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A CONSTRUCTION OF THE BASIC SOLUTION OF THE EQUATION $\Delta u - \partial_t u - \vartheta u = 0$

In this paper we shall construct the basic solution of the equation

(1) $$L(\partial, \partial_t, \vartheta)u = \Delta u - \partial_t u - \vartheta u = 0 \qquad x \in R^N,\ t > 0,$$

(1') $$\vartheta = \vartheta(x,t), \quad \vartheta : [0, T] \xrightarrow{\text{continuous}} L^q(R^N), \quad 1 < q < \infty \quad \text{and} \quad q > \frac{N}{2}$$

and

(1") $$\|\vartheta\| = \sup_s \|\vartheta(\cdot, s)\|_q < \infty$$

and we shall investigate its regularity.

We postulate (cf.[1]) the following form of the basic solution of the equation (1)

(2) $$G(x,y,t) = G_0(x-y,t) + \int_0^t \int_{R^N} G_0(x-\xi,t-s)\vartheta(\xi,s) \times G(\xi,y,s)\, d\xi\, ds,$$

where

(3) $$G_0(x,y,t) = \begin{cases} (4\pi t)^{-N/2} \exp\left(-\frac{|x-y|^2}{4t}\right) & 0 < t < \infty \\ 0 & t \leq 0 \end{cases}$$

is the basic solution of the conduction equation

$$\Delta u - \partial_t u = 0.$$

In view of the assumption about the function ϑ: $\vartheta \in L^{c}(R^N \times [0,T])$ and by Theorem 7, p.119, in [2] we infer that there exists a function $u(x,t) \in L_0^{p,m,1}(S_T)$ satisfying the equation (1) almost everywhere in the strip S_T.

To the class $L_0^{p,m,1}(S_T)$ defined in [2] there belong functions defined and integrable in the strip $S_T \overset{\text{def}}{=} R^N \times [0,T]$, which have weak derivatives up to order m with respect to the space variable x, and the weak first-order derivative with respect to the time variable t. Moreover, the functions u in the class satisfy the condition $u(x,t) = 0$ in $S_\sigma = R^N \times (0,\sigma)$ for some $\sigma > 0$.

It is easy to verify by a direct calculation that the function (2) satisfies the condition (1). In fact, we have

$$(4) \quad L(\partial, \partial_t, \vartheta)G = \delta(x-y)\delta(t) - \vartheta(x,t)G_0(x-y,t) + \vartheta(x,t)G(x,y,t) - \vartheta(x,t)\int_0^t \int_{R^N} G_0(x-\xi,t-s)\vartheta(\xi,s)\cdot G(\xi,y,s)\,d\xi\,ds = \delta(x,y)\delta(t) + \vartheta(x,t)G(x,y,t) - \vartheta(x,t)\cdot \left[G_0(x-y,t) + \int_0^t \int_{R^N} G_0(x-\xi,t-s)\vartheta(\xi,s)\cdot G(\xi,y,s)\,d\xi\,ds\right] = \delta(x-y)\delta(t).$$

Now we are going to solve the integral equation (2). We are looking for a function G represented in the form of a product (see [1], p.8).

$$(5) \qquad G(x,y,t) = G_0(x,y,t)\omega(x,y,t)$$

Substituting the expression (5) to the equation (2) we obtain an integral equation which is satisfied by the function ω:

$$(6)\qquad \omega(x,y,t) = 1 + G_0(x-y,t)^{-1}\int_0^t\int_{R^N} G_0(x-\xi,t-s)\cdot$$

$$\cdot\, \vartheta(\xi,s)G_0(\xi-y,s)\omega(\xi,y,s)d\xi ds.$$

Let $\varphi\in L^{\infty}(R^{2N}\times[0,\ T])$.

We define an operator A in the following way

$$(7)\qquad (A\varphi)(x,y,t) = G_0^{-1}(x,y,t)\int_0^t\int_{R^N} G_0(x-\xi,t-s)\vartheta(\xi,s)\,\cdot$$

$$\cdot\, G_0(\xi-y,s)\varphi(\xi,y,s)d\xi ds.$$

We shall prove the following lemma.

