DEMONSTRATIO MATHEMATICA
Vol. XI No 4 1978

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ORIENTABILITY OF n-DIMENSIONAL PROJECTIVE GEOMETRY

By geometry I shall mean Klein geometry whose basic notions were defined in [1] and [2]. The notions of orientability and orientation of Klein geometry were introduced in [4], [5], [6]. In this paper I shall use the definition of orientation given in [5] which is equivalent with Z. Moszner's definition given in [6] and which seems to me more convenient in applications. The particular case of Klein geometry is n-dimensional projective geometry over the field of real numbers. In the paper it is proved that such geometry of odd dimension is orientable but of even dimension is not orientable. Orientability of projective geometry of an odd dimension was shown by another method in [3] and [4]. In those papers was also shown the construction of orientation in such geometry. Thus the results of this paper are completion of papers [3] and [4]. Of course, the fact of orientability or non-orientability of correspondingly dimensional projective geometry is well known but the proofs given in this paper are new. Therefore I think that they are worth while to publish them. The publication of below considerations is justified also by two following facts. The first - the notions of orientability and orientation are difficult and good knowledge of their different properties is of no little account. The second - these considerations are the example of application of abstract notions of Klein geometry and their properties given in [1] and [2].

Since papers [3] and [5] are in press, in the first place I shall make known to main notions and properties which will be necessary within this paper. For shortness of considerations we shall avoid the repetition of proofs within the range of possibility.

In section 1 I recall to mind the notions related with Klein geometry. The examples of geometries are given in section 2. Particularly in the second section I define the projective geometry which is basic in the paper. The important notion of the reper is given in the third section. The fourth section contains the definition of s-orientability and s-orientation of geometry and their properties. Particularly there is given relation between the orientability of geometry and orientability of the group which determinates this geometry. The fifth section contains the examples illustrating previous considerations. The last section contains the proof of the main theorem on orientability or non-orientability of n-dimensional projective geometry of respective dimension.

1. Klein geometry and its basic properties

According to the definition given in [1], [2], the Klein geometry is the triple

$$(M,\ G,\ f), \tag{1.1}$$

where M is an arbitrary set, G is an arbitrary group and f denotes an effective operation of the group G on the set M, i.e. f is a mapping of the Cartesian product $M \times G$ into M satisfying the conditions

$$\bigwedge_{x \in M} \bigwedge_{g_1, g_2 \in G} f\left(f(x,g_1),g_2\right) = f(x,g_2 \cdot g_1), \tag{1.2}$$

$$\bigwedge_{x \in M} f(x,e) = x, \tag{1.3}$$

$$\bigwedge_{x \in M} f(x,g) = x \Longrightarrow g = e \tag{1.4}$$

In these relations e denotes the neutral element of the group G. The geometry is called transitive if the operation f is transitive, i.e. if it satisfies the condition

$$(1.5) \qquad \bigwedge x_1, x_2 \in M \ \bigvee g \in G \ \text{ such that } \ f(x_1, g) = x_2$$

If in (1.5) there exists only one element g, then the operation f is called simply transitive.

Let us consider two geometries

$$(1.6) \qquad (M_1,\ G_1,\ f_1) \quad \text{and} \quad (M_2,\ G_2,\ f_2).$$

By a morphism of the first geometry into the second one we understand a pair (h, φ) where $h : M \to M$ is a mapping of M into M and φ is a homomorphism G_1 into G_2. This pair must satisfy the condition

$$(1.7) \qquad \bigwedge x \in M_1 \ \bigwedge g \in G_1 \quad f_2\Big(h(x),\ \varphi(g)\Big) = h\Big(f_1(x,g)\Big).$$

If h is a bijection and φ is an isomorphism then geometries (1.6) are called equivalent.

By an object of the geometry (1.1) we understand a triple

$$(1.8) \qquad (\mathfrak{M},\ G,\ F),$$

where $\mathfrak{M}$ is an arbitrary set, G is the same group which appears in the definition of geometry (1.1) and F is an operation of G onto $\mathfrak{M}$ not necessarily effective i.e. $F : \mathfrak{M} \times G \to \mathfrak{M}$ satisfies conditions (1.2) and (1.3).

