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PIECEWISE MONOTONIC TRANSFORMATIONS OF THE DISCRETE STOCHASTIC PROCESSES

Let (Ω, S, P) be a probability space, $T = \langle 0, \infty)$ and let $X: \Omega \times T \to \mathcal{X} \subset R^1$ be a stochastic process. We assume that $\mathcal{X} = \{x_1, x_2, \ldots\}$.

Let $\mathbf{t}_{n-1} = (t_1, t_2, \ldots, t_{n-1})$, $\mathbf{x}_{n-1} = (x_1, x_2, \ldots, x_{n-1})$, $\mathbf{X}(t_{n-1}) = (X(t_1), X(t_2), \ldots, X(t_{n-1}))$ and let

$$P_1(\mathbf{t}_{n-1}, \mathbf{x}_{i_{n-1}}, t_n, \{x_{i_n}\}) = P(X(t_n) \in \{x_{i_n}\} \,|\, \mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}}),$$

where $t_1, t_2, \ldots, t_n \in T$, $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$, $x_{i_1}, x_{i_2}, \ldots, x_{i_n} \in \mathcal{X}$, be the transition probabilities of the process X.

We assume that the process X satisfies the conditions:

1°.

$$(1) \qquad \lim_{t_n \to t_{n-1}} P(X(t_1) = x_{i_1}, \ldots, X(t_n) = x_{i_n}) =$$

$$= \lim_{t_n \to t_{n-1}} P(\mathbf{X}(t_n) = \mathbf{x}_{i_n}) = \begin{cases} 0 & \text{for } x_{i_n} \neq x_{i_{n-1}} \\ P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}}) & \text{for } x_{i_n} = x_{i_{n-1}} \end{cases}$$

and the convergence is uniform with respect to $x_{i_{n-1}}, x_{i_n}$.

2^{o}. There exists a finite limit

$$(2)\quad \lim_{t_n \to t_{n-1}} \frac{P(X(t_n) = x_{i_n}) - \chi_{\{x_{i_n}\}}(x_{i_{n-1}})P(X(t_{n-1}) = x_{i_{n-1}})}{(t_n - t_{n-1})P(X(t_{n-1}) = x_{i_{n-1}})} =$$

$$= q(t_{n-1}, x_{i_{n-1}}, \{x_{i_n}\}),$$

and the convergence is uniform with respect to t_{n-1}, $x_{i_{n-1}}$, $\{x_{i_n}\}$.

3^{o}. We assume that q is a continuous function with respect to t_{n-1}.

From (1) it follows that

$$(3)\quad \lim_{t_n \to t_{n-1}} P_1(t_{n-1}, x_{i_{n-1}}, t_n, \{x_{i_n}\}) = \chi_{\{x_{i_n}\}}(x_{i_{n-2}})$$

and the convergence is uniform with respect to $x_{i_{n-1}}$, $\{x_{i_n}\}$, i.e. process X satisfies the continuity condition.

From (2) it follows that

$$(4)\quad \lim_{t_n \to t_{n-1}} \frac{P_1(t_{n-1}, x_{i_{n-1}}, t_n, \{x_{i_n}\}) - \chi_{\{x_{i_n}\}}(x_{i_{n-1}})}{t_n - t_{n-1}} =$$

$$= q(t_{n-1}, x_{i_{n-1}}, \{x_{i_n}\})$$

and the convergence is uniform with respect to t_{n-1}, $x_{i_{n-1}}$, $\{x_{i_n}\}$.

D e f i n i t i o n. We say that a process X is a discrete stochastic process if its phase space $\mathcal{X}$ is a denumerable set and conditions $2^0, 3^0$ are satisfied.

Let us consider a stochastic process $Z = f(X)$, where f is a piecewise monotonic function. Denote by $\mathcal{Z}$ the phase space of the process Z and the transition probabilities of the process Z by $P_2(t_{n-1}, z_{j_{n-1}}, t_n, \{z_{j_n}\})$.

We shall prove the following theorem.

T h e o r e m. If X is a discrete stochastic process, f is a piecewise monotonic function, then $Z = f(X)$ is a discrete stochastic process.

