DEMONSTRATIO MATHEMATICA
Vol. XI No 4 1978

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BAYESIAN APPROACH TO THE PREDICTION PROBLEM IN GAMMA POPULATION

1. Introduction

Prediction problem, which is receiving much attention recently, has been viewed mostly in two directions. One is the classical approach based on the independence of statistics and their exact distributions. Such is the case in Lawless [9], Faulkenberry [8], Kaminsky, Luks and Nelson [7] and also in Lingappaiah [10], [11]. But, another method, is the Bayes approach with posterior distributions and suitable priors. Such works are seen in Bancroft and Dunsmore [1], Aitcheson and Dunsmore [2] and Dunsmore [3], [4]. Our development here is based on last of these results Dunsmore [4] and Lingappaiah [11]. Our main motivation, here, is about what can be done where more than a pair of samples are available. What we have done here is to consider the posterior distribution at a certain stage as the prior for the next stage, on the lines of Khan [12] and in so doing, we have developed the predictive distribution for an order statistic at the sth stage ((s+1)th sample) and also for the difference of two statistics at this stage. We have discussed the variance in each of these two cases in relation to number of stages. Also, we have evaluated the probability integral for both the situations and particular cases are considered for illustrations.

2. Prediction of $y_s = x_{s(k_s)}$

(2a). Consider $(s+1)$ samples each drawn from a gamma population

$$(1) \qquad f(x) = e^{-\theta x}\theta^{\alpha} x^{\alpha-1}/\Gamma(\alpha);\ x,\theta,\alpha, > 0.$$

Let y_i denote the k_ith order statistics in the sample of size N_i at the stage i ((i+1)th sample). That is, $y_i = x_{i(k_i)}$ and let the sample size at stage 0 (sample 1) be n. Also, let α be a known parameter taking an integer value. In this case, $\sum_{i=1}^{n} x_i = x$ from the first sample is sufficient for θ and x has the gamma distribution with the parameters $n\alpha$ and θ. Let the prior for θ be gamma with parameters h and g. Then it follows, that the posterior distribution is

$$(2) \qquad f(\theta|x) = e^{-\theta H} H^{G} \theta^{G-1}/\Gamma(G),$$

where $H = h + x$, $G = n\alpha + g$.

Now the distribution of $y_1 = x_{1(k_1)}$, that is, the k_1th order statistic in a sample of size N_1 from the second sample or at stage 1, is given by, on the lines of Lingappaiah [11] as.

$$(3)\ f(y_1|\theta) = C_1 \sum_{r_1=0}^{k_1-1} \binom{k_1-1}{r_1} (-1)^{r_1} \sum_{t_1=0}^{(\alpha-1)d_1} a_{t_1}(\alpha, d_1) e^{-\theta y_1 b_1} \theta^{\alpha+t_1} y_1^{\alpha+t_1-1},$$

where $1 \le k_1 < N_1$, $C_1 = [1/B(N_1-k_1+1, k_1)\Gamma(\alpha)]$, $d_1 = b_1 - 1$ and $b_1 = N_1-k_1+r_1+1$ and $a_t(\alpha, d_1)$ is the coefficient of x^t in the expansion of $\left(\sum_{k=0}^{\alpha-1} x^k/k!\right)^{d_1}$. The recurrence relation satisfied by a_t's is given in Lingappaiah [11]. From (2) and (3), we have the predictive distribution at the first stage as

$$(4)\qquad f(y_1|x) = C_1\Sigma_1 A_1 B_1 \left[\frac{y_1^{\alpha+t_1-1}\ \Gamma(G+\alpha+t_1)H^G}{\Gamma(G)\ (H+b_1y_1)^{G+\alpha+t_1}}\right],$$

where

$$(5)\quad \Sigma_1 = \sum_{r_1}\sum_{t_1} \quad\text{and}\quad A_1 = \binom{k_1-1}{r_1}(-1)^{r_1} \quad\text{and}\quad B_1 = a_t(\alpha,d_1).$$

