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A CONTACT PROBLEM IN THE THEORY OF ELASTICITY

1. Formulation of the problem

Let on the plane be given contours l_k $(k=0,1,\dots,n)$ having no points in common. Assume that l_0 encloses contours l_k $(k=1,2,\dots,n)$ which lie each outside the others. Let D_0 denote a multiply connected domain with the boudary $\bigcup_{k=0}^{n} l_k$ and D_k the domain bounded by l_k $(k=1,2,\dots,q)$ respectively, where $0<q\leq n$. Furthermore let $D = D_0 \cup (\bigcup_{k=0}^{q} \bar{D}_k)$ where $\bar{D}_k$ denotes the closure of D_k. Assume that the contours l_k $(k = 0, q+1,\dots,n)$ consist of finite number of arcs $l'_{k_i} = \widehat{a_{k_i} b_{k_i}}$ and $l''_{k_i} = \widehat{b_{k_i} a_{k_{i+1}}}$ where $i = 1,2,\dots,m$, $a_{k_{m+1}} = a_{k_1}$, a_{k_i}, $b_{k_i} \in l_k$ and let

$$l'_k = \bigcup_{i=1}^{m} l'_{k_i} \quad l''_k = \bigcup_{i=1}^{m} l''_{k_i} \quad (l_k = l'_k \cup l''_k).$$

The case of l'_k or l''_k being an empty set for some k is not excluded.

The elastic constants of an anisotropic medium of the domain D_k $(k = 0,1,2,\dots,q)$ are denoted by $\overset{(k)}{A}{}_{rs}^{je}$ $(r,s,j,e = 1,2)$ respectively. The operator of statics of an anisotropic bodies and the stress operator have the form

$$\overset{(k)}{A}\left(\frac{\partial}{\partial x}\right) \equiv \sum_{r,s=1}^{2} \overset{(k)}{A}_{rs} \frac{\partial^2}{\partial \xi_r \, \partial \xi_s} \quad \text{for} \quad x \in D_k \qquad (k=0,1,2,\dots,q),$$

where

$$\overset{(k)}{A}_{rs} = \left\| \overset{(k)}{A}{}^{je}_{rs} \right\|_{j=1,2}^{e=1,2}, \qquad x = x(\xi_1, \xi_2)$$

and

$$\overset{(k)}{T} \equiv \sum_{r,s=1}^{2} \overset{(k)}{A}_{rs} \cos(n_x, \xi_s) \frac{\partial}{\partial \xi_r} \quad (k=0,1,\dots,q),$$

where n_x denotes the exterior normal at the point $x \in l_k$.

The problem consists in the following: determine in D a displacement vector $u(x) = \overset{(k)}{u}(x)$ for $x \in D_k$ $(k=0,1,2,\dots,q)$ of the class $C^2(D_k) \cap C^1(\bar{D}_k)$ satisfying the system of equations

$$(1) \qquad \overset{(k)}{A}\left(\frac{\partial}{\partial x}\right) u(x) = \overset{(k)}{F}(x,u) \quad \text{for} \quad x \in D_k \quad (k=0,1,2,\dots,q)$$

subject to the boundary conditions

$$(2) \qquad \begin{aligned} &\{u(y_0)\}_{D_0} = \overset{(k)}{f}{}^1(y_0) && y_0 \in l_k' \\ &\left\{\overset{(0)}{T} u(y_0)\right\}_{D_0} = \overset{(k)}{f}{}^2(y_0) && y_0 \in l_k'' \quad k = 0, q+1, \dots, n \end{aligned}$$

and the contact conditions

$$\{u(x_0)\}_{D_0} = \{u(x_0)\}_{D_k}$$

(3)

$$\left\{\overset{(o)}{T} u(x_0)\right\}_{D_0} = \left\{\overset{(k)}{T} u(x_0)\right\}_{D_k} + \overset{(k)}{g}(x_0,u) \qquad x_0 \in l_k \quad k=1,2,3,\dots,q,$$

where $\{\ \}_{D_k}$ denotes the limit of a vector when x tends to the boundary from the interior of D_k.

The above problem is a generalization of the problem investigated in [4] to a finite number of insertions from different materials and to a multiply connected domain.

Due to the symmetry conditions $\overset{(k)}{A}{}^{je}_{rs} = \overset{(k)}{A}{}^{re}_{js} = \overset{(k)}{A}{}^{ej}_{sr}$ which are satisfied by the elastic constants, the system (1) is a strongly elliptic system in Vishik's sense.

It is assumed that:

1° The contours l_k $(k=0,1,\dots,n)$ satisfy the Lapunov conditions and there exists at least one arc l'_{k_i} which contains at least three points not lying on one straight line.