2. Examples of Klein geometries

1. Let G be an arbitrary group. Let us denote by l a left-hand translation in G, i.e. the mapping $l : G \times G \to G$ defined as follows: $\bigwedge x \in G$ and $\bigwedge g \in G$ $l(x,g) = g.x$. It is easy to verify that the triple

$$(2.1) \qquad (G,\ G,\ l)$$

is a Klein geometry in the sense of the previous definition. It is a transitive geometry, moreover it is simply transitive.

2. Let H be an arbitrary subgroup of G. Let us denote by G/H the set of the left-hand classes of abstractions with respect to the subgroup H. An element of the set G/H which contains $a \in G$ will be denoted by $[a]$. Then $[a] = a \cdot H = \{a \cdot x : x \in H.\}$ If we shall define an operation F of the group G on the set G/H as follows

$$(2.2) \qquad [x] \in G/H \wedge g \in G \wedge F([x],g) := g[x] = [gx]$$

then the triple

$$(2.3) \qquad (G/H,\ G,\ F)$$

represents an object of the geometry (1.1). The operation F may not be effective. If it is effective then (2.3) represents also some transitive geometry. The operation F is effective if and only if H does not contain non trivial subgroups invariant with respect to G. In general each transitive object of the geometry (1.1) is equivalent with some object (2.3). If G is a Lie group and H is a Lie subgroup then the triple $(G/H, G, F)$ is called a homogeneous space (or Klein space or transitive group of Lie transformations - (see [7], p. 40).

3. Let us denote by $GA(n,R)$ an affine group, i.e. the set of pairs (A,a) where $A \in GL(n,R)$, $a \in R^n$ with the operation $\wedge (A,a) \in GA(n,R)$ and $\wedge (B,b) \in GA(n,R)$, $(B,b) \circ (A,a) = (B \cdot A,\ B \cdot A + b)$. The triple

$$(2.4) \qquad (R^n,\ GA(n,R),\ f), \quad f(x,A,a) := A \cdot x + a$$

is a Klein geometry which is called an n-dimensional affine geometry.

4. In the set $R_*^{n+1} := R^{n+1} - \{(0,\ldots,0)\}$ we introduce a relation of proportionality $\sim$: $\wedge \xi \in R_*^{n+1} \ \wedge \varrho \in R_*^{n+1}$

$\eta \sim \xi \Leftrightarrow \vee \lambda \in R \quad \eta = \lambda\xi$. The quotient set $P^n : R_*^{n+1}/\sim$ is called an n-dimensional projective space. Let us denote by $GP(n,R) := GL(n+1, R)/\{\rho E;\ \rho \neq 0,\ \rho \in R\}$ the quotient group of the linear group $GL(n+1,R)$ by its centre. The elements of $GP(n,R)$ represent the sets of non-singular proportional matrices. An element containing the matrix A will be denoted by the symbol $<A>$. It has the following form $<A> := \{\rho A : A \in GL(n+1,R),\ \rho \neq 0\}$. The operation f of the group $GP(n,R)$ onto P^n is defined as follows: $\wedge [\xi] \in P^n \wedge <A> \in GP(n,R),\ f\big([\xi],\ <A>\big) := [A\cdot\xi]$. Then the triple

$$(P^n,\ GP(n,R),\ f) \tag{2.5}$$

represents a geometry which is an called n-dimensional projective geometry.

3. Repers in Klein geometry

Reper is the basic notion of geometry. It is also the basis for the definition of orientation of geometry. Now we shall define a reper in a Klein geometry and we shall give its most important properties. Let us consider the Klein geometry (1.1). With every subset $P \subset M$ we can connect some subgroup $H(P) \subset G$, which we define as follows: $H(P) := \{g \in G : \wedge\ x \in P\ \ f(x,g) = x\}$ and we call it the subgroup of non-effectiveness with respect to P. The operation is effective if and only if $H(M) = \{e\}$. The set P is called a reper if $H(P) = \{e\}$. Hence it follows that according to the above definition the whole set M is a reper. We are interested in finding repers which have as little points as possible. During the course of considerations we shall restrict ourselves to finite repers, i.e. containing the finite number of points. In this case it is more convenient to consider the sequences of points instead of sets. Therefore we assume the following definition.

