DEMONSTRATIO MATHEMATICA
Vol. XI No 4 1978

Julian Klukowski, Cecylia Łapińska

POST POSETS AS A GENERALIZATION OF POST ALGEBRAS

Generalized Post algebras were investigated by many authors (see [2], [3], [9], [10] and [12]). The known generalizations usually consist in a weakening of the conditions concerning the chain of constants. In this paper the generalization takes another direction - instead of distributive lattices with 0 and 1, what we call d-posets are considered. In a d-poset one only assumes that least upper bounds exist for disjoint elements and one weakens the condition of distributivity. Assuming that a chain of constants exists in a d-poset and imposing conditions analogous to those of Post algebras, one defines the Post posets and studies their basic properties. The Boolean center of a Post algebra is replaced in the Post poset by a Boolean orthoposet. Then a Post poset is a generalization of a Post algebra on one hand of a Boolean orthoposet on the other. Post algebras can be characterized as Post posets in which any two complemented elements have a greatest lower bound. Theorems about the uniqueness of the chain of constants, uniqueness of monotone representation also hold in the larger class of Post posets. An example of a Post poset which is not a Post algebra is given.

In this paper the usual lattice notation is employed. The least upper bound (l.u.b) of x and y is denoted by $x \vee y$ and the greatest lower bound (g.l.b.) by $x \wedge y$, or more briefly by xy. The symbols $\bigvee_{i\in I} x_i$ and $\bigwedge_{i\in I} x_i$ denote, respectively, the supremum and infimum of the x_i over a specified set of indices. The symbols $x \vee_P y$ and $x \wedge_P y$ emphasize that the supremum and infimum are taken in the poset P. If x has a (unique) complement, it is denoted by x .

1. d-posets

Definition 1.1.: Let $(P, \leq)$ be a partially ordered set with the greatest element 1 and the least element 0; P is said to be a d-poset if the following conditions hold:

(i) $\forall a,b \in P$, if $a \wedge b = 0$ then $a \vee b$ exists in P

(ii) $\forall a,b,c \in P$, $a \wedge b = 0$ implies that if any two of the elements $(a \vee b) \wedge c$, $a \wedge c$, $b \wedge c$ exist, then the third also exists and the following equality holds:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c).$$

It is easy to see that if $a_1, a_2, \ldots, a_n$ are mutually disjoint elements of a d-poset P then $\left(\bigvee_{\substack{i=1\\ i\neq k}}^{n} a_i\right) a_k = 0$ and $\left(\bigvee_{i=1}^{n} a_i\right) \wedge b = \bigvee_{i=1}^{n} (a_i \wedge b)$ provided $a_i \wedge b$ exists in P for every $i = 1,2,\ldots,n$.

Let us remark that the class of d-posets is rather wide - every bounded poset can be transformed into a d-poset by adjoining a new zero-element.

Lemma 1.2: Let P be a d-poset. For every $x, \bar{x}, y \in P$ if $x \wedge y = 0$, $\bar{x} \wedge y = 0$ and $x \vee y = \bar{x} \vee y$, then $x = \bar{x}$.

Proof: Since $\bar{x} \leq \bar{x} \vee y = x \vee y$ then $(x \vee y) \wedge \bar{x}$ exists. Consequently by condition (ii), $x \wedge \bar{x}$ exists and $\bar{x} = (x \vee y) \wedge \bar{x} = (x \wedge \bar{x}) \vee (y \wedge \bar{x}) = x \wedge \bar{x}$, so that $\bar{x} \leq x$. Similarly $x \leq \bar{x}$. Hence $x = \bar{x}$.

Definition 1.3: If for some $x \in P$ there exists $x' \in P$ such that $x \wedge x' = 0$ and $x \vee x' = 1$, then

x' is called a complement of x and x is said to be complemented.

Lemma 1.2 implies the following corollary.

Corollary 1.4: In any d-poset every element a has at most one complement a' and $a'' = a$ provided a' exists.

From now on, let P be a d-poset and let B denote the set of all complemented elements of P.

Lemma 1.5: For every $a \in P$ and $b \in B$, $ab = 0$ iff $a \leq b'$.

The easy proof is omitted.

