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BOUNDEDNESS AND STABILITY OF SOLUTIONS OF SOME HYPERBOLIC EQUATIONS

Introduction

In the present paper the properties of solutions of a mixed problem for some non-linear hyperbolic equations with linear boundary conditions with n space variables have been investigated. The present paper continues and generalizes results of the paper [3] and in part [4].

1. Notations and statements of the problem

In the paper functions are denoted by minuscules and constants are denoted by capitals. Let $R = (-\infty,\infty)$, $\bar{R}_+ = \langle 0,\infty)$, Ω be the closure of a bounded domain, the boundary of which is a piecewise smooth surface Γ, $\Omega \subset R^n$.

We consider functions $u: \Omega \times \bar{R}_+ \longrightarrow R$; a_{ij}, h_1, h_2, r, p: $\Omega \longrightarrow R$, $i,j = 1,\dots,n$ and operators α, β : $C^2(\Omega \times \bar{R}_+) \longrightarrow C^0(\Omega \times \bar{R}_+)$. $\alpha[u]$, $\beta[u]$ are values of operators α, β for the function $u \in C^2(\Omega \times \bar{R}_+)$ and $\alpha[u](x,t)$, $\beta[u](x,t)$ are their values at the point (x,t).

In this paper we investigate the properties of solutions of the equation

$$(1)\qquad u_{tt} + \alpha[u]\, u_t + ru = \sum_{j,i=1}^{n} (a_{ij}u_{x_i})_{x_j} + \beta[u]$$

with the initial conditions

(IC) $$u\Big|_{t=0} = h_1, \quad u_t\Big|_{t=0} = h_2$$

and the boundary condition

(BC) $$\left[\sum_{i,j=1}^{n} a_{ij}u_{x_j}\cos\nu_i + pu\right]\Bigg|_{\Gamma} = 0,$$

where ν_i is the angle between the axis Ox_i and the unit vector normal to Γ and pointing out of Ω .

We suppose that the following compatibility condition is satisfied

$$\left[\sum_{i,j=1}^{n} a_{ij}h_{1x_j}\cos\nu_i + ph_1\right]\Bigg|_{\Gamma} = 0.$$

We also suppose that there exists a classical solution u of the problem (1), (IC), (BC), defined on $\Omega\times\bar{R}_+$. There are many papers dealing with the existence of solutions of non--linear problems, e.g. J. Lions [2], K.I. Chudawierdijew [1].

Let the functions describing problem (1), (IC), (BC) satisfy the conditions

(2) $$u \in C^2(\Omega \times \bar{R}_+),\ r,p \in C^0(\Omega),\ h_2 \in C^2(\Omega),$$

$$a_{ij}, h_1 \in C^1(\Omega), \qquad i,j = 1,\dots,n$$

and

A1. There exist constants L_1, L_2 such that for every $u \in C^2(\Omega \times \bar{R}_+)$ the inequalities

$$0 < L_1 \leq \alpha[u]\ (x,t) \leq L_2$$

are satisfied.

A2. There exists a positive constant B such that for every $u \in C^2(\Omega \times \bar{R}_+)$ we have

$$|\beta[u](x,t)| \leq B .$$

A3. There exist constants R_1, R_2 such that for every $x \in \Omega$ we have

$$0 < R_1 \leq r(x) \leq R_2 .$$

A4. There exist constants P_1, P_2 such that for every $x \in \Omega$ we have

$$0 < P_1 \leq p(x) \leq P_2 .$$

A5. The form a_{ij} is symmetrical, i.e. $a_{ij} = a_{ji}$ for $i,j = 1,\dots,n$ and there exists a positive constant A such that for every $x \in \Omega$ and every $(\xi_1,\dots,\xi_n) \in R^n$ the next inequality is satisfied

$$\sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\xi_j \geq A \sum_{i=1}^{n} \xi_i^2 .$$

In order to define boundedness, exponential convergence to zero, stability and asymptotic stability of the solutions of the problem (1), (IC), (BC) we introduce the space $H_1(\Omega \times \bar{R}_+)$ consisting of functions $u \in C^1(\Omega \times \bar{R}_+)$, with the norm (if the assumption A 5 is satisfied) defined by the formula

$$\|u(\cdot,t)\| = \left\{ \int_\Omega \left[u^2(x,t) + u_t^2(x,t) + \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i}(x,t)\, u_{x_j}(x,t) \right] dx + \int_\Gamma u^2(s,t)\, ds \right\}^{\frac{1}{2}} ,$$

where t is considered as a parameter.