L e m m a 1. If

1° $\vartheta(x,t)$ satisfies the assumptions (1′) and (1″),

2° $\sup_{x,y\in R^N}|\varphi(x,y,t)|\le C_\varphi t^\alpha$, $\alpha\ge 0$ and $q>N/2$,

then the following estimation holds

$$(8)\qquad \sup_{x,y\in R^N}|(A\varphi)(x,y,t)|\le C_\varphi\, C\,\frac{\Gamma\left(\alpha+1-\frac{N}{2q}\right)\, t^{\alpha+1-\frac{N}{2q}}}{\Gamma\left(\alpha+2\left(1-\frac{N}{2q}\right)\right)},$$

where:

I. $C = (4\pi)^{-N/2q}\, p^{-N/2p}\,\Gamma\left(1-\frac{N}{2q}\right)\sup_{t\in[0,T]}\|\vartheta(\cdot,\ t)\|_q$.

II. $\frac{1}{p}+\frac{1}{q}=1$.

III. The one-argument function Γ is defined in the standard way

$$x>0:\ \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt.$$

P r o o f . By the definition of A and by the assumptions of Lemma 1 we obtain

$$(9)\quad \left|(A\varphi)(x,y,t)\right| \leq C_\varphi\, G_0^{-1}(x,y,t) \int_0^t\int_{R^N} G_0(x-\xi,t-s)\,\cdot$$

$$\cdot\, \vartheta(\xi,s)G_0(\xi-y,s)\; s^\alpha d\xi ds \leq$$

$$\leq C_\varphi G_0^{-1}(x,y,t)\int_0^t\left\{\int_{R^N}\left[G_0(x-\xi,t-s)G_0(\xi-y,s)\right]^p d\xi\right\}^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_{R^N}\left|\vartheta(\xi,s)\right|^q d\xi\right)^{\frac{1}{q}} s^\alpha ds \leq C_\varphi \sup_{t\in[0,T]}\left\|\vartheta(\cdot,t)\right\|_q\,\cdot$$

$$\cdot\, G_0^{-1}(x,y,t)\int_0^t\left\{\int_{R^N}\left[G_0(x-\xi,t-s)G_0(\xi-y,s)\right]^p d\xi\right\}^{\frac{1}{p}} s^\alpha ds.$$

It is easy to see the function under the integral with respect to the space variable can be written in the following way

$$(10)\quad \left[G_0(x-\xi,t-s)G_0(\xi-y,s)\right]^p = p^{-N}\left[4\pi(t-s)4\pi s\right]^{-\frac{N}{2}(p-1)}\cdot$$

$$\cdot\, G_0(x,\xi,\tfrac{t-s}{p})\; G_0(\xi,\,y,\,\tfrac{s}{p}).$$

From this we obtain

$$(11)\qquad J = \int_{R^N}\left[G_0(x-\xi,t-s)\cdot G_0(\xi-y,s)\right]^p d\xi =$$

$$= \left[4\pi(t-s)4\pi s\right]^{\frac{-Np}{2}}\int_{R^N}\exp\left(-\frac{p}{4}\,\frac{|x-\xi|^2}{t-s} - \frac{p}{4}\,\frac{|\xi-y|^2}{s}\right)d\xi.$$

After changing the variables of integration according to the formula

$$\eta_j = \left[\frac{p}{4}\frac{t}{(t-s)s}\right]^{\frac{1}{2}}(\xi_j - y_j) + \left[\frac{p}{4}\frac{s}{(t-s)t}\right]^{\frac{1}{2}}(y_j - x_j)$$

for $j = 1,2,\dots,N$

we transform the integral J defined by (11) to the following form

$$(12)\qquad J = \left[4\pi(t-s)4\pi s\right]^{-\frac{Np}{2}} \exp\left(-\frac{p}{4}\frac{|x-y|^2}{t}\right)s^{\frac{N}{2}} \cdot$$

$$\cdot\left(\frac{4}{p}\cdot\frac{t-s}{t}\right)^{\frac{N}{2}}\int_{R^N}\exp(-\eta^2)d\eta = p^{-\frac{N}{2}}(4\pi)^{-Np+\frac{N}{2}}\cdot$$

$$\cdot\left[(t-s)s\right]^{-\frac{Np}{2}+\frac{N}{2}} t^{-\frac{N}{2}} \exp\left(-\frac{p}{4}\frac{|x-y|^2}{t}\right);$$

This implies

$$(13)\qquad J^{\frac{1}{p}} = p^{-\frac{N}{2p}}\left[4\pi(t-s)4\pi s\right]^{-\frac{N}{2q}}(4\pi t)^{-\frac{N}{2p}}\cdot$$

$$\cdot\exp\left(-\frac{1}{4}\frac{|x-y|^2}{t}\right) = p^{-\frac{N}{2p}}\left[4\pi(t-s)s\right]^{-\frac{N}{2q}}\cdot$$

$$\cdot\, t^{-\frac{N}{2p}+\frac{N}{2}}G_o(x,y,t).$$

Hence the inequality (9) takes the form

$$(14)\qquad |(A\varphi)(x,y,t)| \leqslant C_\varphi \sup_{t\in[0,T]}\|\vartheta(\cdot,t)\|_q\, p^{-\frac{N}{2p}}\cdot$$

$$\cdot(4\pi)^{-\frac{N}{2q}}\, t^{\frac{N}{2q}}\int_0^t (t-s)^{-\frac{N}{2q}} s^{-\frac{N}{2q}+\alpha}\, ds.$$