If $\alpha = 1$, (4) reduces to (2.1) of Dunsmore [4]. Now suppose we take the posterior for the first stage, that is, $f(\theta|x,y_1)$ as the prior for the second stage and considering a similar expression for $y_2 = x_{2(k_2)}$ and (3), that is, the pdf of the k_2th order statistic in the sample of size N_2 at the second stage (3rd sample), we have the predictive distribution at stage 2 as

$$(6)\quad f(y_2|x,y_1) = \frac{c_2\Sigma_1\Sigma_2\left[\prod_{i=1}^{2} A_iB_iy_i^{\alpha+t_i-1}\right]\Gamma(G+t_1+t_2+2\alpha)/Q_2^{G+t_1+t_2+2\alpha}}{\Sigma_1A_1B_1y_1^{t_1+\alpha-1}\ \Gamma(G+t_1+\alpha)/Q_1^{G+t_1+\alpha}},$$

where $c_2 = \left[1/B(N_2-k_2+1,k_2)\Gamma(\alpha)\right]$, $b_2 = N_2-k_2+r_2+1, d_2 = b_2-1$, and $Q_1 = H+b_1y_1$, $Q_2 = Q_1+b_2y_2$ and Σ_1, Σ_2 are each two-fold. Continuing on this line and considering the posterior at the stage (s-1) as the prior for stage s, we have the predictive distribution for the sth stage as,

$$(7)\quad f(y_s|y_1,\ldots,y_{s-1}) = \frac{c_s\Sigma_1\ldots\Sigma_s\left[\prod_{i=1}^{s} A_iB_iy_i^{\alpha+t_i-1}\right]\Gamma(u^*+t_s+\alpha)/Q_s^{u^*+t_s+\alpha}}{(L)},$$

$$(7a)\quad \text{where}\quad (L) = \Sigma_1\ldots\Sigma_{s-1}\left[\prod_{i=1}^{s-1} A_iB_iy_i^{t_i+\alpha-1}\right]\Gamma(u^*)/Q_{s-1}^{u^*}$$

with $u^* = G + (t_1 + \dots + t_{s-1}) + (s-1)\alpha$,

$$Q_{s-1} = H + b_1 y_1 + \dots + b_{s-1} y_{s-1}, \quad Q_s = Q_{s-1} + b_s y_s,$$

c_s, b_i, d_i, A_i and B_i, similar to definition above. Now, if we set $\alpha = 1$, in (7), we get

$$(8) \quad f(y_s \mid x, y_1 \dots y_{s-1}) = \frac{c'_s \Sigma_1 \dots \Sigma_s \left[\prod_{i=1}^{s} A_i\right] \Gamma(G'+s)/Q_s^{G'+s}}{\Sigma_1 \dots \Sigma_{s-1} \left[\prod_{i=1}^{s-1} A_i\right] \Gamma(G'+s-1)/Q_{s-1}^{G'+s-1}},$$

where $G' = n+g$ and c'_s is c_s with $\alpha = 1$. Further if we set $\alpha = 1$, and $k_i = 1$, $i = 1,\dots,s$ in (7), (that is, we are predicting the first order statistic at the sth stage in terms of the first order statistics at the earlier stages from 1 to s-1), we get

$$(9) \qquad f(z_s \mid x, z_1 \dots z_{s-1}) = \frac{N_s \Gamma(G'+s)/(Q_s^o)^{G'+s}}{\Gamma(G'+s-1)/(Q_{s-1}^o)^{G'+s-1}},$$

where $Q_{s-1}^o = H + N_1 z_1 + \dots + N_{s-1} z_{s-1}$, $Q_s^o = Q_{s-1}^o + N_s z_s$ with $z_i = x_{i(1)}$, $i = 1,\dots,s$. If we set $s = 1$ in (9), we get (2.2) of Dunsmore [4] with $Q_0^o = H$ and $n = k$. Since (7) is a pdf, we have

$$(10) \qquad c_s \Sigma_s A_s B_s \Gamma(t_s+\alpha)/b_s^{t_s+\alpha} = 1$$

and if $\alpha = 1$, (10) reduces to

$$(10a) \qquad \sum_{r=0}^{k-1} \binom{k-1}{r} (-1)^r / b = B(N-k+1, k),$$

where $b = N-k+r+1$.