D e f i n i t i o n s . A solution u of the problem (1), (IC), (BC) defined on $\Omega \times \bar{R}_+$ is said to be

a) H_1-bounded if there exists a positive constant M such that for every $t \in \bar{R}_+$ we have

$$\|u(.,t)\| \leqslant M .$$

b) H_1-exponentially convergent to zero if there exists a positive constant K such that

$$\lim_{t\to\infty} \|u(.,t)\| \, e^{Kt} = 0 .$$

c) H_1-stable if for every $\eta > 0$ there exists $\Delta > 0$ such that for every classical solution v (defined on $\Omega \times \bar{R}_+$) of the considered equation, satisfying given boundary condition the next inequality

$$\|u(.,0) - v(.,0)\| < \Delta \text{ implies } \|u(.t) - v(.,t)\| < \eta \quad \text{for} \quad t \in \bar{R}_+ .$$

d) H_1-asymptotically stable if it is stable and in notations of the last definition we have

$$\lim_{t\to\infty} \|u(.,t) - v(.,t)\| = 0 .$$

In the sequel we use the following Green formula

$$(3) \quad \int_\Omega v \sum_{i,j=1}^{n} (a_{ij}u_{x_j})_{x_i} \, dx = - \int_\Omega \sum_{i,j=1}^{n} a_{ij}u_{x_j}v_{x_i} \, dx +$$

$$+ \int_\Gamma v \sum_{i,j=1}^{n} a_{ij}u_{x_j} \cos \nu_i \, ds .$$

We define a function of Lapunov type for the solution u of the problem (1), (IC), (BC) in the following form

(4) $$k(t)=\int_{\Omega}\Big[\frac{1}{2}u_t^2(x,t)+\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(x)\,u_{x_i}(x,t)\,u_{x_j}(x,t)+$$
$$+\frac{1}{2}r(x)\,u^2(x,t)+\varepsilon u(x,t)u_t(x,t)\Big]dx+\int_{\Gamma}p(s)u^2(s,t)\,ds,$$

where $t\in\bar{R}_+$ and the constant ε satisfies the condition

(5) $$\varepsilon=\frac{1}{2}\min\left(\sqrt{R_1},\ \frac{4L_1R_1}{L_2^2+4R_1},2\right).$$

We introduce an auxiliary function

(6) $$l(t)=\int_{\Omega}\Big[u_t^2(x,t)+\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}(x,t)u_{x_j}(x,t)+$$
$$+u^2(x,t)\Big]dx+\int_{\Gamma}u^2(x,t)\,ds.$$

By the assumption A5 every term in (6) is non-negative. This means that we can put $\|u(\cdot,t)\|=\sqrt{l(t)}$ for $t\in\bar{R}_+$.

2. The auxiliary theorems

L e m m a 1. If the assumptions A3, A4 and the condition (5) are satisfied then there exist positive constants M_1, M_2 such that for $t\in\bar{R}_+$ we have

(7) $$M_1l(t)\leqslant k(t)\leqslant M_2l(t).$$

P r o o f. In virtue of the assumptions A3, A4 we have

$$k(t)\leqslant\int_{\Omega}\Big[\frac{1}{2}u_t^2+\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}+\frac{1}{2}R_2u^2+\frac{1}{2}\varepsilon(u^2+u_t^2)\Big]dx+$$

$$+ P_2 \int_{\Gamma} u^2 \, ds \leqslant M_2 l(t) \quad \text{for} \quad t \in \bar{R}_+ ,$$

where $M_2 = \frac{1}{2} \max (2, R_2 + 1, 2P_2)$.

