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Kazimierz Cegiełka

SOME NOTE ON CHANGES OF RIEMANNIAN METRIC IN GEOMETRY OF SUBMANIFOLDS

Let $(\overline{M},\overline{G})$ be a $\overline{m}$-dimensional Riemannian manifold. We denote by $\overline{\nabla}$ the Levi-Civita connection on $(\overline{M},\overline{G})$. Let $M \subset \overline{M}$ be a m-dimensional submanifold of $\overline{M}$. If $G = \iota^* \overline{G}$, then (M,G) is the Riemannian submanifold of $(\overline{M},\overline{G})$, where $\iota: M \subset \overline{M}$ is the inclusion map. For $p \in M$ the tangent space $T_p\overline{M}$ is the direct sum of $\iota_{*p}(T_pM)$ and $(\iota_{*p}(T_pM))^N$ - the orthogonal component of $\iota_{*p}(T_pM)$ in $T_p\overline{M}$ (with respect to $\overline{G}$), i.e.

$$T_p\overline{M} = \iota_{*p}(T_pM) \oplus (\iota_{*p}(T_pM))^N \quad \text{for} \quad p \in M.$$

If $v \in T_p\overline{M}$ and $v = v^T + v^N$, where $v^T \in \iota_{*p}(T_pM)$, $v^N \in (\iota_{*p}(T_pM))^N$, then v^T and v^N is called the tangential component and the normal component of v, respectively.

Let $W(M,\overline{M})$ denote a $C^\infty(M)$-module of all smooth vector fields along $\iota: M \subset \overline{M}$. We set

$$W(M,\overline{M})^N = \left\{ \xi \in W(M,\overline{M}) : \xi(p) \in (\iota_{*p}(T_pM))^N \quad \text{for} \quad p \in M \right\}.$$

It is obvious that $(W(M,\overline{M}),\overline{G}\circ\iota)$ is a Riemannian module [2] and $W(M,\overline{M})^N$ is a $C^\infty(M)$-submodule of $W(M,\overline{M})$. We observe that $W(M,\overline{M})^N$ with the restriction of $\overline{G}\circ\iota$ to $W(M,\overline{M})^N$

is also the Riemannian module. Those modules have a local vector basis. Namely, the following theorem is true [1].

T h e o r e m 1. Under the above suppositions, for any point $p \in M$ there exist a neighbourhood U of p in M, an orthonormal basis $X_1,\dots,X_m$ of $W(M) = W(M,M)$ on U (with respect to G) and an orthonormal basis $X_{m+1},\dots,X_{\overline{m}}$ of $W(M,\overline{M})^N$ on U (with respect to $\overline{G}\circ\imath$) such that $\imath_*\circ X_1,\dots,\imath_*\circ X_m,X_{m+1},\dots,X_{\overline{m}}$ is an orthonormal basis of $W(M,\overline{M})$ on U (with respect to $\overline{G}\circ\imath$).

Let $\overline{\nabla}^{\imath}$ denote the covariant derivative in $W(M,\overline{M})$ induced by $\overline{\nabla}$ and $\imath$. For any $X \in W(M)$, $\xi \in W(M,\overline{M})^N$ we denote by $-\overline{A}_\xi(X)$ and $\overline{\nabla}^N_X\xi$ the tangential component and the normal component of $\overline{\nabla}^{\imath}_X\xi$, i.e.

$$\overline{\nabla}^{\imath}_X\xi = -\overline{A}_\xi(X) + \overline{\nabla}^N_X\xi \quad \text{for} \quad X \in W(M),\ \xi \in W(M,\overline{M})^N,$$

where

$$\overline{A}_\xi(X)(p) \in \imath_{*p}(T_pM) \quad \text{for} \quad p \in M \quad \text{and} \quad \overline{\nabla}^N_X\xi \in W(M,\overline{M})^N.$$

It is know [1] that $\overline{A} : W(M)\times W(M,\overline{M})^N \longrightarrow W(M,\overline{M})$ is a tensor field and $\overline{\nabla}^N : W(M)\times W(M,\overline{M})^N \longrightarrow W(M,\overline{M})^N$ is a covariant derivative in $W(M,\overline{M})^N$. Moreover, if $v_1,\dots,v_{\overline{m}-m}$ is an orthonormal basis of $(\imath_{*p}(T_pM))^N$, then the vector $\overline{H}(p) = \frac{1}{m}\sum_{k=1}^{\overline{m}-m} \operatorname{tr}\overline{A}_{v_k}(\cdot)(p)v_k$ does not depend on the choice of the basis $v_1,\dots,v_{\overline{m}-m}$. The vector field $\overline{H}$ is called the mean curvature vector field of the submanifold M in $(\overline{M},\overline{G})$ [1]. It follows from Theorem 1 that $\overline{H} \in W(M,\overline{M})^N$. The submanifold M is called minimal in $(\overline{M},\overline{G})$ when $\overline{H} = 0$.

