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A LOCAL PROPERTY OF THE SUBSPACES OF EUCLIDEAN DIFFERENTIAL SPACES

1. Preliminares

Let $M \neq \emptyset$ be a set and C an arbitrary set of real functions defined on M. We denote by τ_C the weakest topology on M such that all functions belonging to C are continuous. For any set A contained in M we denote by $C|A$ the set of functions of the form $\alpha|A$ where $\alpha \in C$. We denote by C_A the set of all real functions on A such that for any point p of A there exists in τ_C an open neighbourhood U of p and a function $\alpha \in C$, such that $\beta|A \cap U = \alpha|A \cap U$. It is easy to verify that, for any set $A \subset M$, we have $\tau_{C_A} = \tau_{C|A} = \tau_C|A$. In particular $\tau_{C_M} = \tau_C$. We denote by scC the set of all real functions of the form $\omega(\alpha_1,\ldots,\alpha_n)$, where $\omega \in \mathcal{E}_n$, $\alpha_1,\ldots,\alpha_n \in C$ and n belongs to the set of all positive integers $\mathcal{N}$ and $\mathcal{E}_n$ is the set of all real C^∞-functions on n-dimensional Euclidean space E^n. An ordered pair (M,C) such that $C_M = C = scC$ is said to be a differential space. The set C is called the differential structure of this differential space [1], [2], [6].

For a set C of real functions defined on M, the set $(scC)_M$ is the smallest differential structure on M including the set C. $(M,(scC)_M)$ is called the differential space generated by C.

If (M,C) is a differential space and $A \subset M$, then (A,C_A) is also a differential space called the differential subspace of (M,C) [1]. It is easy to see that $C_A = (C|A)_A$.

By a vector tangent to a differential space (M,C) at a point p of M we mean any linear mapping $v : C \to E$ which fulfils Leibniz's condition at the point p:

$$v(\alpha \beta) = v(\alpha)\beta(p) + \alpha(p)\, v(\beta) \quad \text{for all } \alpha, \beta \in C.$$

We shall denote by $(M,C)_p$ or M_p a linear space of all vectors tangent to (M,C) at a point $p \in M$.

Any real C^∞-manifold M will be identified with the differential space $(M,C^\infty(M))$, where $C^\infty(M)$ is the set of all smooth real functions on M. In particular, we denote $C^\infty(E^n)$ by $\mathcal{E}_n$ and we call the pair $(E^n,\mathcal{E}_n)$ the n-dimensional Euclidean differential space.

It is easy to verify that for $\emptyset \neq M \subset E^n$, $n \in \mathcal{N}$, $\mathcal{E}_{nM} = (sc\{\pi^i|M;\ i=1,\dots,n\})_M$ where $\pi^i(x^1,x^2,\dots,x^n)) = x^i$ for any $(x^1,x^2,\dots,x^n) \in E^n$. The topological space $(M,\tau_{\mathcal{E}_{nM}})$ is a subspace of the topological space $(E^n,\tau_{\mathcal{E}_n})$.

In the sequel the symbol τ_M will be used instead of $\tau_{\mathcal{E}_{nM}}$. Using a partition of unity it may be proved that $\mathcal{E}_{nM}$ is the set of all functions of the form $\alpha|M$, where α is a C^∞-function on an open set U in E^n including M.

The basic result of this paper consists in the following theorem.

T h e o r e m 1. For any $p \in M \subset E^n$ the integer $m = \dim(M,\mathcal{E}_{nM})_p$ is the smallest one such that there exists in τ_M an open neighbourhood U of the point p which is included in an m-dimensional C^∞-surface of E^n.

2. The proof of basic result

From now on we fix the integer $k > 0$, and the non empty set $M \subset E^k$. For brevity we write $\mathcal{E} := \mathcal{E}_k$, $C := \mathcal{E}_M$, $M_p := (M,C)_p$, $E^k_p := (E^k,\mathcal{E}_k)_p$.

The mappings $L_1 : M_p \to E_p^k$ and $L_2 : E_p^k \to E^k$ defined by

(1) $(L_1(v))(f) := v(f|M)$ for $v \in M_p$ and $f \in \mathcal{E}$,

(2) $L_2(\bar{v}) := (\bar{v}(\pi^1),\ldots,\bar{v}(\pi^k))$ for $\bar{v} \in E_p^k$,

are respectively a linear monomorphism and a linear isomorphism of suitable linear spaces (c.f. [1]).