Lemma 1.6: For all $a,b \in B$, if ab exists in P then the elements $a'b$, ab', $a'b'$, $a \vee b$, $a' \vee b$, $a \vee b'$, $a' \vee b'$ also exist in P and the following equality holds:

$$a \vee b = a'b \vee ab \vee ab' .$$

Proof: Since $b = (a \vee a')b = ab \vee a'b$, then $a'b$ exists by condition (ii). Consequently, ab' and $a'b'$ also exist. It remains to show that $a \vee b$ exists and $a \vee b = a'b \vee ab \vee ab'$. The elements $a'b$, ab, ab' are mutually disjoint, hence the right-hand side of the above equality exists. If $a \leq c$ and $b \leq c$ for certain $c \in P$, then $(a'b \vee ab) \vee ab' = b \vee ab' \leq c$. Therefore $a'b \vee ab \vee ab'$ is the supremum of a and b. The existence of the remaining elements follows from the just proved part of the lemma.

Lemma 1.7: For all $a,b \in B$, if ab exists in P then $(ab)' = a' \vee b'$ and $(a \vee b)' = a'b'$.

Proof: The existence of $a' \vee b'$, $a \vee b$ and $a'b'$ follows from Lemma 1.6. We have $1 = b \vee b' = (ba \vee ba') \vee b' \leq ba \vee a' \vee b'$, so that $ab \vee (a' \vee b') = 1$. Similarly, $ab \wedge (a' \vee b') = ab \wedge (a'b' \vee a'b \vee ab') = (ab \wedge a'b') \vee (ab \wedge a'b) \vee (ab \wedge ab') = 0$. Thus we have proved that $a' \vee b'$ is the complement of ab. By a similar argument one can prove that $a'b'$ is the complement of $a \vee b$.

Remark: Lemmas 1.6 and 1.7 cannot be formulated dually; that is for $a,b \in B$ the infimum $a \wedge b$ need not exist in P even when $a \vee b$ exists in P, which will be shown in Example 1.9.

Lemma 1.8: The set B of all complemented elements of a d-poset P is an orthomodular poset (with the same ordering as in P).

Proof: We check the conditions of the definition of an orthomodular poset (see [5]):

1. For every $a \in B$, $a'' = a$ (by Corollary 1.4).
2. For every $a,b \in B$, $a \leq b'$ iff $b \leq a'$, since $a \leq b'$ iff $a \wedge b = 0$ (by Lemma 1.5).
3. For every $a,b \in B$, if $a \leq b'$ then $a \vee b$ exists in B. This is true since $a \leq b'$ implies $a \wedge b = 0$ and $a \vee b$ exists in P by condition (i); $a \vee b$ is complemented by Lemma 1.7.
4. For every $a,b \in B$, if $a \leq b$ then $a \vee (b' \vee a)' = b$. Since $a \leq b$ implies that ab exists in P, then, by Lemma 1.6, $a'b$ exists in P and, by Lemma 1.7, $a'b$ is complemented. Clearly, $a \wedge a'b = 0$. Hence $(a \vee a'b) \wedge b = ab \vee a'b = (a \vee a') \wedge b = b$ so that $b \leq a \vee a'b$. On the other hand $a \vee a'b \leq b$, thus $b = a \vee a'b = a \vee (b' \vee a)'$.

Notice that if B is regular in P (i.e. for every $x,y \in B$ the supremum and infimum of x and y exist in B iff they exist in P, and they are equal if they exist), then B is a Boolean orthoposet because of Lemma 1.5. In particular, if $a \wedge b$ exists in P for every $a,b \in B$, then B is a Boolean algebra. The orthomodular poset B need not be regular in P, which is shown by the following example.

Example 1.9: Let X be a closed circle in the plane. We place the origin of a polar coordinate system $Or\varphi$ in the centre of the circle X. Let φ_1, φ_2 be real numbers and $\varphi_1 < \varphi_2$; by $<\varphi_1, \varphi_2>$ we shall understand the closed sector lying between the radii $\varphi = \varphi_1$ and $\varphi = \varphi_2$. Now let $p_1 = <0, \pi>$, $p_2 = <\frac{\pi}{2}, \frac{3}{2}\pi>$, $p_3 = <\pi, 2\pi>$, $p_4 =$

$= < \frac{3}{2}\pi , \frac{5}{2}\pi >$ and $t_k^i = < \frac{\pi}{2} \frac{(i+1)\ 2^{k-1} - 1}{2^{k-1}}, \frac{\pi}{2} \frac{(i+1)\ 2^k - 1}{2^k} >$ for $i = 0,1,2,3$; $k = 1,2,\dots$. Let Y denote the family consisting of all the above sectors and the empty set, ordered by the inclusion. Now we take the poset P constructed as follows: $x \in P$ iff $x = \bigcup_{i=1}^{n} x_i$ for some elements $x_1,x_2,\dots$ $\dots,x_n \in Y$ such that $x_i \wedge_Y x_j = 0$ for $i \neq j$. It is easy to see that P, ordered by the inclusion, is a d-poset with the unit X. The set B of complemented elements of P consists only of the elements $\emptyset$, X, p_1, p_2, p_3, p_4. The infimum of p_1 and p_2 does not exist in P, however $p_1 \vee p_2 = X$ and $p_1 \wedge_B p_2 = \emptyset$.