Let ∇ be another covariant derivative on $\overline{M}$. For any $X \in W(M)$, $\xi \in W(M,\overline{M})^N$ we denote by $-A_\xi(X)$ and $\nabla^N_X\xi$ the tangential component and the normal component of $\nabla^{\imath}_X\xi$, i.e.

$$\nabla^{\imath}_X\xi = -A_\xi(X) + \nabla^N_X\xi \quad \text{for} \quad X \in W(M) \quad \text{and} \quad \xi \in W(M,\overline{M})^N,$$

where

$$A_\xi(X)(p) \in \imath_{*p}(T_pM) \quad \text{for} \quad p \in M \quad \text{and} \quad \nabla^N_X\xi \in W(M,\bar{M})^N.$$

Under the above suppositions we shall prove the following theorem.

T h e o r e m 2. $A : W(M) \times W(M,\bar{M})^N \longrightarrow W(M,\bar{M})$ is a tensor field and $\nabla^N : W(M) \times W(M,\bar{M})^N \longrightarrow W(M,\bar{M})^N$ is a covariant derivative in $W(M,\bar{M})^N$.

P r o o f . It is obvious that A and ∇^N are an R-2-linear mapping. Let $\alpha, \beta \in C^\infty(M)$, $X,Y \in W(M)$ and $\xi \in W(M,\bar{M})^N$. Then

$$\nabla^\imath_{\alpha X}\beta\xi = -A_{\beta\xi}(\alpha X) + \nabla^N_{\alpha X}\beta\xi .$$

Comparing this with

$$\nabla^\imath_{\alpha X}\beta\xi = \alpha X(\beta)\xi + \alpha\beta\nabla^\imath_X\xi = \alpha X(\beta)\xi + \alpha\beta(-A_\xi(X) + \nabla^\imath_X\xi) =$$

$$= -\alpha\beta A_\xi(X) + (\alpha X(\beta)\xi + \alpha\beta\nabla^N_X\xi)$$

we obtain $A_{\beta\xi}(\alpha X) = \alpha\beta A_\xi(X)$ for the tangential component and $\nabla^N_{\alpha X}\beta\xi = \alpha X(\beta)\xi + \alpha\beta\nabla^N_X\xi$ for the normal component.

T h e o r e m 3. If ∇ is Riemannian with respect to $\bar{G}$, then ∇^N is also Riemannian with respect to $\bar{G}\circ\imath$.

P r o o f . It is obvious that $\nabla^\imath$ is Riemannian with respect to $\bar{G}\circ\imath$. Hence we have

$$X(\bar{G}\circ\imath)(\xi,\eta) = (\bar{G}\circ\imath)(\nabla^\imath_X\xi,\eta) + (\bar{G}\circ\imath)(\xi,\nabla^\imath_X\eta) =$$

$$= (\bar{G}\circ\imath)(-A_\xi(X) + \nabla^N_X\xi,\eta) + (\bar{G}\circ\imath)(\xi,-A_\eta(X) + \nabla^N_X\eta) =$$

$$= (\bar{G}\circ\imath)(\nabla^N_X\xi,\eta) + (\bar{G}\circ\imath)(\xi,\nabla^N_X\eta) \quad \text{for } X \in W(M),\ \xi,\eta \in W(M,\bar{M})^N.$$

T h e o r e m 4. Let ∇ be a covariant derivative on $\overline{M}$ Riemannian with respect to $\overline{G}$ and with the torsion tensor T. Then

$$(\overline{G}\circ\iota)(T(\iota_*\circ X,\iota_*\circ Y),\xi) + (\overline{G}\circ\iota)(\iota_*\circ X,A_\xi(Y)) = (\overline{G}\circ\iota)(\iota_*\circ Y,A_\xi(X))$$

for $X,Y \in W(M)$, $\xi \in W(M,\overline{M})^N$. In particular, if the vector field $T(\iota_*\circ X,\iota_*\circ Y)$ is tangent to M for every $X,Y \in W(M)$, i.e.

$$T(\iota_*\circ X,\iota_*\circ Y)(p) \in \iota_{*p}(T_pM) \quad \text{for } X,Y \in W(M) \text{ and } p \in M,$$

then A is self-adjoint with respect to $\overline{G}\circ\iota$.