P r o o f. Denote

$$A_n = \left\{ (i_1, \ldots, i_n) : f(x_{i_1}) = z_{j_1}, \ldots, f(x_{i_n}) = z_{j_n} \right\},$$

then we have

$$(5) \qquad P(Z(t_n) = z_{j_n}) = \sum_{A_n} P(X(t_n) = x_{i_n})$$

and passing to the limit under sum sign, we get

$$(6) \qquad \lim_{t_n \to t_{n-1}} P(Z(t_n) = z_{j_n}) = \sum_{A_n} \lim_{t_n \to t_{n-1}} P(X(t_n) = x_{i_n}) =$$

$$= \begin{cases} 0 & \text{if } \bigwedge_{i_n \in A_n} x_{i_n} \neq x_{i_{n-1}} \\ \sum_{A_n} P(X(t_{n-1}) = x_{i_{n-1}}) & \text{if } \bigvee_{i_n \in A_n} x_{i_n} = x_{i_{n-1}} \end{cases} =$$

$$= \begin{cases} 0 & \text{if } z_{j_n} \neq z_{j_{n-1}} \\ P(Z(t_{n-1}) = z_{j_{n-1}}) & \text{if } z_{j_n} = z_{j_{n-1}} \end{cases}.$$

and the convergence is uniform with respect to $z_{j_{n-1}}$, $\{z_{j_n}\}$. Therefore the process Z satisfies condition 1^o.

Now we shall show that the process Z satisfies condition 2^o.

Taking into account (5) we can write

$$(7)\quad \lim_{t_n \to t_{n-1}} \frac{P(\mathbf{Z}(t_n) = \mathbf{z}_{j_n}) - \chi_{\{z_{j_n}\}}(z_{j_{n-1}})P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})}{(t_n - t_{n-1})P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} =$$

$$= \frac{1}{P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} \lim_{t_n \to t_{n-1}} \frac{1}{t_n - t_{n-1}} \times$$

$$\times \left[\sum_{A_n} P(\mathbf{X}(t_n) = \mathbf{x}_{i_n}) - \chi_{\{z_{j_n}\}}(z_{j_{n-1}}) \sum_{A_{n-1}} P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}}) \right].$$

Since

$$\chi_{\{z_{j_n}\}}(z_{j_{n-1}}) = \sum_{\{i_n : f(x_{i_n}) = z_{j_n}\}} \chi_{\{x_{i_n}\}}(x_{i_{n-1}})$$

we can write (7) in the form

$$(8)\quad \lim_{t_n \to t_{n-1}} \frac{P(\mathbf{Z}(t_n) = \mathbf{z}_{j_n}) - \chi_{\{z_{j_n}\}}(z_{j_{n-1}})P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})}{(t_n - t_{n-1})P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} =$$

$$= \frac{1}{P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} \lim_{t_n \to t_{n-1}} \frac{1}{t_n - t_{n-1}} \left[\sum_{A_n} P(\mathbf{X}(t_n) = \mathbf{x}_{i_n}) - \right.$$

$$\left. - \sum_{\{i_n : f(x_{i_n}) = z_{j_n}\}} \chi_{\{x_{i_n}\}}(x_{i_{n-1}}) \sum_{A_{n-1}} P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}} \right] =$$

$$= \frac{1}{P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} \lim_{t_n \to t_{n-1}} \frac{1}{t_n - t_{n-1}} \times$$

$$\times \sum_{A_n} \left[P(\mathbf{X}(t_n) = \mathbf{x}_{i_n}) - \chi_{\{x_{i_n}\}}(x_{i_{n-1}}) P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}}) \right] =$$

$$= \frac{1}{P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} \times$$

$$\times \sum_{A_n} \lim_{t_n \to t_{n-1}} \frac{P(\mathbf{X}(t_n) = \mathbf{x}_{i_n}) - \chi_{\{x_{i_n}\}}(x_{i_{n-1}}) P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}})}{t_n - t_{n-1}} .$$