2° The components of the vectors $\overset{(k)}{F}(x,u)$ and $\overset{(k)}{g}(x_0,u)$ defined and bounded in the regions

$$\left\{x \in D_k,\ \left|\overset{(k)}{u}_j(x)\right| < +\infty\right\},\quad \left\{x_0 \in l,\ \left|\overset{(k)}{u}_j(x_0)\right| < +\infty\right\}$$

respectively satisfy for every k the conditions

$$\left|\overset{(k)}{F}_j(x,u_1,u_2) - \overset{(k)}{F}_j(x',u'_1,u'_2)\right| \leqslant K'_F\ |xx'|^{\alpha} + \frac{1}{2} K_F \sum_{i=1}^{2} |u_i - u'_i|$$

$$\left|\overset{(k)}{g}_j(x_0,u_1,u_2)-\overset{(k)}{g}_j(x_0',u_1',u_2')\right|\leqslant K_g'\left|x_0x_0'\right|^{\alpha}+\frac{1}{2}K_g\sum_{i=1}^{2}\left|u_i-u_i'\right|\quad j=1,2$$

with $\alpha\in(0,1)$ and K_f',K_F,K_g',K_g being some positive constants.

3^0 $\overset{(k)}{f}{}^1(y_0)\in C_\alpha^1(l_k')$, $\overset{(k)}{f}{}^2(y_0)\in C\,(l_k'')$.

2. Reduction of the problem to functional equations

Let us join the successive points b_{k_i} and $a_{k_{i+1}}$ with arcs $\tilde{l}_{k_i}$ lying entirely in the exterior of D in such a way that $\hat{l}_k=l_k'\cup\tilde{l}_k$ is a Lapunov contour where $\tilde{l}_k=\bigcup_{i=1}^{m}\tilde{l}_{k_i}$. Let us denote by $\hat{D}$ a multiply connected domain bounded by $\hat{l}=\bigcup_{k=0}^{n}\hat{l}_k$. Obviously $D\subset\hat{D}$ and $l_k''\subset\hat{D}$ with the exception

$k=1,2,\dots,q$

of the end points $b_{k_i},a_{k_{i+1}}$. For the first boundary value problem in $\hat{D}$, the Green tensor $\hat{G}(x,y)$ can be constructed under the assumption that $\hat{D}$ is homogeneous and consists of medium with elastic constants $\overset{(o)}{A}{}_{rs}^{je}$. The construction follows from the existence and uniqueness of the solution of the first boundary value problem for multiconnected domain

$$\overset{(o)}{A}\left(\frac{\partial}{\partial x}\right)\hat{G}(x,y)=0\quad x\in\hat{D},\quad\left\{\hat{G}(y_0,y)\right\}_{\hat{D}}=0\quad y_0\in\hat{l}.$$

Making use of $\hat{G}(x,y)$ we can now repeat the arguments of the paper [3] to construct the Green tensor $\overset{(o)}{G}(x,y)$ for a mixed boundary value problem in D i.e.

$$\overset{(o)}{A}\left(\frac{\partial}{\partial x}\right)\overset{(o)}{G}(x,y)=0 \quad x\in D, \left\{\overset{(o)}{G}(y_0,y)\right\}_D=0 \quad y_0\in l_k', \left\{\overset{(o)}{T}\overset{(o)}{G}(y_0,y)\right\}_D=0 \quad y_0\in l_k'',$$

$k=0,q+1,\dots,n$ and to prove its properties needed in the subsequent development.

Taking into account Betti's formulae and the properties of $\overset{(o)}{G}(x,y)$ one can prove similarly as in the case of isotropic bodies [1] the following fundamental theorem.

T h e o r e m 1. The boundary value problem (1)-(3) is equivalent to the system of functional equations of the form

$$\begin{aligned}\beta(x)\, u(x) = {} & \sum_{p=0}^{q}\iint_{D_p}\overset{(o)}{G}(x,y)\overset{(p)}{F}(y,u)\,d\tau - \sum_{p=1}^{q}\iint_{D_p}u(y)\overset{(k)}{A}\left(\frac{\partial}{\partial y}\right)\overset{(o)}{G}(x,y)\,d\tau + \\ & + \sum_{p=1}^{q}\int_{l_p}\overset{(o)}{G}(x,y)\overset{(p)}{g}(y,u)\,dl + \sum_{p=1}^{q}\int_{l_p}\left[\tilde{T}\overset{(o)}{G}(x,y)\right]^{*}u(y)\,dl + \\ & + \sum_{\substack{p=0\\ p\neq 1,2,\dots,q}}^{n}\int_{l_p'}\left[\overset{(o)}{T}\overset{(o)}{G}(x,y)\right]^{*}\overset{(o)}{f^1}(y)\,dl + \\ & - \sum_{\substack{p=0\\ p\neq 1,2,\dots,q}}^{n}\int_{l_p''}\overset{(o)}{G}(x,y)\overset{(o)}{f^2}(y)\,dy \qquad x\in D_k,\end{aligned} \tag{4}$$

where $\beta(x)=\overset{(k)}{\beta}$ for $x\in D_k$ is a constant matrix, $\tilde{T}=\overset{(k)}{T}-\overset{(o)}{T}$ and an asterisk denotes the transpose of a matrix.