Let $\partial_i f(p)$ denote i-th partial derivative of the function $f \in \mathcal{E}$ at the point $p \in E^k$, $i = 1,\ldots,k$. If we denote $f_{|h}(p) := h^i \partial_i f(p)$ where $h = (h^1,\ldots,h^k) \in E^k$, we have

(3) $\bar{v}(f) = \partial_i f(p)\bar{v}(\pi^i) = f_{|L_2(\bar{v})}(p)$ for $\bar{v} \in E_p^k$,

(the sumation convention is used here). Let $L := L_2 \circ L_1 : M_p \to E^k$ and

(4) $$\bar{M}_p = \left\{ L(\bar{v}) \in E^k;\ \bar{v} \in M_p \right\} .$$

It is easy to see that the mapping $L : M_p \to \bar{M}_p$ makes these linear spaces isomorphic to each other. We have

(5) $$\begin{cases} L(v) = (v(\pi^1|M),\ldots,v(\pi^k|M)) & \text{for } v \in M_p, \\ v(f|M) = f_{|L(v)} & \text{for } v \in M \text{ and } f \in \mathcal{E}_k. \end{cases}$$

L e m m a 1. For $p \in M$, $h \in E^k$, $k \in \mathcal{N}$ the following properties are equivalent:

(a) $h \in \bar{M}_p$,

(b) there exists a mapping $\bar{v} : \mathcal{E}|M \to E$ such that

$$\bar{v}(f|M) = f_{|h}(p) \quad \text{for} \quad f \in \mathcal{E} .$$

P r o o f . The implication (a)$\Rightarrow$(b) follows immediately from (4) and (5) by putting $\bar{v} := v|(\mathcal{E}|M)$, $v \in M_p$ and $h = L(v)$.

In order to prove the implication (b) $\Rightarrow$ (a) let us suppose that h fulfils (b) and consider the set of functions $\mathcal{E}|M$. From (b) it follows that $\bar{v}$ is the linear mapping of $\mathcal{E}|M$ into E fulfilling the Leibniz's condition at the point p:

$$\bar{v}(\alpha\beta) = \bar{v}(\alpha)\beta(p) + \alpha(p)\bar{v}(\beta) \qquad \text{for} \qquad \alpha,\beta \in \mathcal{E}\,|\,M.$$

By using this condition and linearity of $\bar{v}$ one can easy verify that $\bar{v}(\alpha) = 0$ for each function $\alpha \in \mathcal{E}|M$ equal to 0 at an open neighbourhood of the point p. As a consequence of this the mapping $v : C \to E$ defined

$$v(\alpha) := \bar{v}(f|M) \qquad \text{for} \qquad \alpha \in C,$$

where $f \in \mathcal{E}$ is a function such that $f|U = \alpha|U$ for some set $U \in \tau_M$ including the point p, is well defined. The function v is linear and fulfils Leibniz's condition so it belongs to M_p. For $i = 1,\ldots,k$ we have $v(\pi^i|M) = \pi^i_{|h}(p) = \bar{v}(\pi^i|M) = h^i$, where $h = (h^1,h^2,\ldots,h^k)$, so from (5) we have: $L(v) = (h^1,h^2,\ldots,h^k) = h$. The Lemma is proved.

Lemma 2. For $h \in E^k$ and $p \in M$ the following conditions are equivalent:

(a) $h \in \bar{M}_p$,

(b) $f_{|h}(p) = 0$ for any $f \in \mathcal{E}$ equal to 0 on M.

Proof. It is easy to see that the conditions (b) in Lemmas 1 and 2 are equivalent to each other.

For any $f \in \mathcal{E}$ and $p \in E^k$ we denote $\operatorname{grad} f(p) := (\partial_1 f(p),\ldots,\partial_k f(p))$.

Lemma 3. Let $p = (0,0,\ldots,0) \in M \subset E^k$ and $e_i = (0,0,\ldots,0,1,0,\ldots,0)$ (1 in the i-th position), $1 \leq i \leq k$. If $m := \dim M_p$, $1 \leq m \leq k-1$ and $e_1,\ldots,e_m \in \bar{M}_p$ then there exist functions $f^{m+1},\ldots,f^k \in \mathcal{E}$ equal to 0 on M and such that $\partial_i f^j(p) = \delta^j_i$, where $\delta^j_i = 1$ for $i = j$ and $\delta^j_i = 0$ for $f \neq j$.