And if $k_i = 1$, $i = 1,\ldots,s$ in (10), we have

$$\text{(10b)} \qquad \sum_{t=0}^{(\alpha-1)(N-1)} a_t(\alpha, N-1)\Gamma(\alpha+t)/N^t = N^{\alpha-1}\Gamma(\alpha)$$

and if $N = 2$ in (10b), we get

$$\text{(10c)} \qquad \sum_{t=0}^{\alpha-1} \binom{t+\alpha-1}{t} \frac{1}{2^t} = 2^{\alpha-1}.$$

(2b). Discussion on the variance. Our main aim in developing (7) is to show that the variance goes on decreasing as the number of stages increase, which makes sense intuitively too. The same thing can also be achieved, in addition, by a proper choice of parameters as we see shortly and we have quite a few parameters to deal with such as, h, g, n, N_i, k_i, $i = 1,\ldots,s$, and s itself. In general, we have from (7),

$$\text{(11)} \qquad \mu'_{r(s)} = \frac{1}{(L)} c_s \Sigma_1 \cdots \Sigma_s \left(\prod_{i=1}^{s-1} A_i B_i y_i^{t_i+\alpha-1} \right) \cdot A_s B_s \Gamma(t_s+\alpha+r)\,\Gamma(u^*-r)/Q_{s-1}^{u^*-r}\, b_s^{t_s+\alpha+r},$$

where $\mu'_{r(s)}$ denote rth raw moment of y_s. Obviously, if $r = 0$, (11) reduces to (10). Though it is slightly complex to find variance, in general, however if $\alpha = 1$, $k_i = 1$, $i = 1,\ldots,s$ in (11), we have

$$\text{(12)} \qquad \mu'_{r(s)} = \left(\frac{Q^0_{s-1}}{N_{s-1}}\right) \left[\Gamma(r+1)\Gamma(u_0-r)/\Gamma(u_0)\right],$$

where Q^0_{s-1} is as above and $u_0 = u^*$ with $\alpha = 1$. From (12) we have the variance at the stage s as

$$\text{(13)} \qquad \mu_{2(s)} = \left(\frac{Q^0_{s-1}}{N_s}\right)^2 \cdot E,$$

where $E = u_0/(u_0-1)^2(u_0-2)$ which is fast decreasing as s increases for given G'. Though Q^o_{s-1} increases as s increases (each time by a term), $\mu_{2(s)}$ can be made small as s increases and also by a proper choice of N_s and n,g. Again, if $k_i = 1$, $i = 1,\ldots,s$, $\alpha = 2$, $N_1 = 3$, $s = 1$ in (11), we get

$$(14) \qquad \mu_{2(1)} = \frac{H^2}{81}\left[\frac{278G + 398}{(G-1)^2(G-2)}\right].$$

(2c). <u>Probability Content</u>: Now from (7), again, we have $F(a) = p(y_s \le a)$ as

$$(15) \quad F(a) = \frac{1}{(L)} c_s \Sigma_1 \cdots \Sigma_s \left[\prod_{i=1}^{s-1} A_i B_i y_i^{t_i+\alpha-1}\right] A_s B_s \frac{\Gamma(u^*+t_s+\alpha)}{Q_{s-1}^{u^*}\, b_s^{t_s+\alpha}} \cdot$$

$$\cdot \int_0^{a^*} \frac{w^{t_s+\alpha-1}\, dw}{(1+w)^{u^*+t_s+\alpha}},$$

where $a^* = ab_s/Q_{s-1}$ and if u^* is an integer (with g as an integer), we can express the integral part of (15) in terms of the Binomial cumulative probabilities as

$$(15a) \quad B(u^*-1, t_s+u^*+\alpha-1, 1/(1+a^*))/(t_s+\alpha) \binom{t_s+u^*+\alpha-1}{u^*-1},$$

where $B(k,n,p) = \sum_{x=0}^{k} b(x,n,p)$ and

$$(16) \quad \int_0^a \left[w^{m-1}/(1+w)^{m+n}\right]dw = B(n-1, m+n-1, 1/1+a)/m \binom{m+n-1}{n-1}.$$