D e f i n i t i o n 3.1. By a reper of the order s in the geometry (1.1) we mean a finite sequence of s different points from M, $\{p_1, p_2, \ldots, p_s\}$, $p_i \neq p_j$, $i \neq j$, $i,j = 1,2,\ldots,s$ such that $H(\{p_1, p_2, \ldots, p_s\}) = \{e\}$.

The following questions appear: Do there exist repers of finite order in each geometry? How to find them? Do there exist the minimal repers and how to find them? I do not know answers to these problems. Some results on this field have been obtained by Z.Moszner and J.Tabor. These results are not published so far. It seems that there are very interesting subsets having the property that addition of points from those subsets does not change the group of non-effectiveness.

Using the reper of order s we shall define s-orientability and s-orientation of Klein geometry. With this aim first we shall construct the productive object in the geometry (1.1). This object is represented by the triple

$$(M^s, G, f^s), \tag{3.1}$$

where $M^s := M \times M \times \ldots \times M$, $\bigwedge x=(x_1, x_2, \ldots, x_s) \in M^s$ and $\bigwedge g \in G$ $f^s(x,g) := (f(x_1,g), \ldots, f(x_s,g)) \in M^s$.

The elements x from M^s are finite sequence of s elements from M. Particularly, the repers of order s belong to M^s. Let us denote by M_s the set of all repers of order s. We find that M_s is an invariant subset in the productive object (3.1) (cp. [4], p.364, Hilfssatz 1.1). Then we can construct the partial object (cp. [1] or [2], p.383, def. 1.4), which has the form

$$(M_s, G, f_s), \tag{3.2}$$

where $f_s := f^s \mid M_s \times G$ is the contraction f^s to the product $M_s \times G$. The object (3.2) may not be transitive. Let us denote by $\mathfrak{M}_s$ an arbitrary transitive fibre of the object (3.2). Of course it is an invariant subset and we can construct the next partial object which is transitive

(3.3) $$(\mathfrak{M}_s, G, F_s), \quad F_s := f_s \big| \mathfrak{M}_s \times G.$$

Using this last object we shall define s-orientability of geometry (1.1).

D e f i n i t i o n 3.2. The geometry (1.1) is called s-orientable if there exists an invariant decomposition of the fibre $\mathfrak{M}_s$ of the object (3.3) exactly at two subsets $\mathfrak{M}_s^+$ and $\mathfrak{M}_s^-$, i.e. if these subsets satisfy the conditions:

(3.4) $$\mathfrak{M}_s^+ \neq \emptyset \wedge \mathfrak{M}_s^- \neq \emptyset \wedge \mathfrak{M}_s^+ \cap \mathfrak{M}_s^- = \emptyset,$$

(3.5) $$\mathfrak{M}_s^+ \cup \mathfrak{M}_s^- = \mathfrak{M}_s,$$

(3.6) $$\bigwedge \omega_1, \omega_2 \in \mathfrak{M}_s^{\varepsilon} (\varepsilon = \pm 1) \bigwedge g \in G \bigvee \eta = \pm 1$$

such that

$$F_s(\omega_1, g), \quad F_s(\omega_2, g) \in \mathfrak{M}_s^{\eta}.$$

The pair of subsets $\mathfrak{M}_s^+$ and $\mathfrak{M}_s^-$ is called an s-orientation and the geometry (1.1) with chosen one of these orientations we call s-oriented geometry.

4. Properties of s-orientation

In the definition of s-orientability there appears the transitive fibre $\mathfrak{M}_s$ of the object (3.2). Therefore it seems that s-orientability may depend on the choice of the considerated transitive fibre. As it appears this is so i.e. s-orientability does not depend on the choice of transitive fibre $\mathfrak{M}_s$, because the following theorem is true.

T h e o r e m 4.1. If the geometry (1.1) is s-orientable with respect to some transitive fibre $\mathfrak{M}_s$ of the object (3.2) then it is s-orientable with respect to every other transitive fibre of this object.