Using the same assumptions A3, A4 we obtain

$$k(t) \geqslant \frac{1}{2} \int_{\Omega} \left[u_t^2 - 2\varepsilon |u| \, |u_t| + R_1 u^2 + \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \right] dx +$$

$$+ P_1 \int_{\Gamma} u^2 ds \quad \text{for} \quad t \in \bar{R}_+ .$$

By the condition (5) the quadratic form $\xi^2 - 2\varepsilon\xi\eta + R_1 \eta^2$ is positive definite, therefore there exists a positive constant N, such that $\xi^2 - 2\varepsilon\xi\eta + R_1 \eta^2 \geqslant N(\xi^2 + \eta^2)$.

Consequently we have for $t \in \bar{R}_+$

$$k(t) \geqslant M_1 l(t), \text{ where } M_1 = \frac{1}{2} \min (N, 1, P_1).$$

Thus Lemma 1 is proved.

L e m m a 2. If the assumption A2 holds then there exists a positive constant M_3 such that for every $t \in \bar{R}_+$ we have

$$k_1(t) = \int_{\Omega} (u_t \, \beta[u] + \varepsilon u \, \beta[u]) dx \leqslant M_3 \sqrt{k(t)} .$$

P r o o f. Making use of Schwarz's inequality, the assumption A2 and the formulae (6), (7) we obtain for $t \in \bar{R}_+$

$$k_1(t) \leqslant B \int_{\Omega} |u_t| \, dx + \varepsilon B \int_{\Omega} |u| \, dx \leqslant$$

$$\leqslant B\Big(\int_{\Omega} dx\Big)^{1/2} \Big(\int_{\Omega} u_t^2 \, dx\Big)^{1/2} + \varepsilon B\Big(\int_{\Omega} dx\Big)^{1/2}\Big(\int_{\Omega} u^2 \, dx\Big)^{1/2} \leqslant$$

$$\leqslant M_3 \sqrt{k(t)} ,$$

where

$$M_3 = \frac{|\Omega|^{1/2}\, 2B}{M_1} \quad \text{and} \quad |\Omega| = \int_\Omega dx .$$

This ends the proof of Lemma 2.

L e m m a 3. If the assumptions A1, A3, A4 and the condition (5) are satisfied then there exists a positive constant M_4 such that for every $t \in \bar{R}_+$ we have

$$k_2(t) = \int_\Omega \Big[u_t^2(\alpha[u] - \varepsilon) + \varepsilon u\, u_t\, \alpha[u] + \varepsilon r\, u^2 + + \varepsilon \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \Big] dx + \varepsilon \int_\Gamma p(s)\, u^2\, ds \geqslant 2M_4\, k(t).$$

P r o o f. In virtue of assumptions A1, A3, A4 we have for $t \in \bar{R}_+$

$$k_2(t) \geqslant \int_\Omega \Big[u_t^2(L_1 - \varepsilon) - \varepsilon L_2\, |u|\, |u_t| + \varepsilon R_1 u^2 + + \varepsilon \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \Big] dx + \varepsilon P_1 \int_\Gamma u^2\, ds.$$

By the condition (5) the quadratic form $\xi^2(L_1 - \varepsilon) - \varepsilon L_2 \xi \eta + + \varepsilon R_1 \eta^2$ is positive definite and so there exists a positive constant N_1 such that

$$\xi^2(L_1 - \varepsilon) - \varepsilon L_2 \xi \eta + \varepsilon R_1\, \eta^2 \geqslant N_1(\xi^2 + \eta^2).$$

Therefore

$$k_2(t) \geqslant 2M_4\, k(t) \quad \text{for} \quad t \in \bar{R}_+ ,$$

where

$$M_4 = \frac{\min\, (N_1, \varepsilon\, , \varepsilon P_1)}{2M_2} .$$

Hence Lemma 3 is proved.