Again with $F(\infty) = 1$, (15) reduces to (10). Now, if we set $\alpha = 1$, $k_i = 1$, $i = 1,\ldots,s$ in (15), we get

$$(17) \qquad F(a_0) = (G'+s-1) \int_0^{a_0 N_s/Q^0_{s-1}} \left[dw/(1+w)^{G'+s}\right],$$

and if $s = 1$, in (17), we get (23) of Dunsmore [4], which is

$$(18) \qquad a_0 = \frac{H}{N_1}\left[(1-\delta)^{-1/G'}-1\right] \text{ where } F(a_0) = \delta .$$

From (15) using (15a), we can calculate $F(a_0)$ for values of $\alpha \neq 1$, which gives the probability below a_0 for different integral values of α. For example, if we set $s = 1$, $N_1 = 2$, we have from (15), (15a) and (18),

$$(19) \qquad F(a_0) = \frac{1}{2^{\alpha-1}} \sum_{r=0}^{\alpha-1} \binom{\alpha+r-1}{r} \frac{1}{2^r} B(G-1, G+\alpha+r-1, p),$$

where $p = (20)^{-1/G'}$ with $= .95$. Here again $G' = n+g$ and $G = n\alpha+g$.

3. Predictive distribution of the distance $T_s = x_{s(k_s)}-x_{s(k_s')}$ (3a). Now, we develop to get the predictive distribution of the difference $T_s = x_{s(k_s)}-x_{s(k_s')}$ where $1 \leq k_s' < k_s \leq N_s$, that is, the difference between the k_sth and the k_s'th order statistics in the sample of size N_s at the stage s. Obviously, if $k_s' = 1$ and $k_s' = N_s$, we are dealing with R_s, the range at stage s. Now from Lingappaiah [11], we have that

$$T = x_{(k)}-x_{(k')}, \quad 1 \leq k' < k \leq N,$$

$$(20) \qquad f(T|\theta) = c^0 \sum \Omega A^0 B^0 e^{-\theta T\delta} T^{\nu-1} \Gamma(u)\theta^{\nu}/w^{u}, \quad T > 0,$$

where $A^o = a_t(\alpha, i+j)$, $B^o = b_p(\alpha, \delta-1)$,
$c^o = N!/(N-k)!(k-k'-1)!(k'-1)!\Gamma^2(\alpha)$,

$$(21)\quad \begin{cases} i \text{ is from } 0 \text{ to } k'-1, & u = t+\alpha+m, \\ j \text{ is from } 0 \text{ to } k-k'-1 = \lambda, & \nu = p+\alpha-m, \\ t \text{ is from } 0 \text{ to } (\alpha-1)(i+j) \text{ and } & w = N-k'+i+1, \\ p \text{ is from } 0 \text{ to } (\alpha-1)(\delta-1), & \delta = N-k'+j, \\ m \text{ is from } 0 \text{ to } (p+\alpha-1), & \end{cases}$$

$$\Omega = \binom{k'-1}{i}\binom{\lambda}{j}\binom{p+\alpha-1}{m}(-1)^{i+j+\lambda}$$

and $a_t(\alpha, i+j)$ is the coefficient of x^t in the expansion of $\left(\sum_{k=0}^{\alpha-1} x^k/k!\right)^{i+j}$ and $b_p(\alpha, \delta-1)$ is the coefficient of $(x+T)^p$ in the expansion of $\left(\sum_{k=0}^{\alpha-1}(x+T)^k/k!\right)^{\delta-1}$. Now from (2) and (20), we get the predictive distribution of T_1 at stage 1, as developed in section 2,

$$(22)\quad f(T_1|x) = c_1^o\ \Sigma_1\ \frac{\Omega_1 A_1^o B_1^o H^G \Gamma(G+\nu_1)T_1^{\nu_1-1}\Gamma(u_1)}{\Gamma(G)(H+T_1\delta_1)^{G+\nu_1} w_1^{u_1}},\ T_1 > 0,$$