2. Post posets

D e f i n i t i o n 2.1: Let $(P, \leqslant)$ be a d-poset with $0 \neq 1$; P is called a Post poset (of order n) if it satisfies the following conditions:

(p_1) There exists a chain $0 = e_0 \leqslant e_1 \leqslant \dots \leqslant e_{n-1} = 1$ in P such that for every $x \in P$ and $1 \leqslant i \leqslant n-1$

a) xe_i exists and if $xe_1 = 0$ then $x = 0$;

b) $x \vee e_i$ exists and if $x \vee e_{i-1} = e_i$ then $x = e_i$.

(p_2) For every x P, there exists a sequence $C_0(x), C_1(x), \dots, C_{n-1}(x)$ of elements of P such that

c) $C_i(x)\ C_j(x) = 0$ for every $i \neq j$

d) $\bigvee_{i=0}^{n-1} C_i(x) = 1$

e) $x = C_1(x)e_1 \vee \dots \vee C_{n-1}(x)e_{n-1}$

From now on, let P denote a Post poset.

L e m m a 2.2: a) If $x \in P$ and $xe_i = 0$ for some $i > 1$, then $x = 0$. b) If $x \vee e_i = e_j$ for some $i < j$ then $x = e_j$.

P r o o f : Immediate from condition (p_1).

L e m m a 2.3: If $x,y \in P$ and $xy = 0$ then

$$\bigvee_{i=1}^{n-1} C_i(x) \wedge \bigvee_{i=1}^{n-1} C_i(y) = 0.$$

P r o o f : Suppose that $C_k(x)C_l(y) = 0$ does not hold for some k and l. Then there exists $z \in P$, $z \neq 0$, $z \leqslant C_k(x)$ and $z \leqslant C_l(x)$. Let $k \leqslant l$. Thus $ze_k \leqslant x$ and $ze_k \leqslant ze_l \leqslant y$, which contradicts $xy = 0$. Therefore $C_i(x)\, C_j(y) = 0$ for every $i,j = 1,\dots,n-1$ and by applying condition (ii) of Def.1.1 we obtain $\bigvee_{i=1}^{n-1} C_i(x) \wedge \bigvee_{i=1}^{n-1} C_i(y) = 0$.

T h e o r e m 2.4: An element $x \in P$ is complemented iff $x = C_i(y)$ for some i and some $y \in P$.

P r o o f : If $x = C_i(y)$ then $x' = \bigvee_{\substack{j=0 \\ j\neq i}}^{n-1} C_j(y)$. Indeed, it follows directly from Definitions 1.1 and 2.1 that $\bigvee_{j\neq i} C_j(y)$ exists and $x \wedge \bigvee_{j\neq i} C_j(y) = 0$, $x \vee \bigvee_{j\neq i} C_j(y) = 1$. On the other hand, suppose x has a complement x'. Since $xx' = 0$ then $x'C_{n-1}(x) = 0$ from Lemma 2.3. Hence $x' \vee C_{n-1}(x)$ exists. Since $x \leqslant e_{n-2} \vee C_{n-1}(x)$ by condition (p_2) then $1 = x \vee x' \leqslant e_{n-2} \vee C_{n-1}(x) \vee x'$. Then $x' \vee C_{n-1}(x) = 1$ from (p_1). Therefore $C_{n-1}(x)$ is a complement of x' and $C_{n-1}(x) = x$ since the complements are unique (Corollary 1.4).

Thus we have shown that the set B of all complemented elements of a Post poset P is exactly the set $\{C_j(x) : x \in P,\ j = 0,1,\dots,n-1\}$.

L e m m a 2.5: If b is a complemented element of P $(b \in B)$ and $be_j \leqslant be_i$ for some $i < j$, then $b = 0$.