P r o o f . Let $p \in M$ be any point of M. Then for every $X,Y \in W(M)$ there exist a neighbourhood U of p in M, a neighbourhood V of p in $\overline{M}$ and tangent vector fields $\overline{X}$, $\overline{Y}$ on V such that $U \subset V$ and

$$(1) \qquad \iota_*\circ X|U = \overline{X}\circ\iota|U, \quad \iota_*\circ Y|U = \overline{Y}\circ\iota|U.$$

Hence we get

$$\iota_*\circ[X,Y]|U = [\overline{X},\overline{Y}]\circ\iota|U.$$

Differentiating covariantly the equality

$$(\overline{G}\circ\iota)(\iota_*\circ Y,\xi) = 0 \quad \text{for } Y \in W(M),\ \xi \in W(M,\overline{M})^N$$

with respect to ∇^ι in the direction $X \in W(M)$ we have

$$(\overline{G}\circ\iota)(\nabla^\iota_X\iota_*\circ Y,\xi) + (\overline{G}\circ\iota)(\iota_*\circ Y,\nabla^\iota_X\xi) = 0$$

by virtue of Theorem 3. Next, it follows from (1) that

$$(\overline{G}\circ\iota)(\nabla^\iota_X(\overline{Y}\circ\iota),\xi)|U + (\overline{G}\circ\iota)(\iota_*\circ Y,-A_\xi(X) + \nabla^N_X\xi)|U = 0.$$

Since $(\bar{G} \circ \imath)(\imath_* \circ Y, \nabla^N_X \xi) = 0$, we get

$$(\bar{G} \circ \imath)(\nabla_{\imath_* \circ X}\bar{Y}, \xi)\big|U = (\bar{G} \circ \imath)(\imath_* \circ Y, A_\xi(X))\big|U.$$

We observe that

$$(\nabla_{\imath_* \circ X}\bar{Y} - \nabla_{\imath_* \circ Y}\bar{X})\big|U = (\nabla_{\bar{X}}\bar{Y} - \nabla_{\bar{Y}}\bar{X}) \circ \imath\big|U = T(\bar{X},\bar{Y}) \circ \imath\big|U +$$

$$+ [\bar{X},\bar{Y}] \circ \imath\big|U = T(\imath_* \circ X, \imath_* \circ Y)\big|U +$$

$$+ \imath_* \circ [X,Y]\big|U.$$

Thus

$$(\bar{G} \circ \imath)(\imath_* \circ [X,Y], \xi) + (\bar{G} \circ \imath)(T(\imath_* \circ X, \imath_* \circ Y), \xi) =$$

$$= (\bar{G} \circ \imath)(\imath_* \circ Y, A_\xi(X)) - (\bar{G} \circ \imath)(A_\xi(Y), \imath_* \circ X).$$

Hence it follows that

$$(\bar{G} \circ \imath)(T(\imath_* \circ X, \imath_* \circ Y), \xi) + (\bar{G} \circ \imath)(\imath_* \circ X, A_\xi(Y)) = (\bar{G} \circ \imath)(\imath_* \circ Y, A_\xi(X)).$$

We now assume that ∇ is any covariant derivative on $\bar{M}$. Let $u_1, \dots, u_{\bar{m}-m}$ be an orthonormal basis of $(\imath_{*p}(T_pM))^N$ $(p \in M)$. Let $A^k = A_{\xi_k}(\cdot)(p) : T_pM \longrightarrow T_pM$, where $\xi_k \in W(M,\bar{M})^N$ is a vector field such that $\xi_k(p) = u_k$ for $k = 1, \dots, \bar{m}-m$. By a simple calculation we verify that the vector $H(p) = \frac{1}{m} \operatorname{tr} A^k u_k$ is independent of the choice of the orthonormal basis $u_1, \dots, u_{\bar{m}-m}$. It follows from Theorem 1 that H is smooth, i.e. $H \in W(M,\bar{M})^N$. We say that the vector field H is a mean curvature of the submanifold M in $(\bar{M},\bar{G})$ with respect to the covariant derivative ∇.

Let E be the deformation tensor of ∇ and $\overline{\nabla}$, i.e.

$$E_X Y = \nabla_X Y - \overline{\nabla}_X Y \quad \text{for} \quad X,Y \in W(\overline{M}).$$

We set

$$E_X Y = E^T_X Y + E^N_X Y \quad \text{for} \quad X,Y \in W(\overline{M}),$$

where

$$E^N_X Y \in W(M,\overline{M})^N \quad \text{and} \quad (E^T_X Y)(p) \in \iota_{*p}(T_pM) \quad \text{for} \quad p \in M.$$