The above theorem was proved in [5]. The proof is based on the notion of orientability of the group and on the relation between the orientability of the group and the orientability of the geometry determined by this group. This relation is given in two lemmas which we shall quote below. First we shall recall the definition of orientability of a group.

D e f i n i t i o n 4.1. The group G we call orientable if it contains subgroup H with index 2. Then we also say that G is oriented by the subgroup H.

The following lemmas are true.

L e m m a 4.1. If the geometry (1.1) is s-orientable then the group G is orientable.

L e m m a 4.2. If G is orientable and if in the geometry (1.1) there exist repers of order s, then every transitive fibre of the object (3.2) has an invariant decomposition into exactly two non-empty disjoint sets and so the geometry (1.1) is s-orientable.

Using the above lemmas we obtain immediately a proof of Theorem 4.1. These lemmas show also that the orientability of a geometry reduces itself to the orientability of the group which determines the geometry.

Z.Moszner and J.Tabor in [8] have given some conditions for the orientability of a group. We shall repeat those of them which will be useful below.

L e m m a 4.3. A subgroup $H \subset G$ orients G if and only if $H \neq G$ and $\bigwedge a \in G-H$ and $\bigwedge b \in G-H$ $a \cdot b \in H$ (cp. [8], p.324).

To formulate the next lemma let us denote by $C(G)$ the set of squares of elements of the group G. This set is defined as follows: $C(G) := \{x \in G : \bigvee a \in G$ such that $x = a^2\}$.

L e m m a 4.4. If G is oriented by the subgroup $H \subset G$ then $C(G) \subset H$ (cp. [8], p.324).

5. Examples of the set of repers and of an orientation of a geometry

1. Repers in the geometry (2.1) consist of points. Hence they are of the first order. The set of repers of the order one is identical with G when it is transitive. If G is orientable by a subgroup H then the geometry (2.1) is 1-orientable. An invariant decomposition has the form (H, G -H). Each subset H and G - H is a 1-orientation of this geometry.

2. In the n-dimensional affine geometry (2.4) there exist repers of the order n+1. They are composed by n+1 points $(p_o, p_1, \dots, p_n)$ satisfying the condition

$$\det(p_1 - p_o,\ p_2 - p_o, \dots, p_n - p_o) \neq 0$$

The set of repers of (n+1) -st order $\mathfrak{M}_{n+1}$ is transitive. There exists an invariant decomposition of $\mathfrak{M}_{n+1}$ at exactly two non-empty subsets:

$$\mathfrak{M}^{+}_{n+1} := \left\{(p_o, p_1, \dots, p_n) : \det(p_1 - p_o, \dots, p_n - p_o) > 0\right\}$$

and

$$\mathfrak{M}^{-}_{n+1} := \left\{(p_o, p_1, \dots, p_n) : \det(p_1 - p_o, \dots, p_n - p_o) < 0\right\}.$$

The geometry (2.4) is (n+1)-orientable. The group of this geometry is oriented by the subgroup $H := \left\{(A, a) : A \in GL(n,R)\ \det A > 0,\ a \in R^n\right\}$.

3. In the n-dimensional projective geometry (2.5) there exist repers of the order (n+2). They are the systems of n+2 projective points $([\xi_o], [\xi_1], \dots, [\xi_{n+1}])$ with the following property: every determinant of the (n+1) order formed from the matrix (ξ^i_α) $i = 1, 2, \dots, n+1$, $\alpha = 0, 1, \dots, n+1$ is non-zero. Geometrically this means that none of the n+1 points of the given system lie on one n-1 dimensional hyperplane. The set $\mathfrak{M}_{n+2}$ of considered repers is transitive. In the next section

we shall see that this set may not have a two-elements invariant decomposition. Such situation is when n is an odd number. The corresponding invariant decompositions are given in [4] or [5]. In this case the geometry is (n+2)-orientable. When n is an even number the projective geometry is not orientable.

6. Orientability of projective geometry

Now we shall consider the orientability of the projective geometry. Using properties given above we shall prove the following theorem.

T h e o r e m 6.1. If n is an odd number then the n-dimensional projective geometry is (n+2)-orientable. If n is an even number then the n-dimensional projective geometry is not orientable.