P r o o f : Since $e_i \leqslant e_j$ then $e_j = (b' \vee b)e_j = b'e_j \vee be_j \leqslant b'e_j \vee be_i \leqslant b'e_j \vee e_i \leqslant e_j$ so that $b'e_j \vee e_i = e_j$. Hence $b'e_j = e_j$ by Lemma 2.2. Thus $be_j = 0$; consequently $b = 0$ by Lemma 2.2.

L e m m a 2.6: If $a,b \in B$ and $i = 1,2,\dots,n-1$ then $ae_i \leqslant be_i$ implies $a \leqslant b$.

P r o o f : If $a \nleqslant b$ then $ab' \neq 0$ or ab' does not exist by Lemma 1.5. Then there exists $c \in P$, $c \neq 0$, $c \leqslant a$ and $c \leqslant b'$, so that $cb = 0$ (by Lemma 1.5). Hence $ce_i \leqslant ae_i$, $ce_i \wedge be_i = 0$ and, by Lemma 2.2, $ce_i \neq 0$. Therefore $ae_i \nleqslant be_i$ and this completes the proof.

T h e o r e m 2.7: If P is a Post poset then the set B of all complemented elements of P is a Boolean orthoposet (with the same ordering as in P).

P r o o f : Using Lemmas 1.5 and 1.8 it suffices to show that if $a \wedge_B b = 0$ then also $a \wedge_P b = 0$ for any $a,b \in B$. Suppose $a \wedge_B b = 0$ but $a \wedge_P b = 0$ does not hold. Then there is a non-zero element $x \in P$ such that $x \leq a$ and $x \leq b$. Let $x = C_1(x)e_1 \vee \dots \vee C_{n-1}(x)$. Take the greatest k such that $C_k(x) \neq 0$. Thus $C_k(x)e_k \leq a$, $C_k(x)e_k \leq b$ and also $C_k(x)e_k \leq ae_k$, $C_k(x)e_k \leq be_k$. Therefore by Lemma 2.6 $C_k(x) \leq a$ and $C_k(x) \leq b$, which contradicts $a \wedge_B b = 0$ since $C_k(x) \in B$.

L e m m a 2.8: Let $x,y \in P$, $x = C_1(x)e_1 \vee \dots \vee C_{n-1}(x)$ and $y = C_1(y)e_1 \vee \dots \vee C_{n-1}(y)$. Then $x \leq y$ iff $\bigvee_{i=k}^{n-1} C_i(x) \leq \bigvee_{i=k}^{n-1} C_i(y)$ for every $k = 1,2,\dots,n-1$.

P r o o f : Let $x \leq y$ and suppose $\bigvee_{i=k}^{n-1} C_i(x) \nleq \bigvee_{i=k}^{n-1} C_i(y)$ for some k. Then, by the disjunctivity of Boolean orthoposet B (see [2] and [6]), there is a non-zero element $b \in B$ such that $b \leq \bigvee_{i=k}^{n-1} C_i(x)$ and $b \wedge \bigvee_{i=k}^{n-1} C_i(y) = 0$. Hence

$$be_k \leq \bigvee_{i=k}^{n-1} C_i(x)e_k \leq x \leq y, \text{ so that } be_k = be_k y =$$

$$= be_k \bigvee_{i=1}^{n-1} C_i(y)e_i = be_k \bigvee_{i=1}^{k-1} C_i(y)e_i \leq be_k e_{k-1} = be_{k-1}.$$

Therefore it follows from Lemma 2.5 that $b = 0$; a contradiction. On the other hand, if $\bigvee_{i=k}^{n-1} C_i(x) \leq \bigvee_{i=k}^{n-1} C_i(y)$ for $k = 1,2,\dots,n-1$ then $C_k(x) \leq \bigvee_{i=k}^{n-1} C_i(y)$ and $C_k(x)e_k \leq$

$$\leq \bigvee_{i=k}^{n-1} C_i(y)e_k \leq \bigvee_{i=k}^{n-1} C_i(y)e_i \leq y. \text{ Therefore } x \leq y.$$

T h e o r e m 2.9: For any given $x \in P$ there is only one sequence of elements $C_0(x),\dots,C_{n-1}(x)$ satisfying condition (p_2).