It is a straightforward matter to verify the identities

$$\overset{(k)}{A}\left(\frac{\partial}{\partial x}\right)=a_k \sum_{r,s=1}^{2} \overset{(k)}{\tau}_{rs} \frac{\partial^2}{\partial \xi_r \partial \xi_s},$$

(5)

$$\overset{(k)}{T} = \overset{(k)}{a}\overset{(o)}{T} + a_k \sum_{r,s=1}^{2} \overset{(k)}{\tau}_{rs} \cos(n_x, \xi_s) \frac{\partial}{\partial \xi_r},$$

where $\overset{(k)}{a} = \frac{a_k}{a_o}$, $\overset{(k)}{\tau}_{rs} = \left\| \overset{(k)}{\tau}{}^{je}_{rs} \right\| \begin{matrix} e=1,2 \\ j=1,2 \end{matrix}$,

$$\overset{(k)}{\tau}{}^{je}_{rs} = \frac{1}{a_k} \overset{(k)}{A}{}^{je}_{rs} - \frac{1}{a_o} \overset{(o)}{A}{}^{je}_{rs}$$

and

(6) $$a_k = \overset{(k)}{A}{}^{je}_{rs} + \overset{(k)}{A}{}^{\gamma\delta}_{\alpha\beta} \quad \text{for} \quad r+s+j+e=6, \qquad r=j$$

and $\alpha+\beta+\gamma+\delta=6$, $\alpha\neq\gamma$.

In the case of isotropic medium the constants $\frac{1}{a_k} A^{je}_{rs}$ are reduced to the Poisson constants of the k-th medium.

By inserting (5) into (4) we see that $u(x)$ satisfies the following system of functional equations

(7) $$2\pi E \overset{(k)}{a} u(x) + \sum_{p=1}^{q} (1-\overset{(p)}{a}) \int_{l_p} \left[\overset{(o)}{T}\overset{(o)}{G}(x,y)\right]^* u(y)\,dl = \overset{(k)}{P}(x,u)$$

$$x \in D_k, \quad k=0,1,\dots,q,$$

where

$$\overset{(k)}{L}(x,u) = \sum_{p=0}^{q} \iint_{D_p} \overset{(o)}{G}(x,y)\, \overset{(p)}{F}(y,u)\, d\tau + \tag{8}$$

$$+ \sum_{p=1}^{q} \Big(a_p \sum_{r,s=1}^{2} \overset{(p)}{\tau}_{rs} \int_{l_p} u(y)\, \frac{\partial \overset{(o)}{G}(x,y)}{\partial \eta_r} \cos(n_y, \eta_s)\, dl\Big) +$$

$$- \sum_{p=1}^{q} a_p \Big(\sum_{r,s=1}^{2} \overset{(p)}{\tau}_{rs} \iint_{D_p} u(y)\, \frac{\partial^2 \overset{(o)}{G}(x,y)}{\partial \eta_r\, \partial \eta_s}\, d\tau\Big) +$$

$$+ \sum_{p=1}^{q} \int_{l_p} \overset{(o)}{G}(x,y)\, \overset{(p)}{g}(y,u)\, dl +$$

$$+ \sum_{\substack{p=0 \\ p \neq 1,2,\ldots,q}}^{n} \Big(\int_{l'_p} \big[\overset{(o)}{T}\, \overset{(o)}{G}(x,y)\big]^{*}\, \overset{(p)}{f^{1}}(y)\, dl\Big) +$$

$$- \sum_{\substack{p=0 \\ p \neq 1,2,\ldots,q}}^{n} \int_{l''_p} \overset{(o)}{G}(x,y)\, \overset{(p)}{f^{2}}(y)\, dl - a_k \sum_{r,s=1}^{2} \overset{(k)}{\tau}_{rs} C_{rs} u(x),$$

$$x \in D_k,$$

where C_{rs} are constant matrices dependent on $\overset{(k)}{A}{}^{je}_{rs}$, E denotes the unity matrix and the integrals appearing in the third term of $\overset{(k)}{P}(x,u)$ are understood in the Cauchy principal value sense.