P r o o f . Because in the n-dimensional projective geometry there exist repers of the order (n+2), the orientability of such geometry, according lemmas 4.1 and 4.2 reduces to the orientability of the projective group $GP(n,R)$.

Let n be an odd number. Then in $GP(n,R)$ we can define a subgroup G^+ as follows:

$$G^+ := \left\{ \langle A\rangle \ : A \in GL(n+1, R),\ \det A > 0 \right\}.$$

The set G^+ is well defined. Indeed, if $\det A > 0$ then for every $\varrho \neq 0$ $\det \varrho A = \varrho^{n+1} \det A > 0$ because on account of the fact that n is an odd number, we have $\varrho^{n+1} > 0$. Hence it follows that the condition which defines G^+ does not depend on the choice of an element from the class of abstraction $\langle A\rangle$. It is easy to verify that G^+ is a subgroup of the group $GP(n,R)$. Obviously G^+ is different from $GP(n,R)$ and it satisfies the condition:

$$\bigwedge_{\langle A\rangle \in GP(n,R) - G^+} \quad \text{and} \quad \bigwedge_{\langle B\rangle \in GP(n,R) - G^+} \langle B\rangle \langle A\rangle \in G^+$$

By virtue of Lemma 3.4 G^+ orients $GP(n,R)$. Thus $GP(n,R)$ is orientable and consequently the projective geometry is orientable, moreover n+2 orientable (Lemma 4.2).

Now let n be an even number. First we shall prove that in this case every element of $GP(n,R)$ can be represented in the form of a product of squares of some elements from the same group $GP(n,R)$. We shall prove this fact using a lemma of E.Artin ([9], p.152). To formulate this lemma let us denote by $E_{ij}(\alpha)$, $i \neq j$, $i,j = 1,2,\dots,n$ the matrix which we obtain if we put the number α instead of zero into i-th row and j-th column of the unit matrix. These matrices fulfil the relation: $E_{ij}(\alpha) \cdot E_{ij}(\beta) = E_{ij}(\alpha+\beta)$. Hence they are squares of nonsingular matrices $E_{ij}(\alpha) = \left[E_{ij}\left(\frac{\alpha}{2}\right)\right]^2$. Now we shall give the above mentioned lemma:

L e m m a 6.1. Every nonsingular matrix A can be represented as the product

$$A = B \cdot D, \tag{6.1}$$

where B is the product of the matrices $E_{ij}(\alpha)$ and D is the diagonal matrix. All elements of the principal diagonal of matrix D are equal to one except the last one which is equal to $\mu = \det A$.

If $\det A > 0$ then $\mu > 0$ and the diagonal matrix D is also a square of some nonsingular matrix. Thus we obtain

C o r o l l a r y 6.1. Every matrix with the positive determinant is a product of squares of nonsingular matrices.

From this corollary it follows that if n is an even number then every element of the group $GP(n,R)$ can be represented as a product of squares of elements of this group. Indeed, let $\langle A \rangle \in GP(n,R)$. Since n+1 is an odd number, we can assume that $\det A > 0$. Otherwise it suffices to multiply A by (-1). Then we have $\det(-1)A = (-1)^{n+1} \det A = -\det A > 0$. Because $\det A > 0$, from Corollary 6.1 it follows that the matrix A can be represented as a product of squares of nonsingular matrices. Therefore $\langle A \rangle$ is a product of squares of elements of $GP(n,R)$.

Now we shall prove that the group GP(n,R) (n is an even number) is not orientable. Suppose, on the contrary that H is a subgroup GP(n,R) which orients it. On account of Lemma 4.4 it follows that $C(GP(n,R)) \subset H$. This means that every square of an element from GP(n,R) belongs to H. Hence it follows that every product of squares of elements from GP(n,R) also belongs to H, because H is a subgroup. Above we have shown that every element of GP(n,R) is such a product. Hence $GP(n,R) \subset H$ i.e. GP(n,R) = H. The last equality is impossible because H has an index 2. Hence it follows that in GP(n,R) there exists no subgroup with index 2. The group GP(n,R) and consequently the n-dimensional projective geometry is not orientable if n is an even number. This completes the proof of Theorem 6.1.

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Received November 5, 1977.