P r o o f : Suppose that $x = C_1(x)e_1 \vee \dots \vee C_{n-1}(x) = \bar{C}_1(x)e_1 \vee \dots \vee \bar{C}_{n-1}(x)$. Thus $C_{n-1}(x) \leq \bar{C}_{n-1}(x)$ and $\bar{C}_{n-1}(x) \leq C_{n-1}(x)$ by Lemma 2.8, hence $C_{n-1}(x) = \bar{C}_{n-1}(x)$. Since $\bigvee_{i=1}^{n-2} C_i(x)e_i \vee C_{n-1}(x) = \bigvee_{i=1}^{n-2} \bar{C}_i(x)e_i \vee C_{n-1}(x)$ and $\bigvee_{i=1}^{n-2} C_i(x)e_i \wedge C_{n-1}(x) = 0$, $\left(\bigvee_{i=1}^{n-2} \bar{C}_i(x)e_i\right) C_{n-1}(x) = 0$ then $\bigvee_{i=1}^{n-2} C_i(x)e_i = \bigvee_{i=1}^{n-2} \bar{C}_i(x)e_i$ by Lemma 1.2. Using again Lemma 2.8 we obtain $\bar{C}_{n-2}(x) = C_{n-2}(x)$. Reiterating the above argumentation we have $C_i(x) = \bar{C}_i(x)$ for $i = 1,2,\dots,n-1$; also $C_o(x) = \left(\bigvee_{i=1}^{n-1} C_i(x)\right)' = \left(\bigvee_{i=1}^{n-1} \bar{C}_i(x)\right)' = \bar{C}_o(x)$ and this completes the proof.

T h e o r e m 2.10: In any Post poset the elements e_i, $i = 0,1,\dots,n-1$, are distinct and unique.

P r o o f : It follows from Theorem 2.9 that $C_j(e_j) = 1$ and $C_i(e_j) = 0$ for $i \neq j$. If $i \neq j$ and $e_i = e_j$ then $C_i(e_i) = 1 = C_j(e_j) = C_j(e_i)$, which contradicts condition (p_2). If there is another sequence $0 = \bar{e}_o \leq \bar{e}_1 \leq \dots \leq \bar{e}_{n-1} = 1$ satisfying (p_1) and (p_2) then $e_i = \bigvee_{k=1}^{n-1} C_k(e_i)\bar{e}_k = \bar{e}_i$ so that $e_i = \bar{e}_i$ for $i = 1,\dots,n-2$. Thus the elements e_i, $i = 0,1,\dots,n-1$ are both unique and distinct.

T h e o r e m 2.11: Every Post poset P is pseudocomplemented; that is, for any $x \in P$, there exists $x^* \in P$ such that $xy = 0$ iff $y \leq x^*$; moreover $x^* \vee x^{**} = 1$.

P r o o f : We shall show that $x^* = C_o(x)$. By condition (p_2), $xC_o(x) = 0$ and $xy = 0$ if $y \leq C_o(x)$. Conversely, if $xy = 0$ then by Lemma 2.3, $\bigvee_{i=1}^{n-1} C_i(x) \wedge \bigvee_{i=1}^{n-1} C_i(y) = 0$. Since for every $i = 1,2,\dots,n-1$ it is $C_i(x) \in B$, we have $\bigvee_{i=1}^{n-1} C_i(y) \leq \left(\bigvee_{i=1}^{n-1} C_i(x)\right)' = C_o(x)$, hence $y \leq \bigvee_{i=1}^{n-1} C_i(y) \leq C_o(x)$. Thus we have also $x^* \vee x^{**} = C_o(x) \vee C_o^*(x) = C_o(x) \vee C_o'(x) = 1$.

T h e o r e m 2.12: For any non-increasing sequence $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{n-1}$ of complemented elements of a Post poset P, the supremum $b_1e_1 \vee \ldots \vee b_{n-1}$ exists in P and conversely, every $x \in P$ has a monotone representation; that is, for every $x \in P$, there is exactly one sequence of complemented elements $D_1(x) \geqslant D_2(x) \geqslant \ldots \geqslant D_{n-1}(x)$ such that $x = \bigvee_{i=1}^{n-1} D_i(x)e_i$.