Proof. Let the assumptions of the Lemma be satisfied. Then

(6) $$\bar{M}_p = \mathrm{Lin}(e_1,\dots,e_m),$$

where $\mathrm{Lin}(e_1,\dots,e_m)$ is the linear subspace of E^k spanned by $e_1,\dots,e_m$. We put $K := \{h \in E^k;\ h = \mathrm{grad}\ f(p),\ f \in \mathcal{E}$ and $f = 0$ on $M\}$. K is a linear subspace of E^k and $e_i \perp K$ with respect to the canonical scalar product in E^k so $K \subset \mathrm{Lin}(e_{m+1},\dots,e_k)$ (see Lemma 2). We shall prove more, namely that $K = \mathrm{Lin}(e_{m+1},\dots,e_k)$. If the above equality is not satisfied, then there exists a non-zero vector $h \in \mathrm{Lin}(e_{m+1},\dots,e_k)$, such that $K \perp h$. Hence $f_{|\underline{h}}(p) = \mathrm{grad}\ f(p)\cdot h = 0$ for $f = 0$ on M, and $h \in \bar{M}_p$ (Lemma 2), but this contradicts (6). From above equality we obtain the existence of functions $f_{m+1},\dots,f_k \in \mathcal{E}_k$ equal to zero on M, such that $\mathrm{grad}\ f_j(p) = e_j$ or equivalently $\partial_i f^j(p) = \delta_i^j$. The Lemma is proved.

P r o p o s i t i o n 1. Let $p \in M \subset E^k$. If $0 < m := \dim M_p \leq k$ then there exist non empty sets: U open in τ_M and V open in $\tau_{\mathcal{E}_m}$, and regular 1-1 C^∞-mapping $\phi : V \to E^k$ such that

$$p \in U \subset \left\{\phi(u) \in E^k;\ u \in V\right\}.$$

P r o o f . If $m = k$, the proposition evidently holds. We suppose that $1 \leq m < k$. We can assume, without loss of generality, that $p = (0,\dots,0) \in E^k$ and $\bar{M}_p = \mathrm{Lin}(e_1,\dots,e_m)$. We denote $q = (x^1,\dots,x^k) = (u,w)$ where $u = (x^1,\dots,x^m)$ and $w = (x^{m+1},\dots,x^k)$. Let f^j, $j = m+1,\dots,k$, are functions as in Lemma 3. We define a mapping $F : E^k \to E^{k-m}$ by

$$F(q) := (f^{m+1}(q),\dots,f^k(q)) \quad \text{for} \quad q \in E^k.$$

This mapping has the following properties:

(a) $F(q) = F(u,w) = 0$ for $q = (u,w) \in M$,

(b) F is C^∞-mapping,

(c) F is regular at the point $p = (\bar{u},\bar{w})$.

From the inverse mapping theorem it follows that there exists:

(d) a set $U' \in \tau_{E^k}$ such that $p \in U'$,

(e) a set $V \in \tau_{E^m}$ such that $\bar{u} \in V$,

(f) a C^∞-mapping $\psi : V \to E^{k-m}$ such that for any $u \in V$ we have $F(u,\psi(u)) = 0$,

(g) if $F(q) = 0$ and $q = (u,w) \in U'$ then $u \in V$ and $w = \psi(u)$.

It is evident that $U := U' \cap M$, V and $\phi(u) := (u,\psi(u))$ for $u \in V$ fulfil conditions of Proposition 1.

Now, we examine the case of $\dim M_p = 0$, which was not considered above.

P r o p o s i t i o n 2. Let $p \in M \subset E^k$. If $\dim M_p = 0$ then the point p is isolated in M.

P r o o f . Let us set $|q| := \sqrt{(x^1)^2 + \ldots + (x^k)^2}$ for any $q = (x^1,\ldots,x^k) \in E^k$.

Let us assume the point p is not isolated. Then there exists a sequence (p_i) of points of M different from p and convergent to p. For the sequence $h_n := \frac{p_n - p}{|p_n - p|}$, $n \in \mathcal{N}$ of points of S^{k-1} we can find a subsequence h_{n_i} covergent to a point $h \in S^{k-1}$. One can easy see that for any $f \in \mathcal{E}$

$$\lim_{i\to\infty} \frac{f(p_{n_i}) - f(p)}{|p_{n_i} - p|} = f_{|h}(p).$$

It easy to see that left side of this equality defines mapping $\bar{v} : \mathcal{E}|M \to E$ such that $\bar{v}(f|M) = f_{|h}(p)$. From Lemma 1 $h \in \bar{M}_p$ so $\dim M_p \neq 0$, which ends of the proof.