L e m m a 4. If $k \in C^1(\bar{R}_+)$, $k(t) \geqslant 0$ for $t \in \bar{R}_+$, M_3, M_4 are positive constants, then the inequality

$$\dot{k}(t) \leqslant -2M_4\, k(t) + M_3 \sqrt{k(t)} \tag{8}$$

implies the following estimation

$$k(t) \leqslant \left(\sqrt{k(0)} + \frac{M_3}{2M_4}\right)^2 \quad \text{for} \quad t \in \bar{R}_+ . \tag{9}$$

P r o o f. In virtue of the assumption (8) we obtain for $t \in \bar{R}_+$

$$\frac{d}{dt}\left(e^{2M_4 t}\, k(t)\right) = e^{2M_4 t}\left(\dot{k}(t) + 2M_4\, k(t)\right) \leqslant M_3\, e^{2M_4 t} \sqrt{k(t)} .$$

Taking $K_3(t) = \exp(2M_4 t)\, k(t)$ we get

$$\dot{k}_3(t) \leqslant M_3\, e^{M_4 t} \sqrt{k_3(t)} \quad \text{for} \quad t \in \bar{R}_+ .$$

For any positive integer n we have

$$\dot{k}_3(t) \leqslant M_3\, e^{M_4 t} \sqrt{k_3(t) + \frac{1}{n}} \quad \text{for} \quad t \in \bar{R}_+ .$$

Therefore

$$\frac{\dot{k}_3(t)}{\sqrt{k_3(t) + \frac{1}{n}}} \leqslant M_3\, e^{M_4 t} \quad \text{for} \quad t \in \bar{R}_+ .$$

Integrating both parts of this inequality from 0 to t we obtain

$$\sqrt{k_3(t) + \frac{1}{n}} \leqslant \sqrt{k_3(0) + \frac{1}{n}} + \frac{M_3}{2M_4}\left(e^{M_4 t} - 1\right) .$$

It follows that if $n \to \infty$ then

$$\sqrt{k_3(t)} \leqslant \sqrt{k_3(0)} + \frac{M_3}{2M_4^2}\left(e^{M_4 t} - 1\right)$$

and finally

$$\sqrt{k(t)} \leqslant \sqrt{k(0)}\; e^{-M_4 t} + \frac{M_3}{2M_4^2} - \frac{M_3}{2M_4^2}\, e^{-M_4 t} .$$

The constants M_3, M_4 being positive and $t \in \overline{R}_+$ we obtain the estimation (9).

3. Theorems

T h e o r e m 1. If the assumptions A1-A5 are satisfied then every solution u of the problem (1), (IC), (BC) is H_1-bounded.

P r o o f . Let u be an arbitrary solution of (1) with conditions (IC) and (BC). Let the function of Lapunov type for this solution be of the form (4).

In virtue of (2), the integrands in (4) are of class C^1 with respect to t and of class C^0 with respect to x. Thus we may interchange integration and differentiation with respect to t. After differentiation of k with respect to t we obtain for $t \in \overline{R}_+$

$$\dot{k}(t) = \int_\Omega \left[u_t\, u_{tt} + \sum_{i,j=1}^{n} a_{ij} u_{x_i t} u_{x_j} + r u u_t + \varepsilon u_t^2 + \varepsilon u u_{tt} \right] dx + \\ + \int_\Gamma p(s)\; u\; u_t\; ds .$$

Computing u_{tt} from (1) and introducing it into the previous formula we see that the last equality becomes

$$\dot{k}(t) = \int_\Omega \Big[u_t \sum_{i,j=1}^{n} (a_{ij}u_{x_i})_{x_j} + u_t\,\beta[u] - \alpha[u]\,u_t^2 +$$

$$+ \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j} + \varepsilon u_t^2 + \varepsilon u \sum_{i,j=1}^{n} (a_{ij}u_{x_i})_{x_j} +$$

$$+ \varepsilon u\,\beta[u] - \varepsilon u\,\alpha[u]u_t - \varepsilon r(x)u^2\Big]dx + \int_\Gamma p(s)u\,u_t\,ds\,.$$

Making use of Green's formula (3) and the boundary conditions (BC) after some routine transformations we get for $t \in \bar{R}_+$

$$-\dot{k}(t) = \int_\Omega \Big[u_t^2(\alpha[u] - \varepsilon) + \varepsilon u\,u_t\,\alpha[u] + \varepsilon r u^2 +$$

$$+ \varepsilon \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}\Big]dx + \varepsilon\int_\Gamma p(s)u^2\,ds - \int_\Omega (u_t\,\beta[u] + \varepsilon u\,\beta[u])dx.$$

From Lemmas 2 and 3 we obtain the following estimation

(10) $$\dot{k}(t) \leqslant -2M_4\,k(t) + M_3\sqrt{k(t)} \quad \text{for} \quad t \in \bar{R}_+\,.$$

By the definition of the auxiliary function l given by (5), the assumption A5 and Lemma 1 we have for $t \in \bar{R}_+$ $k(t) \geqslant 0$.