where c_1^o, i_1, j_1, t_1, m_1, p_1 and u_1, ν_1, w_1, Ω_1, are as in (21) with new subscript 1 and $A_1^o = a_{t_1}(\alpha, i_1+j_1)$ and $B_1^o = b_{p_1}(\alpha, \delta_1-1)$. Now, if we put $N_1 = 2$, $\alpha = 2$, $k_1' = 1$, in (22), we get

$$(23)\quad f(T_1|x) = 2\sum_{m=0}^{1}\binom{1}{m}\frac{\Gamma(G+2-m)\Gamma(2+m)T_1^{1-m}}{\Gamma(G)(H+T_1)^{G+2-m}2^{2+m}}.$$

Now, developing exactly on the lines of section 2, by taking the posterior distribution for stage (s-1) as the prior for stage s and considering the $f(T_s|\theta)$ similar to (20), we have now the predictive distribution of T_s at stage s as

$$(24)\quad f(T_s|x,T_1,\ldots,T_s) = \frac{1}{(M)} c_s^o \Sigma_1 \ldots \Sigma_s \left[\prod_{l=1}^{s} \Omega_l A_l^o B_l^o T_l^{\nu_l - 1} \Gamma(u_l)/w_l^{u_l} \right] \cdot$$

$$\cdot\, \Gamma(\nu_s + \varepsilon) \Big/ Q_s^{\nu_s + \varepsilon},$$

where $M = \Sigma_1 \ldots \Sigma_{s-1} \prod_{l=1}^{s-1} \Omega_l A_l^o B_l^o T_l^{\nu_l - 1} \Gamma(u_l)/w_l^{u_l} \Gamma(\varepsilon)/Q_{s-1}^{\varepsilon}$

with

$$\varepsilon = G + (\nu_1 + \ldots + \nu_{s-1}), \quad G = n\alpha + g$$

(25)

$$Q_{s-1} = H + T_1\delta_1 + \ldots + T_{s-1}\delta_{s-1}, \quad Q_s = Q_{s-1} + T_s\delta_s$$

and c_s^o, ν_l, w_l, u_l as in (21) with new subscripts l and $A_l^o = a_{t_l}(\alpha, i_l + j_l)$, $B_l^o = b_{p_l}(\alpha, \delta_l - 1)$ and again Σ_l is five-fold on i_l, j_l, m_l, t_l, and p_l.

(3b). <u>Probability integral:</u> From (24) we have $P(T_s \leq a) = F(a)$ as

$$(26)\quad F(a) = \frac{1}{(M)} c_s^o \Sigma_1 \ldots \Sigma_s \prod_{l=1}^{s-1} \Omega_l A_l^o B_l^o T_l^{\nu_l - 1} \Gamma(u_l)/w_l^{u_l} \times$$

$$\times \left[\Omega_s A_s^o B_s^o \Gamma(u_s)/w_s^{u_s} \delta_s^{\nu_s} \right] \times \left[\Gamma(\varepsilon + \nu_s) \int_0^{a^{**}} \frac{w^{\nu_s - 1}\, dw}{(1+w)^{\varepsilon + \nu_s}} \right] \Big/ Q_{s-1}^{\varepsilon},$$

where $a^{**} = a\delta_s/Q_{s-1}$ and again as we did in section 2, if ε is an integer (with integer g), we can express the integral part of (26) as

$$(26a) \qquad B(\varepsilon-1,\ \varepsilon+\nu_s-1,\ 1/1+a^{**})/\nu_s \binom{\varepsilon+\nu_s-1}{\varepsilon-1}.$$

Because of $F(\infty)=1$, we have from (26), similar to (10),

$$(27) \qquad c_s^o \Sigma_s \left[\Omega_s A_s^o B_s^o \Gamma(u_s)\Gamma(\nu_s)/w_s^{u_s}\ \delta_s^{\nu_s}\right] = 1.$$