Theorem 1 results easily from Propositions 1 and 2. In that Theorem an non-empty discrete subset of E^n is called a 0-dimensional C^∞-surface in E^n.

3. Collloraries

We say, that differential space (N,D) can be diffeomorphically embeded into the differential space (L,H) if there exists a subset $L' \subset L$ such that $(L', H_{L'})$ and (N,D) are diffeomorphic to each other. In the sequel we shall consider only differential spaces (N,D) such that any point $p \in N$ has a neighbourhood V such that (V, D_V) can be embeded into $(E^{n(p)}, \mathcal{E}_{n(p)})$ for some $n(p) \in \mathcal{N}$. From Theorem 1 we obtain:

C o l l o r a r y 1. For a point p of the differential space (N,D) there exist a set $V \in \tau_D$ and an n-dimensional C^∞-manifold $(\tilde{N}, C^\infty(\tilde{N}))$, $n := \dim N_p$, such that $p \in V \subset \tilde{N}$ and $D_V = C^\infty(\tilde{N})_V$. The inequality

$$\dim M_q \leq \dim M_p$$

is fulfiled for any point $q \in V$.

C o r o l l a r y 2. If (N,D) is a differential space such that (N, τ_D) is separable and if there exists $n \in \mathcal{N}$ such that for any $p \in N$ $\dim(N,D)_p \leq n$, then topological dimension of (N, τ_D) does not exceed n.

P r o o f . This results easily from Corollary 1.

Differential spaces which have tangent spaces of constant dimension are the most interesting. For a differential space (N,D) and $i = 0,1,\ldots$ we shall denote by N^i union of all sets $V \in \tau_D$ such that $\dim(N,D) = i$ for any $q \in V$. If N^i is not empty then (N^i, D_{N^i}) is a differential subspace of (N,D) and for any $q \in N^i$ $\dim(N^i, D_{N^i}) = i$. From Corollary 1 we obtain

C o r o l l a r y 3. For any differential space (N,D) the set $\bigcup_{i=0}^{\infty} N^i$ is open and dense in the topological space (N, τ_D).

P r o o f . For any subset $A \subset N$ we denote its closure in (N, τ_D) by $\bar{A}$. We shall use mathematical induction. Let $p \in N$. It is easy to see, that $\dim(N,D)_p \geq 0$. If $\dim(N,D)_p = 0$ then the point p is isolated in (N, τ_D) and $p \in N^0$, see Corollary 1, so $p \in \overline{\bigcup_{i=0}^{\infty} N^i}$. Suppose, that for any $q \in N$

such that $0 \leq \dim(N,D)_q \leq m-1$ we have $q \in \overline{\bigcup_{i=0}^{\infty} N^i}$. For any point $p \in N$ such that $\dim(N,D)_p = m$ there exists an open neighbourhood V of p such that $\dim(N,D)_q \leq m$ for any point $q \in V$ (Corollary 1). Let $U \in \tau_D$ be a set containing the point p. If for any $q \in U \cap V$ $\dim(N,D)_q = m$, then $p \in N^m$ and $p \in \overline{\bigcup_{i=0}^{\infty} N^i}$. If it is not true then there exists a point $q \in U \cap V$, such that $\dim(N,D)_q \leq m-1$. From the induction hypothesis, the point $q \in \overline{\bigcup_{i=0}^{\infty} N^i}$, so $U \cap \bigcup_{i=0}^{\infty} N^i \neq \emptyset$. This is true for any set $U \in \tau_D$ containing the point p, so we have $p \in \overline{\bigcup_{i=0}^{\infty} N^i}$. The corollary is proved.

By virtue of Corollary 1 any point p of differential space (N,D) such that $\dim(N,D)_p = k$ has a neighbourhood V such that (V,D_V) can be diffeomorphically embedded in $(E^k, \mathcal{E}_k)$. Hence it is interesting to consider the differential subspace $(M,\mathcal{E}_{kM})$ of $(E^k,\mathcal{E}_k)$ for which there exists a point $p \in M$ such that $\dim(M,C)_p = k$.

C o r o l l a r y 4. Let $p \in M \subset E^k$. $\dim(M,\mathcal{E}_{kM})_p = k$ if and only if for any $f \in \mathcal{E}_k$ equal to 0 on M $\partial_i f(p) = 0$ for $i = 1,2,\ldots,m$.