From (2), (5) and (8) we have

$$(9) \quad \lim_{t_n \to t_{n-1}} \frac{P(\mathbf{Z}(t_n) = \mathbf{z}_{j_n}) - \chi_{\{z_{j_n}\}}(z_{j_{n-1}}) P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})}{(t_n - t_{n-1}) P(\mathbf{Z}(t_{n-1}) = \mathbf{z}_{j_{n-1}})} =$$

$$= \frac{\sum_{A_{n-1}} q(\mathbf{t}_{n-1}, \mathbf{x}_{i_{n-1}}, \{x_{i_n}\}) P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}})}{\sum_{A_{n-1}} P(\mathbf{X}(t_{n-1}) = \mathbf{x}_{i_{n-1}})} = Q(\mathbf{t}_{n-1}, \mathbf{z}_{j_{n-1}}, \{z_{j_n}\}).$$

Therefore the process Z satisfies condition 2°. The continuity of the function Q follows from condition 3° and (9). The proof of the theorem is complete.

From (6) and (9) it follows that the transition probabilities of the process Z satisfy the conditions

$$(10) \qquad \lim_{t_n \to t_{n-1}} P_2(\mathbf{t}_{n-1}, \mathbf{z}_{j_{n-1}}, t_n, \{z_{j_n}\}) = \chi_{\{z_{j_n}\}}(z_{j_{n-1}})$$

and the convergence is uniform with respect to $z_{j_{n-1}}, \{z_{j_n}\}$;

$$(11) \qquad \lim_{t_n \to t_{n-1}} \frac{P_2(\mathbf{t}_{n-1}, \mathbf{z}_{j_{n-1}}, t_n \{z_{j_n}\}) - \chi_{\{z_{j_n}\}}(z_{j_{n-1}})}{t_n - t_{n-1}} =$$

$$= Q(\mathbf{t}_{n-1}, \mathbf{z}_{j_{n-1}}, \{z_{j_n}\})$$

and the convergence is uniform with respect to t_{n-1}, $z_{j_{n-1}}$, $\{z_{j_n}\}$.

E x a m p l e. Let the function q have the form

$$q(\mathbf{t}_{n-1}, \mathbf{x}_{i_{n-1}}, \{x_{i_n}\}) = \lambda = \text{const.},$$

then from (9) it follows that

$$Q(t_{n-1}, z_{j_{n-1}}, \{z_{j_n}\}) = \lambda \operatorname{Card}\{i_n:\ f(x_{i_n}) = z_{j_n}\}.$$

It follows from the proved theorem that the class of discrete stochastic processes in closed with respect to piecewise monotinic transformations.

In applications sometimes it is interesting to know the transition probabilities

$$(12) \qquad P_3(t_{n-1}, x_{i_{n-1}}, t_n, \{y\}) = P(Y(t_n) = y \mid X(t_{n-1}) = x_{i_{n-1}})$$

and their properties, if

a) we know the transition probabilities $P_1(t_{n-1}, x_{i_{n-1}}, t_n, \{x_{i_n}\})$,

b) $Y = g(X)$,

c) the process X satisfies condition 2^o and 3^o.

Let us consider the case when function g is a piecewise monotonic function (it is no special case of the previous considerations).

We shall prove that if the process X satisfies condition 2^o, then the transmition probabilities (12) satisfy conditions

$$(13) \qquad \lim_{t_n \to t_{n-1}} P_3(t_{n-1}, x_{i_{n-1}}, t_n, \{y\}) = \chi_{\{y\}}(x_{i_{n-1}}),$$

$$(14) \qquad \lim_{t_n \to t_{n-1}} \frac{P_3(t_{n-1}, x_{i_{n-1}}, t_n, \{y\}) - \chi_{\{y\}}(x_{i_{n-1}})}{t_n - t_{n-1}} =$$

$$= \sum_{\{i_n: g(x_{i_n}) = y\}} q(t_{n-1}, x_{i_{n-1}}, \{x_{i_n}\}),$$

where

(15) $$\chi_{\{y\}}(x_{i_{n-1}}) = \sum_{\{i_n : g(x_{i_n})=y\}} \chi_{\{x_{i_n}\}}(x_{i_{n-1}})$$

and the convergence in (13) is uniform with respect to $x_{i_{n-1}}, \{y\}$, the convergence in (14) is uniform with respect to $t_{n-1}, x_{i_{n-1}}, \{y\}$.