For example, if $s = 1$, $N_1 = 2$, $k_1' = 1$, then (26) reduces to

$$(27a) \qquad F(a) = 2^{-(\alpha-1)} \sum_{m=0}^{\alpha-1} \binom{\alpha+m-1}{m} \frac{1}{2^m} B(G-1, G+\alpha-m-1, 1/(1+\frac{a}{H}))$$

with $Q_0 = H$ and from (27a) we get (10c) again. We can compare (26), that is $P(T_s \le a)$ with $P(T \le a)$ in Lingappaiah [11] which is

$$(28) \qquad P(T \le a) = c^o \Sigma\left[\Omega A^o B^o \Gamma(u)\Gamma(\nu)/\delta^\nu w^u\right] \left[1 - \sum_{k=0}^{\nu-1} e^{-a\delta}(a\delta)^k/k!\right]$$

(28) gives the probability of $T \le a$ in the current sample while (26) gives the same (current sample is not needed now) based on the earlier samples. (26) and (28) are both easy to calculate using either Incomplete Beta and Gamma integral tables or Cumulative Binomial and Poisson tables. Also, further if $\alpha = 1$, $k_1' = 1$, $s = 1$ in (26) we have

$$(29) \qquad F(a_0) = B(G'-1, G', 1/(1+a_0/H))$$

again with $G' = n+g$. From (29), using (16), we get

$$(29a) \qquad a_0 = \left[(1-\delta)^{-1/G'} - 1\right]$$

and as we did in section 2, we can calculate $F(a_0)$ from (26) and (26a) for different values of $\alpha \neq 1$.

(3c). <u>Variance</u>: Again, as in our section 2, we try to show here also, that the variance is affected by the number of stages. For from (24), we have

$$(30)\quad \mu'_{r(s)} = \frac{1}{(M)} c_0^s \sum_1 \ldots \sum_s \prod_{l=1}^{s-1} \Omega_l A_l^o B_l^o T_l^{\nu_l - 1} \Gamma(u_l)/w_l^{u_l}$$

$$\cdot\, \Omega_s\, A_s^o B_s^o \Gamma(u_s)/w_s^{u_s} \cdot \Gamma(\nu_s + r)\Gamma(\varepsilon - r) Q_{s-1}^r \Big/ Q_{s-1}\delta_s^{\nu_s + r} \,.$$

Again if $r = 0$, (30) reduces to (27) and $\mu'_{r(s)}$ is the rth raw moment of T_s.

Suppose $N_1 = 2$, $s = 1$, $k'_1 = 1$. We have from (30)

$$(31)\qquad \mu'_{r(1)} = \frac{2}{\Gamma^2(\alpha)} \sum_{m=0}^{\alpha-1} \binom{\alpha-1}{m} \frac{\Gamma(m+\alpha)\Gamma(\alpha - m + r)\Gamma(G-r)H^r}{2^{\alpha+m}\,\Gamma(G)}$$

and if $r = 0$, we have (10c). Also from (30), we have for $\alpha = 1$, $N_i = 2$, $i = 1,\ldots,s$, $k'_i = 1$, $i = 1,\ldots,s$.

$$(31a)\qquad \mu'_{r(s)} = \Gamma(r+1)\Gamma(G' + s - 1 - r)(Q_{s-1}^o)^r / \Gamma(G' + s - 1)$$

which gives

$$(31b)\qquad \mu_{2(s)} = (Q_{s-1}^o)^2 (G' + s - 1)/(G' + s - 2)^2 (G' + s - 3).$$

(31b) shows that the variance, as in section 2, can be reduced as s increases for given G' despite the fact that Q_{s-1}^o increases as s increases. Now in (31), if $\alpha = 2$, we have

$$(31c)\qquad \mu_{2(1)} = H^2(7G+2)/4(G-1)^2(G-2).$$

Now, we can compare this variance of T_s with that of T in Lingappaiah [11], which gives,

$$(32) \qquad \mu'_{r(T)} = c^o \sum \Omega A^o B^o \Gamma(\nu+r)\Gamma(u)/w^u \delta^{\nu+r}$$

which again for $N = 2$, $\alpha = 2$, $k' = 1$, gives

$$(32a) \qquad \mu'_{r(T)} = \frac{1}{2} \sum_{m=0}^{1} \binom{1}{m} \Gamma(2-m+r)\Gamma(2+m)/2^m.$$