P r o o f . We get this immediately from Lemma 2, as $e_i \in \bar{M}_p$, $i = 1,\ldots,k$.

C o r o l l a r y 5. Let $\emptyset \neq M \subset E^k$. Then $\dim(M,\mathcal{E}_{kM})_p = k$ for any p M if and only if for any $f \in \mathcal{E}_k$ equal to 0 on M all partial derivatives of any order are equal to 0 on M.

P r o o f . This corollary follows easily, by induction, from Corollary 4.

By virtue of above Corollary, any subset $M \subset E^k$ such that $(M,\mathcal{E}_{kM})$ has the constant dimension k, has the same property, as any open set of E^k: the value of the partial derivatives of a function $f \in \mathcal{E}_k$ at a point $p \in M$ are uniquely determined by the values of the function on M.

For a differential space (N,D) a linear mapping $X : D \to D$ such that $X(\alpha\beta) = X(\alpha)\beta + \alpha X(\beta)$ is called a vector field on (N,D) [1]. It easy to see that for any point $p \in N$ the function $X_p : D \to E$ defined by $X_p(\alpha) := (X\alpha)(p)$ for $\alpha \in D$ is a vector belonging to $(N,D)_p$.

C o r o l l a r y 6. Let (N,D) be a differential space. A point p belongs to $\bigcup_{i=0}^{\infty} N^i$ if and only if there exist vector fields $X_1,\dots,X_s$ on (N,D) such that $\{X_{1p},\dots,X_{sp}\}$ is the basis of $(N,D)_p$.

P r o o f . If $X_1,\dots,X_k$ are such vector fields on (N,D) that $X_{1p},\dots,X_{kp}$ is a basis of $(N,D)_p$ then there exists a set $V' \in \tau_D$ such that $p \in V'$ and $X_{1q},\dots,X_{kq}$ are lineary independent for any $q \in V'$ (cf. [1]). As there exists an open neighbourhood V'' of p such that for any $q \in V''$ $\dim(N,D)_q \leq k$ (Corollary 1), for any $q \in V' \cap V''$ $\dim(N,D)_q = k$ and $p \in N^k \subset \bigcup_{i=0}^{\infty} N^i$.

Now we shall prove the other implication. For the point $p \in N^0$ the proof is trivial. Let $p \in N^k$, $k > 0$ and U be such an open neighbourhood of the point p that (U,D_U) is diffeomorphic to $(V,\mathcal{E}_{kV})$ for certain $V \subset E^k$ and $\dim(V,\mathcal{E}_{kV})_q = k$ for any $q \in V$. It is sufficient to prove Corollary for $(V,\mathcal{E}_{kV})$.

For $q \in V$ and $\alpha \in \mathcal{E}_{kV}$ there exists an open neighbourhood V_q of q and a function $f_{\alpha,q} \in \mathcal{E}$ such that $\alpha|V_q = f_{\alpha,q}|V_q$. By virtue of Corollary 5 the functions $X_i : \mathcal{E}_{kV} \to \mathcal{E}_{kV}$, $i = 1,2,\dots,k$, defined for $\alpha \in \mathcal{E}_{kV}$, by

$$(X_i\alpha)(q) = \partial_i(f_{\alpha,q})(q) \qquad \text{for} \quad q \in V$$

are well defined. It can be easily verified that they are vector fields on $(V,\mathcal{E}_{kV})$ and $X_{1q},\dots,X_{kq}$ is the basis of $(V,\mathcal{E}_{kV})_q$ for any $q \in V$.

4. Examples

Example 1. Let $M \subset E^k$ be dense in E^k. Then by Corollary 1 the dimension of $(M, \mathcal{E}_{kM})_p$ is k for any $p \in M$.

Example 2. The graph of the function $f : E \to E$ which is x^2 for $x > 0$ and 0 for $x \leq 0$ has the tangent space of dimension 1 at all points except for the point $(0,0)$, where it has tangent space of dimension 2. It results easily from Corollary 1.

Example 3. The graph of the function $g : E \to E$ of class C^1 which is not of class C^∞ at any point is a differential subspace of $(E^2, \mathcal{E}_2)$ of constant dimension 2. It results easily from Corollary 1.

Example. 4. Let $M \subset E^k$. If topological dimension of any non empty open subset of M is k then $\dim(M, \mathcal{E}_{kM})_p = k$ for any $p \in M$. This follows easily from Corollary 2.

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