P r o o f : If $b_1, b_2, \ldots, b_n \in B$ and $b_1 \geqslant b_2 \geqslant \ldots$ $\ldots \geqslant b_{n-1}$ then $b_i b'_{i+1}$ exists in B for $i = 1, \ldots, n-2$ and $b_i b'_{i+1} \wedge b_j b'_{j+1} = 0$ for $i \neq j$, so that $\bigvee_{i=1}^{n-1} b_i b'_{i+1} e_i$ exists; it is easy to see that $\bigvee_{i=1}^{n-1} b_i b'_{i+1} e_i =$ $= \bigvee_{i=1}^{n-1} b_i e_i$. Now take an arbitrary $x \in P$. Let $x = \bigvee_{i=1}^{n-1} C_i(x)e_i$. Then obviously $x = C_1(x)e_1 \vee C_2(x)(e_1 \vee e_2) \vee \ldots$ $\ldots \vee C_{n-1}(x)(e_1 \vee e_2 \vee \ldots \vee e_{n-1})$ and from the conditions of Definition 1.1 we infer $x = \left(\bigvee_{i=1}^{n-1} C_i(x)\right)e_1 \vee \left(\bigvee_{i=2}^{n-1} C_i(x)\right)e_2 \vee \ldots$ $\ldots \vee C_{n-1}(x)e_{n-1}$. Therefore $x = D_1(x)e_1 \vee \ldots \vee D_{n-1}(x)e_{n-1}$, where $D_k(x) = \bigvee_{i=k}^{n-1} C_i(x)$ and obviously every $D_i(x)$ is complemented as well as $D_1(x) \geqslant \ldots \geqslant D_{n-1}(x)$. The uniqueness of the monotone representation follows directly from Lemma 2.8, which just states that $x \leqslant y \Longleftrightarrow D_i(x) \leqslant D_i(y)$ for $i =$ $= 1, 2, \ldots, n-1$.

L e m m a 2.13: For every $x, y \in P$ the supremum $x \vee y$ (the infimum xy) exists in P iff $D_i(x) \vee D_i(y)$ ($D_i(x) \wedge D_i(y)$, respectively) exists in P for every $i = 1, 2, \ldots, n-1$, and the following equality holds: $x \vee y =$ $= \bigvee_{i=1}^{n-1} \left(D_i(x) \vee D_i(y)\right)e_i$ $\left(xy = \bigvee_{i=1}^{n-1} D_i(x)D_i(y)e_i\right.$, respectively$\left.\right)$.

P r o o f : First, let $D_i(x) \vee D_i(y)$ exist for $i = 1, 2, \ldots, n-1$. The elements $D_i(x) \vee D_i(y)$ form a monotone sequence, hence $z = \bigvee_{i=1}^{n-1} \left(D_i(x) \vee D_i(y)\right)e_i$ exists by Theorem 2.12. Of course, $x \leq z$ and $y \leq z$ by Lemma 2.8. Now let $u \in P$, $u = \bigvee_{i=1}^{n-1} D_i(u)e_i$ and $x \leq u$, $y \leq u$. Applying again Lemma 2.8 we have $D_i(x) \leq D_i(u)$ and $D_i(y) \leq D_i(u)$ for every $i = 1,2,\dots,n-1$. Thus $D_i(x) \vee D_i(y) \leq D_i(u)$ so that $z \leq u$; therefore z is the least upper bound of x and y. To prove the converse, assume $x \vee y = z$ exists. Then $D_i(x) \leq D_i(z)$ and $D_i(y) \leq D_i(z)$ for every $i = 1,\dots,n-1$. Suppose $D_k(x) \vee D_k(y)$ does not exist for some k. Hence for any $c \in B$ such that $D_k(x) \leq c$ and $D_k(y) \leq c$, there exists $d \in B$ such that $D_k(x) \leq d$, $D_k(y) \leq d$ and $c \nleq d$. In particular, there is $d \in B$ such that $D_k(x) \leq d$, $D_k(y) \leq d$ and $D_k(z) \nleq d$. Consider the element $w = e_1 \vee e_2 \vee \dots \vee e_{k-1} \vee de_k \vee \dots \vee d$. It is evident that $x \leq w$ and $y \leq w$ but $z \nleq w$; a contradiction. This completes the proof for the supremum. The proof for the infimum is analogous.

As an immediate consequence of Theorem 2.12 and Lemmas 2.8, 2.13 we obtain the following corollary:

Any Post poset of order n can be considered as a set of all non-increasing sequences $(b_1,b_2,\dots,b_n)$ of elements of some Boolean orthoposet $(B,\leq)$ with the ordering as follows: $(a_1,a_2,\dots,a_{n-1}) \leq (b_1,b_2,\dots,b_{n-1})$ iff $a_i \leq b_i$ for every $i = 1,2,\dots,n-1$.

Of course, every Post algebra is a Post poset but not conversely. Post algebras can be characterized as follows:

T h e o r e m 2.14: A Post poset is a Post algebra iff greatest lower bounds exist in P for any two complemented elements of P.