The integral appearing in the above formula can be determined for $q > \frac{N}{2}$ by means of the following elementary relation

$$\int_0^s u^\alpha (s-u)^\beta \, du = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} s^{\alpha+\beta+1} \tag{15}$$

Then the inequality (14) takes the form

$$|(A\varphi)(x,y,t)| \leq C_\varphi \sup_{t\in[0,T]} \|\vartheta(\cdot,t)\|_q \, (4\pi)^{-\frac{N}{2q}} \cdot$$

$$\cdot\, p^{-\frac{N}{2p}} \Gamma\left(1-\frac{N}{2q}\right) \cdot \frac{\Gamma\left(\alpha+1-\frac{N}{2q}\right)}{\Gamma\left(\alpha+2\left(1-\frac{N}{2q}\right)\right)} t^{\alpha+1-\frac{N}{2q}} \tag{16}$$

which was to be proved.

We shall construct the solution of (6) by the method of successive approximations.

The successive approximations will have the form

$$\begin{aligned}
&\omega_0 \overset{\text{def}}{=} 1 \\
&\omega_1 = \omega_0 + A\omega_0 \\
&\omega_2 = \omega_0 + A\omega_1 = \omega_0 + A\omega_0 + A^2\omega_0 \\
&\vdots \\
&\omega_m = \omega_0 + A\omega_{m-1} = \omega_0 + A\omega_0 + A^2\omega_0 + \ldots + A^m\omega_0
\end{aligned} \tag{17}$$

In the above relations we have assumed the following notation

$$A^i\omega_0 = A^{i-1}(A\omega_0), \qquad i = 2,3,\ldots$$

and

$$A^1 = A.$$

To prove that the following series is convergent

$$S = \sum_{m=0}^{\infty} A^m \omega_o \tag{18}$$

and, consequently, to prove that the sequence ω_m is convergent we shall show that

$$\exists_{H} \ \forall_{r \geqslant H} \ \|(A^r \omega_o)(\cdot,\cdot.t)\|_{\infty} \leqslant \varkappa < 1 \qquad \forall_{t \in [0,T]} \ . \tag{19}$$

First we are going to prove the following formula

$$\sup_{x,y \in R^N} |(A^r \omega_o)(x,y,t)| \leqslant C^r \frac{\Gamma\left(1 - \frac{N}{2q}\right)}{\Gamma\left((r+1)\left(1 - \frac{N}{2q}\right)\right)} t^{r\left(1 - \frac{N}{2q}\right)} \tag{20}$$

$$r = 0,1,2,\ldots$$

The constant C in the formula above has been defined in Lemma 1 and the symbol $A^o \omega_o$ means ω_o.

The proof will go by induction. According to the definition of ω_o, for $r = 0$ the estimation of the left-hand side of (20) takes the form

$$\sup_{x,y \in R^N} |\omega_o(x,y,t)| \leqslant 1.$$

Let us assume that the inequality (20) holds for the index equal to $r - 1$, i.e.

$$\sup_{x,y \in R^N} |(A^{r-1} \omega_o)(x,y,t)| \leqslant C^{r-1} \frac{\Gamma\left(1 - \frac{N}{2q}\right)}{\Gamma\left(r\left(1 - \frac{N}{2q}\right)\right)} t^{(r-1)(1-\frac{N}{2q})}$$

$$r = 1,2,\ldots$$

We shall prove that the inequality (20) holds for the index equal to r. By Lemma 1, taking as φ the function $A^{r-1}\omega_0$, and for the constant C_φ the value

$$C^{r-1}\cdot\frac{\Gamma\left(1-\frac{N}{2q}\right)}{\Gamma\left(r\left(1-\frac{N}{2q}\right)\right)}$$

and taking $\alpha=(r-1)\left(1-\frac{N}{2q}\right)\geqslant 0$ for $q>\frac{N}{2}$ we obtain

$$\sup_{x,y,\in R^N}\left|(A^r\omega_0)(x,y,t)\right| = \sup_{x,y,\in R^N}\left|A(A^{r-1}\omega_0)(x,y,t)\right|\leqslant$$

$$\leqslant C^r\,\frac{\Gamma\left(1-\frac{N}{2q}\right)}{\Gamma\left(r\left(1-\frac{N}{2q}\right)\right)}\cdot\frac{\Gamma\left(r\left(1-\frac{N}{2q}\right)\right)}{\Gamma\left((r+1)\left(1-\frac{N}{2q}\right)\right)}\,t^{r\left(1-\frac{N}{2q}\right)} =$$

$$= C^r\,\frac{\Gamma\left(1-\frac{N}{2q}\right)}{\Gamma\left((r+1)\left(1-\frac{N}{2q}\right)\right)}\,t^{r\left(1-\frac{N}{2q}\right)}$$

which was to be proved.