4. Comments

At this point, we would like to make some comments on our development. We have assumed that α to be known in (1) and also takes only integer values. Otherwise A, B's do not make sense. Further, though few results of Sections 2 and 3, are similar for example, of $F(a)$ and $\mu'_{r(s)}$ for particular values, each one has to be deduced separately, since one cannot be obtained from the other. Also, in the application of the result of Lingappaiah [11], we need just one single sample, which is partly a convenience and partly a loss of information on earlier samples while our present method requires few earlier samples, which of course is a problem of economics. Our result here, is mainly developed to show, how we can make use of earlier information, though, based on the variance, it does tell, that more we sample, better results, we can expect. Also, it is to be noted that we may predict y_s at stage s, based on any order statistic in earlier samples from 2 to s and it may be a matter of study as to which $(s-1)$ tuple $(k_1,\dots,k_{s-1})$ gives the best prediction of y_s. Same is the situation in section 3 also. We could have developed the prediction of T_s based on any order statistic in earlier stages: However, basing T_s on $T_1,\dots,T_{s-1}$ is in one way meaningful and secondly, we may not be able to generalise from stage 1 to stage s as we did in section 3. Finally, it is to be noted that our method of treating the posterior distribution at the stage (s-1) as the prior for stage s is quite logical in the sense, we carry along all the prior information with us all through the development. Again, in our method, the experimenter has quite a choice of selecting any number of samples as he desires. Obviously, more samples, the current result is based on, the better it will be. For example, if he desires to discard first (s_0-1) samples $s_0 < s$, then simply take the s_0th sample as stage 0 and proceed further on, since the samples are completely independent. Incidentally, there are many ways of using the earlier information at the current stage. For example, after a certain number of samples are available, one may wish to pool all this information, by taking the product of densities of order statistics at these stages, and treat this as stage 0 with the current situation as stage 1. However, though the author has not compared these two approaches, it is felt that our present method is more reliable and meaningful in the sense, that at each stage, previous information is filtered for the next stage.

REFERENCES

[1] G.A. Bancroft, T.R. Dunsmore: Predictive distributions in life tests under competing causes of failure, Biometrika, 63 (1976) 195-198.

[2] J. Aitcheson, I.R. Dunsmore: Statistical prediction analysis, Cambridge University Press (1975).

[3] I.R. Dunsmore: Asymptotic prediction analysis, Biometrika, 63 (1976) 627-30.

[4] I.R. Dunsmore: The Bayesian predictive distribution in Life testing models, Technometrics 16 (1974) 455-460.

[5] K.S. Kaminsky, E.M. Luks, P.L. Nelson: An Urn model and the prediction of order statistics, Comm. Statist. 4 (1975) 245-250.

[6] K.S. K a m i n s k y : Comparison of prediction intervals for failure times when life is exponential, Technometrics, 19 (1977) 83-86.

[7] K.S. K a m i n s k y , P.I. N e l s o n : Best linear unbiased prediction of order statistics in Location and Scale families, J. Amer. Statist. Assoc. 70 (1975) 145-150.

[8] G.D. F a u l k e n b e r r y : A method of obtaining prediction intervals, J. Amer. Statist. Assoc. 68 (1973) 433-435.

[9] J.F. L a w l e s s : A prediction problem concerning samples from the exponential distribution with applications in life testing, Technometrics, 13 (1971) 725-730.

[10] G.S. L i n g a p p a i a h : Prediction in exponential life testing, Canad. J. Statist., 1 (1973) 113-117.

[11] G.S. L i n g a p p a i a h : Prediction in samples from the gamma distribution as applied to life testing. Austral. J. Statist., 16 (1974) 30-32.

[12] M.Z. K h a n : Optimum allocation in Bayesian Stratified two-phase sampling when there are m-attributes. Metrika, 23 (1976) 211-219.

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Received July 21st, 1977.