3. Solution of the problem

The proof of existence and uniquenes' of the solution of the given problem is based on the Banach - Cacciopoli theorem. Let us consider a functional space Λ consisting of all continuous vectors $u(x) = \left[u_1(x), u_2(x)\right]$ defined on $\bar{D}$ and of the bounded norm

$$C_\alpha(u,\bar{D}) = C(u,\bar{D}) + H_\alpha(u,\bar{D}), \tag{9}$$

where

$$C(u,\bar{D}) = \max_{\substack{j=1,2\\k=0,1,\dots,q}} \left[\sup_{x\in\bar{D}_k} |u_j(x)|\right],$$

$$H_\alpha(u,\bar{D}) = \max_{\substack{j=1,2\\k=0,1,\dots,q}} \left[\sup_{xx'\in\bar{D}_k} \frac{|u_j(x)-u_j(x')|}{|xx'|^\alpha}\right].$$

The distance $\delta(u,u')$ between two points u, u' is defined as the norm of the difference

$$\delta(u,u') = C_\alpha(u-u',\bar{D}).$$

In the space Λ let us consider the transformation

$$2\pi E \overset{(k)}{a} \tilde{u}(x) + \sum_{p=1}^{q} (1-\overset{(p)}{a}) \int_{l_p} \left[\overset{(o)}{T}\overset{(o)}{G}(x,y)\right]^{*} \tilde{u}(y)\,dl = \overset{(k)}{P}(x,u) \quad x\in D_k \tag{10}$$

which assigns to each point $u(x)$ of the space Λ a point $\tilde{u}$ of the space $\tilde{\Lambda}$, and let

$$\tau = \max_{\substack{r,s,j,e=1,2 \\ k=1,2,\dots,q}} \left| \overset{(k)}{\tau}{}_{rs}^{je} \right| .$$

T h e o r e m 2. If the assumptions 1°– 3° are satisfied and the constants τ, K_F, K_g are sufficiently small to satisfy the condition

$$\tau + K_F + K_g < \frac{1}{m} ,$$

where m depends only on the elastic constants $\overset{(k)}{A}{}_{rs}^{je}$ and the domains of integration in (4), then there exists exactly one solution of the problem (1)-(3).

P r o o f. Limiting process for $x \to x_0 \in l_k$ leads to a system of singular integral equations with the kernel which differs from the kernel appearing in [4] only by the continuous terms. Thus, the system is uniquely solvable provided the right-hand side is of the class $C_\alpha(l)$, $l = \bigcup_{k=1}^{q} l_k$. From the properties of Cauchy integrals on Lapounov contours (comp.1°) it results that $P(x,u) \in C_\alpha(l)$ where $P(x,u) = \overset{(k)}{P}(x,u)$ for $x \in l_k$. Solving the system of integral equations we obtain the estimation of $C_\alpha(\tilde{u}(x_0), l)$.

Considering now the transformation (10) we get the following inequality

$$(11) \qquad C_\alpha(\tilde{u},\bar{D}) \leqslant m\left[C_\alpha(u,\bar{D})\tau + C(g,l) + C(F,\bar{D})\right] + m_1\, C_\alpha(S,\bar{D}),$$

where S denotes the known function on the right-hand side of (10) [comp.3°] and m, m_1 are constants depending on $\overset{(k)}{A}{}_{rs}^{je}$ and on the domains of integration.

Here we have

$$g(x_o,u) = \overset{(k)}{g}(x_o,u) \quad \text{for} \quad x_o \in l_k \quad \text{and} \quad F(x,u) = \overset{(k)}{F}(x,u)$$

$$\text{for} \quad x \in D_k.$$

From (11) it follows that the transformation (10) maps the space Λ onto itself if the constants τ are sufficiently small i.e. if $\tau < \frac{1}{m}$.

Furthermore from (11) we have

$$C_\alpha(u-u',\bar{D}) \leqslant m\left[\tau \cdot C(u-u',\bar{D}) + C(F-F',\bar{D}) + C(g-g',l)\right],$$

where $g' = g(x,u')$, $F' = F(x,u')$ and from the assumption 2^o we obtain

$$C(F-F',\bar{D}) \leqslant K_F\, C(u-u',\bar{D}), \quad C(g-g',l) \leqslant K_g\, C(u-u',l).$$

Finally we have

$$C_\alpha(\tilde{u}-\tilde{u}',\bar{D}) \leqslant m(\tau + K_f + K_g)\, C_\alpha(u-u',\bar{D})$$

and the transformation (10) decreases the distance between every pair of points u, u' if

$$\tau + K_F + K_g < \frac{1}{m}. \tag{12}$$

Under the condition (12) all the assumptions of Banach's fixed point theorem are satisfied and there exists exactly one fixed point u^* of the transformation (10). This implies the existence and uniqueness of the solution of the system of functional equation (4), which in view of Theorem 1, completes the proof.

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