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CERTAIN REFLEXIVE LATTICES OF SUBSPACES OF A HILBERT SPACE

1. Let H be a Hilbert space, let $\mathcal{L}(H)$ be the set of all closed linear subspaces of H, and let $\mathcal{A}(H)$ be the set of all bounded linear operators on H. To each subset $\mathcal{A}$ of $\mathcal{A}(H)$ there corresponds the set $\mathrm{Lat}\,\mathcal{A}$ of all those elements of $\mathcal{L}(H)$ that are invariant under every operator in $\mathcal{A}$. Any subset of $\mathcal{L}(H)$ such that $\mathcal{L} = \mathrm{Lat}\,\mathcal{A}$ for some $\mathcal{A}$ will be called reflexive (cf. [1], p.258). It is obvious that the set $\mathrm{Lat}\,\mathcal{A}$ is always a complete sublattice of the lattice $\mathcal{L}(H)$. (The lattice structure on $\mathcal{L}(H)$ is given by the inclusion, i.e., $L \wedge M = L \cap M$ and $L \vee M$ is the closure of $L + M$, for L,M in $\mathcal{L}(H)$). We will give certain sufficient conditions for a complete sublattice $\mathcal{L}$ of $\mathcal{L}(H)$ to be reflexive. We get the well-known Theorems of Ringrose (cf. [3]) and Harrison (cf. [2]) as special cases of our Theorems. The proofs given in the paper are shorter and simpler than the proofs of Ringrose and Harrison.

2. Let $\mathcal{L}$ be a complete sublattice of the lattice $\mathcal{L}(H)$ and let M be an element of $\mathcal{L}(H)$. Define

$$M^* = \bigvee \left\{L : L \in \mathcal{L} \ \& \ M \not\subset L\right\}$$

$$M^+ = \bigwedge \left\{L^* : L \in \mathcal{L} \ \& \ L \not\subset M\right\}$$

$$M^- = \bigvee \left\{L : L \in \mathcal{L} \ \& \ L \subset M\right\}.$$

It is obvious that each of these operations sends $\mathcal{L}(H)$ into $\mathcal{L}$, and if $M \subset N$, then $M^* \subset N^*$, $M^+ \subset N^+$, $M^- \subset N^-$. Moreover $M^- \subset M$ and $M^+ = (M^-)^+$ for every M in $\mathcal{L}(H)$.

3. Theorem. Let $\mathcal{L}$ be a complete sublattice of $\mathcal{L}(H)$ and let

$$\mathfrak{M}(\mathcal{L}) = \left\{ M \in \mathcal{L}(H) : \forall_{L \in \mathcal{L}(H)} (L \subset M \text{ or } M \subset L^*) \right\}.$$

Then $\mathcal{L}$ is a subset of $\mathfrak{M}(\mathcal{L})$ and $\mathfrak{M}(\mathcal{L}) = \mathrm{Lat}\,\mathcal{A}$, where

$$\mathcal{A} : \left\{ P_L \circ A \circ (1 - P_{L^*}) : A \in \mathcal{A}(H),\ L \in \mathcal{L} \right\}$$

(P_L denotes the orthogonal projection onto L).

Proof. The inclusion $\mathcal{L} \subset \mathfrak{M}(\mathcal{L})$ is obvious. Let M be in $\mathfrak{M}(\mathcal{L})$ and let $B = P_L \circ A(1-P_{L^*})$ for some A in $\mathcal{A}(H)$ and L in $\mathcal{L}$. If $L \subset M$, then $B(M) \subset M \subset L$. If $M \subset L^*$, then $B(M) = 0 \subset M$ (0 denotes the zero subspace). Consequently M is in $\mathrm{Lat}\,\mathcal{A}$.

Let M be in $\mathrm{Lat}\,\mathcal{A}$ and let L be in $\mathcal{L}$. If M is not included in L^*, then there exists an f in $M \setminus L^*$. Since $g = (1-P_{L^*})f \neq 0$, it follows that for every u in L there exists an A in $\mathcal{A}(H)$ such that $u = Ag$. Since f is in M and M is in $\mathrm{Lat}\,\mathcal{A}$, it follows that $u = P_L u = P_L \circ A \circ (1-P_{L^*})f$ belongs to M, and hence L is included in M. Consequently M is in $\mathfrak{M}(\mathcal{L})$.

4. Lemma. Let M be an element of $\mathcal{L}(H)$. Then the following conditions are equivalent:

(1) $M \in \mathfrak{M}(\mathcal{L})$

(2) $M \subset M^+$

(3) there exists in L in $\mathcal{L}$ such that $L \subset M \subset L^+$

Proof. The implications $(1) \Rightarrow (2)$, $(2) \Rightarrow (1)$ and $(2) \Rightarrow (3)$ are obvious. If $L \subset M \subset L^+$ for some L in $\mathcal{L}$, then $L^+ \subset M^+$, and hence $M \subset M^+$.

5. T h e o r e m . The equality $\mathfrak{M}(\mathcal{L}) = \mathcal{L}$ holds if and only if the lattice $\mathcal{L}$ satisfy the following condition

$$\left\{M \in \mathcal{L}(H) : L \subset M \subset L^+\right\} \subset \mathcal{L}$$

for each L in $\mathcal{L}$.

This theorem follows immediately from Lemma 4. As a corollary we get the following theorem.

6. T h e o r e m . Let $\mathcal{L}$ be a complete sublattice of the lattice $\mathcal{L}(H)$. If for every L in $\mathcal{L}$ the set $\left\{M : M \in \mathcal{L}(H) \ \& \ L \subset M \subset L^+\right\}$ is included in $\mathcal{L}$, then $\mathcal{L}$ is reflexive.