Using the formula $\Gamma(x+1)=x\Gamma(x)$ we can write the denominator of (20) in the following form

$$P=\Gamma\left((r+1)\left(1-\frac{N}{2q}\right)\right)=\Gamma\left(r+1-\frac{(r+1)N}{2q}\right)= \tag{21}$$

$$=\left(r-\frac{(r+1)N}{2q}\right)\left(r-1-\frac{(r+1)N}{2q}\right)\cdot\ \ldots\ \cdot$$

$$\cdot\left(r-\left(r-\left[\frac{(r+1)N}{2q}\right]-2\right)-\frac{(r+1)N}{2q}\right)\cdot$$

$$\cdot\,\Gamma\left(r-\left(r-\left[\frac{(r+1)N}{2q}\right]-2\right)-\frac{(r+1)N}{2q}\right).$$

All factors in the above equality are positive. Hence taking into account the elementary inequality of the form

(22) $$V_1 = \left[\frac{(r+1)N}{2q}\right] + 1 > \frac{(r+1)N}{2q}$$

we can estimate the function (21) from below. Namely, we have the following relation

(23) $$P > (r-V_1)(r-1-V_1) \cdot \ldots \cdot (r-(r-V_1-1)-V_1) \cdot$$
$$\cdot \Gamma\left(r - (r-V_1-1) - \frac{(r+1)N}{2q}\right) =$$
$$= (r - V_1)! \cdot \Gamma\left(V_1 + 1 - \frac{(r+1)N}{2q}\right).$$

It is easy to observe that the argument of the function Γ is contained in the interval $(1,2]$, the function Γ takes in this interval positive values not greater than 1. Hence the estimation (20) can be replaced by the inequality

(24) $$\sup_{x,y\in R^N} \left|(A^r\omega_0)(x,y,t)\right| \leq C_\Gamma \frac{C^r\, t^{r\left(1-\frac{N}{2q}\right)}}{(r - V_1)!},$$

where

(25) $$C_\Gamma = \frac{\Gamma\left(1 - \frac{N}{2q}\right)}{\Gamma\left(V_1+1 - \frac{(r+1)N}{2q}\right)}.$$

It is clear that, in view of (24), starting from some place the terms of (18) are, with respect to the norm of $L^\infty(R^N \times R^N)$, less than a positive $\varkappa < 1$ uniformly with respect to $t \in [0,T]$. Consequently, the sum of this series, to be denoted by $f(x,y,t)$, belongs to the space $C([0,T], L^\infty(R^N \times R^N))$, the norm of which we denote by the symbol

(26) $$\|f\| = \sup_s \|f(\cdot, \cdot, s)\|_q.$$

We have to prove still that the function $f(x,y,t)$, which is the limit of the sequence ω_r, is a solution of the integral equation (6). To this aim, we represent f in the form

$$(27)\qquad f(x,y,t) = \omega_m(x,y,t) + Q_m(x,y,t),$$

where Q_m denotes a function of the class $C([0,T],L^\infty(R^N \times R^N))$. Since the sequence ω_m is convergent to f, we infer that

$$(28)\qquad \|f(\cdot,\cdot,t) - \omega_m(\cdot,\cdot,t)\|_\infty \xrightarrow[m\to\infty]{} 0.$$

Hence the norm of Q_m tends to 0 with $m \to \infty$. By (27), taking into account the definition of $\omega_m = \omega_0 + A\omega_{m-1}$, we obtain

$$(29)\qquad f - \omega_0 - Af = Q_m + A\omega_{m-1} - Af,$$

$$(30)\qquad \|f(\cdot,\cdot,t)-\omega_0-Af(\cdot,\cdot,t)\|_\infty < \|Q_m(\cdot,\cdot,t)\|_\infty + \|A(\omega_{m-1}(\cdot,\cdot,t) - f(\cdot,\cdot,t))\|_\infty .$$

In view of (23) and (30) taking limit with $m \to \infty$ we obtain

$$f(x,y,t) - \omega_0 - Af(x,y,t) = 0$$

that is

$$f(x,y,t) = 1 + Af(x,y,t).$$

Hence f is a solution of (6).

The uniqueness of this solution and consequently the uniqueness of the solutions of the equations (2) and (1), follows from Theorem 8, p.125 of [2].

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