Note that

(16) $$\lim_{t_n \to t_{n-1}} P(\mathbf{X}(t_{n-1}) = x_{i_{n-1}}, Y(t_n) = y) =$$

$$= \lim_{t_n \to t_{n-1}} \sum_{\{i_n : g(x_{i_n})=y\}} P(\mathbf{X}(t_n) = x_{i_n}) =$$

$$= \sum_{\{i_n : g(x_{i_n})=y\}} \lim_{t_n \to t_{n-1}} P(\mathbf{X}(t_n) = x_{i_n}) =$$

$$= \begin{cases} 0 & \text{if } \bigwedge_{\substack{i_n \\ g(x_{i_n})=y}} x_{i_n} \neq x_{i_{n-1}}, \\ P(\mathbf{X}(t_{n-1}) = x_{i_{n-1}}) & \text{if } \bigvee_{\substack{i_n \\ g(x_{i_n})=y}} x_{i_n} = x_{i_{n-1}}. \end{cases}$$

Therefore from (12) and (16) we have

(17) $$\lim_{t_n \to t_{n-1}} \frac{P(\mathbf{X}(t_{n-1}) = x_{i_{n-1}}, Y(t_n) = y)}{P(\mathbf{X}(t_{n-1}) = x_{i_{n-1}})} =$$

$$= \lim_{t_n \to t_{n-1}} P_3(\mathbf{t}_{n-1}, \boldsymbol{x}_{i_{n-1}}, t_n, \{y\}) = \begin{cases} 0, & \text{if} \bigwedge\limits_{\substack{i_n \\ g(x_{i_n})=y}} x_{i_n} \neq x_{i_{n-1}}, \\ 1, & \text{if} \bigvee\limits_{\substack{i_n \\ g(x_{i_n})=y}} x_{i_n} = x_{i_{n-1}}. \end{cases}$$

We can write formula (17) in the form

$$(18) \quad \lim_{t_n \to t_{n-1}} P_3(\mathbf{t}_{n-1}, \boldsymbol{x}_{i_{n-1}}, t_n, \{y\}) = \sum_{\{i_n : g(x_{i_n})=y\}} \chi_{\{x_{i_n}\}}(x_{i_{n-1}}).$$

Now we shall seek the intensity function of the process $Y = g(X)$.

Let us denote

$$(19) \quad \sum_{\{i_n : g(x_{i_n})=y\}} \chi_{\{x_{i_n}\}}(x_{i_{n-1}}) = \chi_{\{y\}}(x_{i_{n-1}}).$$

We shall find the limit

$$(20) \quad \lim_{t_n \to t_{n-1}} \frac{P(\boldsymbol{X}(t_{n-1})=\boldsymbol{x}_{i_{n-1}}, Y(t_n)=y) - \chi_{\{y\}}(x_{i_{n-1}}) P(\boldsymbol{X}(t_{n-1})=\boldsymbol{x}_{i_{n-1}})}{(t_n - t_{n-1}) P(\boldsymbol{X}(t_{n-1}) = \boldsymbol{x}_{i_{n-1}})}.$$

Taking into account (16) and (19) we obtain

$$(21) \quad \sum_{\{i_n : g(x_{i_n})=y\}} \lim_{t_n \to t_{n-1}} \frac{P(\boldsymbol{X}(t_n)=\boldsymbol{x}_{i_n}) - \chi_{\{x_{i_n}\}}(x_{i_{n-1}}) P(\boldsymbol{X}(t_{n-1})=\boldsymbol{x}_{i_{n-1}})}{(t_n - t_{n-1}) P(\boldsymbol{X}(t_{n-1}) = \boldsymbol{x}_{i_{n-1}})} =$$

$$= \sum_{\{i_n : g(x_{i_n}) = y\}} q(t_{n-1}, x_{i_{n-1}}, \{x_{i_n}\}).$$

From (21) it follows that relation (14) is thus proved.

Because q is a continuous function with respect to t_{n-1}, concequently the sum on the right-hand side of (21) is a continuous function with respect to t_{n-1}.

Therefore the process $Y = g(X)$ is a discrete stochastic process.

All the consideration in this paper concern non-markovian processes (processes which can be but need not be markovian processes). Some results for non-markovian processes were also obtained in the previous paper of the author, for example [1], [2].

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