P r o o f : The necessity of the condition is obvious, so we prove only the sufficiency. If for any two complemented elements x and y the infimum xy exists in P, then the Boolean orthoposet B of all complemented elements of P is a Boolean algebra (see [5]). Hence for every $x,y \in P$

$D_i(x) \vee D_i(y)$ and $D_i(x) \wedge D_i(y)$ exist for $i = 1,2,\ldots,n-1$; consequently P is a lattice by Lemma 2.13. Since P is d-poset and every $x \in P$ has a finite disjoint representation, then using condition (ii) of 1.1 it is not difficult to prove that the lattice P is distributive. Therefore it is clear by Definition 2.1 that P is a Post algebra.

T h e o r e m 2.15: A d-poset P is a Post poset of order n iff P has a subchain $0 = e_0 \leq e_1 \leq \ldots \leq e_{n-1} = 1$ and there exists a Boolean orthoposet $B \subset P$, regular in P, such that the following conditions hold:

(1) be_i exists in P for every $b \in B$ and $i = 0,1,\ldots,n-1$;

(2) for every $x \in P$ there exists a sequence $b_1 \geq b_2 \geq \ldots \geq b_{n-1}$ of elements of B such that $x = b_1e_1 \vee \ldots \vee b_{n-1}$;

(3) if $a \in B$ and $ae_i \leq e_{i-1}$ then $a = 0$.

P r o o f : First we prove the necessity of the above conditions. For every Post poset P the set B of all complemented elements of P is a Boolean orthoposet, regular in P, by Theorem 2.7. (1) follows from (p_1) and (2) follows from Theorem 2.12. Condition (3) holds, because if $ae_i \leq e_{i-1}$ for $a \in B$, then $ae_i \leq ae_{i-1}$, so that $a = 0$ by Lemma 2.5.

Now, to prove the sufficiency of the above conditions we shall verify axioms (p_1) and (p_2). Let (1), (2), (3) hold and x be an arbitrary element of P, $x = \bigvee_{i=1}^{n-1} b_ie_i$, $b_1 \geq b_2 \geq \ldots \geq b_{n-1}$, $b_i \in B$ for $i = 1,2,\ldots,n-1$. To show that (p_2) holds we put $C_0(x) = b_1'$, $C_i(x) = b_i b_{i+1}'$ for $i = 1,\ldots,n-2$, $C_{n-1}(x) = b_{n-1}$. This definition is correct, since B is regular in P and all the elements $C_i(x)$ exist in B. Clearly, (p_2)c) and (p_2)d) hold; (p_2)e) is also satisfied because by 1.1 we have:

$$x = b_1e_1 \vee \ldots \vee b_{n-1} = b_1(b_2' \vee b_2)e_1 \vee b_2e_2 \vee \ldots \vee b_{n-1} =$$

$$= b_1b_2'e_1 \vee (b_2e_1 \vee b_2e_2) \vee \ldots \vee b_{n-1} =$$

$$= b_1 b'_2 e_1 \vee b_2 e_2 \vee \ldots \vee b_{n-1} = \ldots =$$

$$= b_1 b'_2 e_1 \vee b_2 b'_3 e_2 \vee \ldots \vee b_{n-1} = C_1(x)e_1 \vee \ldots \vee C_{n-1}(x).$$

It remains to check (p_1). Observe that xe_i exists for every $x \in P$ and $i = 1,\ldots,n-1$ because $x = \bigvee_{i=1}^{n-1} b_i e_i = \bigvee_{i=1}^{n-1} C_i(x)e_i$ by the just proved (p_2), so that xe_i is the supremum of the disjoint elements $C_1(x)e_1, C_2(x)e_2, \ldots, \left(\bigvee_{k=i}^{n-1} C_k(x)\right) e_i$, each of which exists by the condition (1). If $xe_1 = 0$, where $x = \bigvee_{i=1}^{n-1} C_i(x)e_i$, $C_i(x) \in B$, then $\left(\bigvee_{i=1}^{n-1} C_i(x)e_i\right)e_1 = \left(\bigvee_{i=1}^{n-1} C_i(x)\right)e_1 = 0$. Hence by (3) we have $\bigvee_{i=1}^{n-1} C_i(x) = 0$ as well as $C_i(x) = 0$ for $i = 1,\ldots,n-1$, so that $x = 0$. Therefore (p_1)a) is proved to hold. Now observe that $x \vee e_i$ also exists for every $x \in P$ and $i = 1,\ldots,n-1$. In fact,

$$x \vee e_i = \bigvee_{k=1}^{n-1} b_k e_k \vee e_i = e_i \vee b_{i+1}e_{i+1} \vee \ldots \vee b_{n-1} =$$