Applying Lemma 4 to k we get for $t \in \bar{R}_+$

$$k(t) \leqslant \left(\sqrt{k(0)} + \frac{M_3}{2M_4}\right)^2.$$

From the above estimation and in virtue of the inequality (7) it follows that

$$\|u(\cdot,t)\| \leqslant M \quad \text{for} \quad t \in \bar{R}_+\,,$$

where $M = \frac{1}{\sqrt{M_1}}\left(\sqrt{k(0)} + \frac{M_3}{2M_4}\right)$. This completes the proof of Theorem 1.

R e m a r k. 1. The H_1-boundedness of u implies the boundedness of u, u_t, $\text{grad}_x u$ in the sense of $L^2(\Omega)$ for $t \in \bar{R}_+$.

T h e o r e m 2. If the assumptions A1, A3, A4, A5 hold then every solution of the homogeneous equation (1) ($\beta[u] \equiv 0$) with conditions (IC), (BC) is H_1-bounded and H_1--exponentially convergent to zero.

P r o o f. The H_1-boundedness of u follows from Theorem 1, as all its hypotheses are satisfied.

H_1-exponential convergence to zero follows from the inequality

$$\dot{k}(t) \leqslant -2M_4\, k(t) \quad \text{for} \quad t \in \bar{R}_+ . \tag{11}$$

One can obtain this inequality directly from (10) putting $M_3 = 0$ because $\beta[u] \equiv 0$.

From the inequality (11) we get the following estimation for $t \in \bar{R}_+$

$$k(t) \leqslant k(0)\; e^{-2M_4 t} . \tag{12}$$

The last inequality implies exponential H_1-convergence to zero of the solution u for $t \to \infty$.

Thus Theorem 2 is proved.

R e m a r k 2. From the estimation (12) and by Lemma 1 we infer that u, u_t, $\text{grad}_x u$ converge to zero in the sense of $L^2(\Omega)$ when $t \to \infty$.

T h e o r e m 3. If the assumptions A1, A3, A4, A5 are satisfied then the zero solution of the homogeneous equation (1) ($\beta[u] \equiv 0$) with conditions (BC) and (IC: $h_1 \equiv h_2 \equiv 0$) is H_1-stable and H_1-asymptotically stable.

P r o o f. According to definition of H_1-stability we must prove that for every $\eta > 0$ there exists $\Delta > 0$ such that if $\|u(\cdot,0)\| < \Delta$ then the solution u of problem (1), (BC), (IC: $h_1 \equiv h_2 \equiv 0$) satisfies the inequality $\|u(\cdot,t)\| < \eta$ for $t \in \bar{R}_+$.

Putting $\Delta = \sqrt{(M_1/M_2)}\,\eta$ in virtue inequalities (7) and (12) we obtain

$$\| u(.,t) \| < \eta e^{-M_4 t} \quad \text{for} \quad t \in \bar{R}_+ .$$

From the above estimation it follows that the zero solution is stable and H_1-asymptotically stable.

Hence the proof of Theorem 3 is completed.

R e m a r k 3. By the following inequality

$$\int_\Gamma v^2(s)\,ds \leqslant \int_\Omega \left[\sum_{i=1}^{n} v_{x_i}^2 + c_\delta v^2 \right] dx ,$$

(where constants δ, c_δ are positive and c_δ depends on Ω and δ only) and by the assumption A 5 we obtain the boundedness, convergence to zero, stability and asymptotic stability of solutions of the problem (1), (IC), (BC) in the energetic norm i.e.

$$\|u(.,t)\|_1 = \left\{ \int_\Omega \left[u^2(x,t) + u_t^2(x,t) + (\mathrm{grad}_x u(x,t))^2 \right] dx \right\}^{\frac{1}{2}} .$$

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