1} h_1(x_o) + \binom{k}{2} 2! \Delta^{k-2} h_2(x_o) + \ldots +$$

$$+ \binom{k}{k} k! h_k(x_o) \quad (k=0,1,\ldots,n-1).$$

This proves our Lemma 5. Now we shall prove the following lemma.

L e m m a 5. Let the functions f_k, h_k $(k=0,1,\dots,n-1)$ satisfy the assumptions of Lemma 4 and satisfy the system (10). Then we have

$$h_k(x) = (k!)^{-1} \sum_{i=0}^{k} (-1)^k \binom{k}{i} \Delta^i f_{k-i}(x) \quad (k=0,1,\dots,n-1). \tag{10a}$$

P r o o f . Substituting the functions h_i $(i=0,1,\dots,k)$ defined by formula (1a) into the right hand side of the k-th equation of system (10) we get

$$P_k = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} i!\Delta^{k-i} \left(\frac{1}{i!} \sum_{s=0}^{i} (-1)^s \binom{i}{s} \Delta^s f_{i-s} \right. =$$

$$= \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} \left(\sum_{s=0}^{i} (-1)^s \Delta^s f_{i-s}) \right) \quad (k=0,1,\dots,n-1).$$

If $k = 0$, then $P_k = f_0$. Let $k \geqq 1$. It can easily be checked that the sum P_k is a linear combination of the functions $\Delta^{k-1} f_l$ $(l=0,1,\dots,k)$ with the coefficients α_l^k $(l=0,1,\dots,k)$ of the form

$$P_k = \alpha_k^k f_k + \sum_{l=0}^{k-1} \alpha_l^k \Delta^{k-l} f_l, \tag{11}$$

where

$$\left\{ \begin{aligned} \alpha_k^k &= \frac{1}{k!} \\ \alpha_l^k &= \frac{1}{k!} \frac{1}{l!} \sum_{j=l}^{k} \binom{k}{j} \binom{j}{j-l} (-1)^{j+l} = \frac{1}{k!l!} \sum_{j=l}^{k} \binom{k}{j} j(j-1)\dots \end{aligned} \right. \tag{11a}$$

$$\dots(j-l+1)\,(-1)^{j+l} \quad (l=0,1,\dots,k-1).$$

In order to prove (10a) it is enough to show that $\alpha_l^k = 0$ $(l=0,1,\dots,k-1)$. Since

$$(12) \qquad (x+b)^k = \sum_{j=0}^{k} \binom{k}{j} x^j\, b^{k-j},$$

we have

$$(12a) \qquad D_x^l (x+b)^k = \sum_{j=l}^{k} \binom{k}{j} k(k-1)\dots(k-l+1) x^{j-l}\, b^{k-j} =$$

$$= k(k-1)\dots(k-l+1)(x+b)^{k-l} \quad \text{for} \quad l=1,2,\dots,k-1.$$

If we substitute $x = -1$, $b = 1$ in (12) and (12a) we obtain

$$0 = \sum_{j=0}^{k} \binom{k}{0} (-1)^j,$$

$$0 = \sum_{j=1}^{k} \binom{k}{j} k(k-l)\dots(k-l+1)(-1)^{j-l} = k!\,l!\,\alpha_l^k = k!\,\alpha_0^k$$

$$\text{for} \quad l=1,2,\dots,k-1.$$

Lemma 6. Let the functions h_i $(i=0,1,\dots,n-1)$ be bounded and measurable in the set E_2 and let

$$I(X) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} h_i(y) D_{x_2}^{\alpha_1} D_{x_2}^{\alpha_2} D_t^{\alpha}(t^i U_1(X,y))\,dy_2,$$

where $\alpha_1, \alpha_2 = 0,1,\dots,2n$; $\alpha = 0,1,\dots,n$; $i = 0,1,\dots,n-1$. Then the integrals $I(X)$ are uniformly convergent in the set

$$W_1 = \left\{ X:\ |x_i| \le a_i,\ t_1 \le t \le t_2 \right\},$$

where a_i $(i=1,2)$, t_i $(i=1,2)$ are positive numbers and $U_1(X,y) = U(X,Y)\big|_{s=0}$.

P r o o f . The integral $I(X)$ is a linear combination of the integrals

$$H(X) = t^{\beta_1} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} h_i(y) r^{\beta_2} (x_1-y_1)^{\beta_3} (x_2-y_2)^{\beta_4} \exp((4t)^{-1} r^2) dy_2,$$

where $\beta_i \geq 0$ $(i=2,3,4)$, $\beta_1 \leq n-2$.

Let us observe that the function $H(X)$ may be written in the form

$$H(X) = t^{m_o} M_1 \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} h_i(y)((t)^{-1} r^2)^{\frac{\beta_2}{2}} \exp((-16t)^{-1} r^2)((t)^{-1} \times$$

$$\times (x_1-y_1)^2)^{\frac{\beta_3}{2}} \exp((-16t)^{-1}(x_1-y_1)^2)((t)^{-1}(x_2-y_2)^2)^{\frac{\beta_4}{2}} \exp((-16t)^{-1} \times$$

$$\times (x_2-y_2)^2) \exp((-8t)^{-1} r^2) dy_2,$$

where $m_o = \beta_1 - \frac{1}{2} \sum_{i/2}^{4} \beta_i$, M_1 is a convenient constant.

Using the inequality (8) and assumptions of lemma 6 we get

$$|H(X)| \leq M_2 t^{m_o} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} \exp((-8t_2)^{-1} r^2) dy_2, \text{ for } X \in W_2.$$

M_2 being the convenient positive constant. It follows from the above inequality that the integral $H(X)$ is uniformly convergent in W_1.

Now we shall prove

T h e o r e m 1. Let the functions f_i, h_i $(i=0,1,\ldots$ $\ldots,n-1)$ be of the class $C^{2n-2i-2}$ and be bounded with their derivatives up to the $2n-2i-2$ order in the set E_2 and satisfy the equations (10a). Then the function $Q(X)$ given by formulas (9), (9a) is the solution of the problem (1), (2), (3).

P r o o f . By Lemmas 1 and 6 we have

$$P^n Q(X) = ((4\pi)^{-1} \sum_{i=0}^{n-1} \iint_{E_2} h_i(y) P^n (t^i U_1(X,y))dy = (4\pi)^{-1} \times$$

$$\times \sum_{i=0}^{n-1} \iint_{E_2} h_i(y) P_X^{n-i-1} (P_X^{i+1}(t^i U_1(X,y)))dy = 0$$

for $X \in W$ and for $i=0,1,\dots,n-1$.

By Lemma 4 and by assumptions of Theorem 1 we obtain

$$\lim_{X \to (x_0,0) \in S} D_t^i Q(X) = f_i(x_0) \quad (i=0,1,\dots,n-1).$$

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