$$= b'_{i+1}e_i \vee b_{i+1}b'_{i+2}e_{i+1} \vee \ldots \vee b_{n-2}b'_{n-1}e_{n-2} \vee b_{n-1}$$

exists as the supremum of disjoint elements. If $x \vee e_{i-1} \leq e_i$ for some i then evidently $x \leq e_i$. We shall show that also $e_i \leq x$. Since $x \vee e_{i-1} = e_{i-1} \vee b_i e_i \vee \ldots \vee b_{n-1} = e_i$, then for $k = 1,\ldots,n-1-i$ $b_{i+k}e_{i+k} \leq e_i$, so that $b_{i+k} = 0$ by condition (3). Hence $e_i = e_{i-1} \vee b_i e_i = b'_i e_{i-1} \vee b_i e_i$ and by applying the properties of a d-poset we infer $b'_i e_i = b'_i e_{i-1} \vee 0 \leq e_{i-1}$. Thus from the condition (3) $b'_i = 0$ and we obtain $b_i = 1$; consequently $e_i \leq x$. Therefore (p_2)b) also holds. This completes the proof.

Finally we present an example of a Post poset which is not a Post algebra. Theorem 2.14 yields that any such poset must be infinite.

E x a m p l e 2.16: Consider, as in Example 1.9, a closed circle X in the plane. Let A be the family consisting of the empty set and all finite unions of closed sectors of X. Then A is a Boolean algebra under the operations $\vee$, $\wedge$, $'$ defined in the following way: $a \vee b = a \cup b$, $a \wedge b = \mathrm{Int}(a \cap b)$ $a' = \overline{-a}$, where $\cup$, $\cap$, $-$ are the set--theoretical operations and $\bar{a}$ denotes the topological closure of a. Now let p_1, p_2 and t_k^i be the same as in Example 1.9. Denote by T the set $\{t_k^i = i = 0,1,2,3;\ k = 1,2,\ldots\}$. Consider, for $i = 1,2$, the Boolean algebra A_i generated by $T \cup \{p_i\}$ in A. It is not difficult to show (see also [8]) that $B = A_1 \cup A_2$ is a Boolean orthoposet with the same ordering and complementation as in A. Now let e_1 be a closed circle, the radius of which is less than the radius of X, and let X and e_1 have the same centre. Consider the set P of the elements of the form $(e_1 \cap b_1) \cup b_2$ where $b_1, b_2 \in B$ and $b_1 \geqslant b_2$. One can prove that P ordered by the inclusion is a Post poset of order 3 with the chain $\phi < e_1 < X$. Since $p_1 \wedge p_2$ does not exist in P, P is not a Post algebra.

BIBLIOGRAPHY

[1] R. B a l b e s , Ph. D w i n g e r : Distributive lattices, University of Missouri Press (1974).

[2] C.C. C h a n g A. H o r n : Prime ideals characterization of generalized Post algebras. Lattice theory, Proc. Symp. Pure Math. 2 (1961) 43-48.

[3] Ph. D w i n g e r : Generalized Post algebras, Bull. Polon. Sci., Sér. Sci. Math. Astron. Phys. 16 (1968) 559-563.

[4] G. E p s t e i n : The lattice theory of Post algebras, Trans. Amer. Math. Soc. (1960) 300-317.

[5] J. K l u k o w s k i : On Boolean orthomodular posets, Demonstratio Math. 8 (1975) 5-14.

[6] J. Klukowski: On the representation of Boolean orthomodular partially ordered sets, Demonstratio Math. 8 (1975) 405-422.

[7] C. Łapińska: Vollstandige Boolesche orthomodulare teilweise geordnete Mengen, Demonstratio Math. 8 (1975) 465-469.

[8] C. Łapińska: On the structure of Boolean orthoposets, Demonstratio Math. 10 (1977) 497-514.

[9] H. Rasiowa: On generalized Post algebras of order ω^+ and ω^+-valued predicate calculi, Bull. Polon. Sci., Sér. Sci. Math. Astron. Phys. 21 (1973) 209-219.

[10] H. Sawicka: On some properties of Post algebras with countable chain of constants, Colloq. Math. 25 (1972) 201-209.

[11] T. Traczyk: Axioms and some properties of Post algebras, Colloq. Math. 10 (1963) 193-210.

[12] T. Traczyk: On Post algebras with uncountable chain of constants. Algebras of homomorphisms, Bull. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 15 (1964) 147-149.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
Received November 14. 1977.