Then, for any $X \in W(M)$, $\xi \in W(M,\overline{M})^N$ and $p \in M$ there exist a neighbourhood U od p in M, a neighbourhood V of p in $\overline{M}$ and $\overline{X},\overline{Y} \in W(\overline{M})$ such that $U \subset V$, $\iota_* \circ X|U = \overline{X} \circ \iota|U$ and $\xi|U = \overline{Y} \circ \iota|U$. Thus, on the set U, we have

$$\nabla^\iota_X \xi = \nabla^\iota_X(\overline{Y} \circ \iota) = \nabla_{\iota_* \circ X}\overline{Y} = \nabla_{\overline{X} \circ \iota}\overline{Y} = (\nabla_{\overline{X}}\overline{Y}) \circ \iota = (\overline{\nabla}_{\overline{X}}\overline{Y} + E_{\overline{X}}\overline{Y}) \circ \iota =$$

$$= \overline{\nabla}_{\overline{X}}\overline{Y} \circ \iota + E_{\overline{X}}\overline{Y} \circ \iota = \overline{\nabla}_{\overline{X} \circ \iota}\overline{Y} + E_{\overline{X}}\overline{Y} \circ \iota = \overline{\nabla}_{\iota_* \circ X}\overline{Y} + E_{\overline{X}}\overline{Y} \circ \iota =$$

$$= \overline{\nabla}^\iota_X(\overline{Y} \circ \iota) + E_{\overline{X}}\overline{Y} \circ \iota = \overline{\nabla}^\iota_X \xi + E_{\overline{X}}\overline{Y} \circ \iota =$$

$$= (-\overline{A}_\xi(X) + E^T_{\overline{X}}\overline{Y} \circ \iota) + (\overline{\nabla}^N_X \xi + E^N_{\overline{X}}\overline{Y} \circ \iota) =$$

$$= (-\overline{A}_\xi(X) + E^T_{\iota_* \circ X}\xi) + (\overline{\nabla}^N_X \xi + E^N_{\iota_* \circ X}\xi).$$

Since p is an arbitrary point of M, we get

$$A_\xi(X) = \overline{A}_\xi(X) + E^T_{\iota_* \circ X} \quad \text{for} \quad X \in W(M) \text{ and } \xi \in W(M,\overline{M})^N.$$

We set

$$B_\xi(X) = E^T_{\iota_* \circ X}\xi \quad \text{for} \quad X \in W(M) \quad \text{and} \quad \xi \in W(M,\overline{M})^N.$$

Let $u_1,\dots,u_{\bar{m}-m}$ be an orthonormal basis of $(\iota_{*p}(T_pM))^N$ $(p \in M)$. It is easy that

$$\bar{B}(p) = \frac{1}{m}\sum_{k=1}^{\bar{m}-m} \operatorname{tr} B_{\xi_k}(\cdot)(p)u_k \quad \text{for} \quad p \in M,$$

is independent of the choice of basis $u_1,\dots,u_{\bar{m}-m}$, where $\xi_k \in W(M,\bar{M})^N$ is such that $\xi_k(p) = u_k$ for $k = 1,\dots,\bar{m}-m$. Hence we obtain the following theorem.

T h e o r e m 5. Under the above hypothesis and notation we have

$$H = \bar{H} + \bar{B}.$$

As an application, we consider a conformal change of Riemannian metric on $\bar{M}$. Let $\tilde{G}$ be the Riemannian metric on $\bar{M}$ given by

$$\tilde{G} = \frac{1}{\alpha}\bar{G} \tag{2}$$

for some positive function $\alpha \in C^\infty(\bar{M})$. Let ∇ be the Levi-Civita connection on $(\bar{M},\tilde{G})$. Then it is known (see for instance [3]) that

$$E_XY = \frac{1}{2\alpha}\left(\bar{G}(X,Y)V - X(\alpha)Y - Y(\alpha)X\right) \quad \text{for} \quad X,Y \in W(\bar{M}), \tag{3}$$

where $V \in W(\bar{M})$ is such that

$$Z(\alpha) = \bar{G}(V,Z) \quad \text{for} \quad Z \in W(\bar{M}).$$

We say that the conformal change (2) is differentially constant on M when

$$\xi(p)(\alpha) = 0 \quad \text{for} \quad \xi \in W(M,\bar{M})^N \quad \text{and} \quad p \in M.$$

In this case it is obvious from (3) that $\bar{B} = 0$. As an immediate consequence of Theorem 5 we obtain the following theorem.

T h e o r e m 6. Let M be a submanifold of $\bar{M}$. Let (2) be the conformal change of Riemannian metric which is differentially constant on M. Then M is minimal in $(\bar{M},\bar{G})$ if and only if M is minimal in $(\bar{M},\tilde{G})$.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW
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