In particular, if $L = L^+$ for each L in $\mathcal{L}$, then $\mathcal{L}$ is reflexive.

7. T h e o r e m . If $\mathcal{L} \subset \mathcal{L}(H)$ is a complete chain, then $L = L^+$ for each L in $\mathcal{L}$.

P r o o f . It is easy to see that a complete sublattice $\mathcal{L}$ of $\mathcal{L}(H)$ is a chain if and only if $L^* \subset L$ for every L in $\mathcal{L}$. If $L \neq L^+$, then immediately from the definition of L^+ it follows that $L^+ \subset (L^+)^*$. Hence, by the assumption, $L^+ = (L^+)^*$, i.e., $L^+ = \bigvee \left\{N \in \mathcal{L} : L^+ \not\subset N\right\}$. Consequently there is an N_0 in $\mathcal{L}$ such that $L^+ \not\subset N_0$ and $N_0 \not\subset L$. Hence $L^+ \subset N_0^*$, by the definition of L^+. Since $N_0^* \subset N$, we get a contradiction.

8. C o r o l l a r y (Ringrose's Theorem, cf. [3]). If $\mathcal{L} \subset \mathcal{L}(H)$ is a complete chain, then $\mathcal{L}$ is reflexive.

9. T h e o r e m . Let $\mathcal{L}$ be a complete sublattice of $\mathcal{L}(H)$ and let $\mathcal{P}$ be the set of all L in $\mathcal{L}$ satisfying the following condition

$$L \not\subset \bigvee \left\{N \in \mathcal{L} : L \not\subset N\right\}$$

(i.e., $L \not\subset L^*$). If every M in $\mathcal{L}$ is the supremum of some subset of $\mathcal{P}$, then $\mathcal{L}$ is reflexive.

P r o o f . We will show that $L = L^+$ for each L in $\mathcal{L}$. Let L be in $\mathcal{L}$. By the assumption $L^+ = \bigvee \mathcal{A}$, where $\mathcal{A}$ is a subset of $\mathcal{P}$. If $L^+ \neq L$, then $L_0 \not\subset L$ for some L_0 in $\mathcal{A}$. Hence $L^+ \subset (L_0)^*$, and consequently $L \subset (L_0)^*$. On the other hand $L_0 \not\subset (L_0)^*$ by the definition of $\mathcal{P}$.

10. C o r o l l a r y (Harrison's Theorem, cf. [2]). Let $\mathcal{L}$ be a complete sublattice of $\mathcal{L}(H)$, distributive in the sense:

$$L \wedge \bigvee \mathcal{S} = \bigvee \left\{ L \wedge S : S \in \mathcal{S} \right\}$$

for each L in $\mathcal{L}$ and each subset $\mathcal{S}$ of $\mathcal{L}$. Let $\mathcal{G}$ be the set of all L in $\mathcal{L}$ satisfying the following condition:

$$L \neq \bigvee \left\{ N \in \mathcal{L} : N \subset L \ \& \ N \neq L \right\}$$

(i.e., L is infinitely-join-irreducible). If every M in $\mathcal{L}$ is the supremum of some subset of $\mathcal{G}$, then $\mathcal{L}$ is reflexive.

P r o o f . It is obvious that if $\mathcal{L}$ is distributive in the above sense, then $\mathcal{G} = \mathcal{P}$.

11. E x a m p l e . Let H be an infinite dimensional Hilbert space. Consider two subspaces M and N_0 of H such that $M \wedge N_0 = 0$, $M \cup N_0 = H$ but $M + N_0 \neq H$. Let $u_1, \dots, u_n$ be vectors in H such that $(M+N) \cap \operatorname{span}\{u_1, \dots, u_n\} = 0$, and let $N_n = \operatorname{span}(N_0 \cup \{u_1, \dots, u_n\})$. Let $\mathcal{N}$ be the set of all those N in $\mathcal{L}(H)$ such that $N_0 \subset N \subset N_n$. It is not difficult to verify that the set $\mathcal{L} = \{0, M, H\} \cup \mathcal{N}$ is a complete lattice. Moreover $0^+ = 0$, $M^+ = M$, $H^+ = H$ and $N^+ = N_n$ for every N in $\mathcal{N}$. Consequently, by the theorem 6, $\mathcal{L}$ is reflexive. (Compare [1], §3(a)).

12. The equality $\mathcal{L} = \mathfrak{M}(\mathcal{L})$ is not necessary condition of reflexivity of $\mathcal{L}$. For example, let $\{0,L,M,N,H\}$ be a double triangle of subspaces of a four dimensional space H, (i.e., $L \vee M = M \vee N = N \vee L = H$, $L \wedge M = M \wedge N = N \wedge L = 0$), and let $\mathcal{L}$ be the smallest reflexive sublattice of $\mathcal{L}(H)$ such that $\{0,L,M,N,H\} \subset \mathcal{L}$ (cf. [1], §3(b)). If K_1, K_2 are non-trivial different elements of $\mathcal{L}$, then $K_1 \cup K_2 = H$ and $K_1 \wedge K_2 = 0$. Hence $K^* = H$ for each nonzero K in $\mathcal{L}$. Consequently $\mathfrak{M}(\mathcal{L}) = \mathcal{L}(H)$, but $\mathcal{L} \neq \mathcal{L}(H)$.

Added in proof. After this note had been submitted for publication, the paper of W.E. Longstaff "Strongly reflexive lattices", J. London Math. Soc. 11 (1975), 491-498, was called to the author's attention, where similar results have been obtained (as in our note, Longstaff's technique is an extension of that of [2]).

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