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ON SOME CLASSES OF β-STARLIKE AND OF QUASI-β-STARLIKE MEROMORPHIC k-SYMMETRIC FUNCTIONS

1. Let $\mathcal{P}(\beta)$ be a family of functions P of the form

$$P(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{1.1}$$

that are holomorphic in the unit disc $K = \{z : |z| < 1\}$ and satisfy

$$\left|\frac{P(z) - 1}{P(z) + 1}\right| < \beta, \tag{1.2}$$

where β is a fixed number in the interval $(0,1]$.

Observe that $\mathcal{P}(\beta) \subset \mathcal{P}$ and $\mathcal{P}(1) \equiv \mathcal{P}$, where $\mathcal{P}$ is the known family of functions of Caratheodory type.

By $\mathcal{P}(\beta,k)$ $k \geqslant 1$, $k \in N$ we denote a subclass of k-symmetric functions of the class $\mathcal{P}(\beta)$ i.e. of functions of the form

$$P(z) = 1 + p_k z^k + p_{2k} z^{2k} + \ldots \tag{1.3}$$

satisfying (1.2) for $z \in K$.

From the definition of the family $\mathcal{P}(\beta,k)$ we have $\mathcal{P}(1,k) = \mathcal{P}_k$, where $\mathcal{P}_k$ is a subclass of k-symmetric functions of the class $\mathcal{P}$. It is easy that if $P \in \mathcal{P}(\beta,k)$ and $0<\beta<1$, then

$$(1.4)\qquad \left|P(z) - \frac{1+\beta^2}{1-\beta^2}\right| < \frac{2\beta}{1-\beta^2}, \qquad z \in K.$$

Similarly as in [6], we can prove the following lemmas:

L e m m a 1. A function P is in $\mathcal{P}(\beta,k)$ if and only if there exists a holomorphic function ω such that

$$(1.5)\qquad P(z) = \frac{1+\beta\omega(z)}{1-\beta\omega(z)}, \qquad z \in K,$$

where $\omega(0) = 0$ and $|\omega(z)| \leqslant |z|^k$.

L e m m a 2. A function P belongs to $\mathcal{P}(\beta,k)$ if and only if there exists a function $p \in \mathcal{P}_k$ such that

$$(1.6)\qquad P(z) = \frac{(1+\beta)\,p(z) + 1-\beta}{(1-\beta)\,p(z) + 1+\beta}, \qquad z \in K.$$

Let z_0 be any fixed point of K.

We define the functional

$$(1.7)\qquad H : \mathcal{P}(\beta,k) \ni P \longrightarrow H(P) = P(z_0).$$

L e m m a 3. The set of values of the functional (1.7) is the closed disc with centre c and radius ϱ, where

$$(1.8)\qquad c = \frac{1+\beta^2 r^{2k}}{1-\beta^2 r^{2k}}, \quad \varrho = \frac{2\beta r^k}{1-\beta^2 r^{2k}}, \qquad r = |z_0|.$$

Let us denote then by $\mathcal{P}_2(\beta,k)$ a subclass of a family $\mathcal{P}(\beta,k)$ consisting of those functions P of the form (1.6) for which the function $p \in \mathcal{P}_k$ is defined

$$(1.9)\qquad p(z) = \frac{1+\lambda}{2}\,p_1(z) + \frac{1-\lambda}{2}\,p_2(z),$$

where

$$(1.10)\qquad p_m(z) = \frac{1+\varepsilon_m z^k}{1-\varepsilon_m z^k}, \quad m = 1,2, \quad |\varepsilon_m| = 1,$$

$$-1 \leqslant \lambda \leqslant 1.$$

L e m m a 4. If $P \in \mathcal{P}_2(\beta,k)$, then for $z = re^{i\varphi}$, $0 \leqslant r \leqslant 1$, $0 \leqslant \varphi \leqslant 2\pi$ we have

$$(1.11)\qquad P(z) = c + \tilde{k}\gamma,$$

where $|\gamma| = 1$, $0 \leqslant \tilde{k} \leqslant \rho$, c, ρ are defined by (1.8).

L e m m a 5. If $P \in \mathcal{P}_2(\beta,k)$, then for $|z| = r < 1$ we have

$$(1.12)\qquad z P'(z) = k\,\frac{P^2(z) - 1}{2} - k\,\frac{\rho^*}{2\rho}\left[\rho^2 - |P(z) - c|^2\right]\eta^*,$$

where c, ρ are defined by (1.8) and

$$(1.13)\qquad \rho^* = \frac{2r^k}{1-r^{2k}}, \qquad |\eta^*| = 1.$$

2. Let $\sum^*(\beta,k)$ be a family of functions F of the form

$$(2.1)\qquad F(z) = \sum_{n=0}^{\infty} a_{nk-1} z^{nk-1}, \quad a_{-1} = 1$$

meromorphic in the unit disc such that

$$(2.2)\qquad \frac{-z F'(z)}{F(z)} = P(z), \quad P \in \mathcal{P}(\beta,k).$$

Obserwe that $\sum^*(1,1) \equiv \sum^*$, where $\sum^*$ is the known class of starlike meromorphic functions which map the ring

$K_0 = \{z : 0 < |z| < 1\}$ onto regions whose complement to the closed plane is a starlike region with respect to the origin.

T h e o r e m 1. If $F \in \sum^*(\beta,k)$ is defined by (2.1) then the coefficients of F satisfy

$$|a_{nk-1}| \leq \frac{2}{nk}, \quad n = 1,2,\ldots . \tag{2.3}$$

The estimate (2.3) is sharp and the equality in (2.3) is realized by the function

$$F^*(z) = \frac{1}{z}\left(1 - \varepsilon\beta z^{nk}\right)^{\frac{2}{nk}}, \qquad |\varepsilon| = 1. \tag{2.4}$$

P r o o f . Let $F \in \sum^*(\beta,k)$. Making use of the definitions of the families $\sum^*(\beta,k)$, $\mathcal{P}(\beta,k)$ and of Lemma 1 we obtain

$$\frac{-z F'(z)}{F(z)} = \frac{1+\beta\omega(z)}{1-\beta\omega(z)}, \qquad z \in K.$$

Hence by (2.1) we have

$$\sum_{m=1}^{\infty} mk\, a_{mk-1}\, z^{mk} = \left(-2 + \sum_{m=1}^{\infty} (mk-2)\, a_{mk-1}\, z^{mk}\right)\beta\omega(z).$$

Applying Clunie's method (c.f. [2]) we get

$$|nk\, a_{nk-1}|^2 \leq 4\beta^2 + \sum_{m=1}^{n-1} \left[(mk-2)^2\beta^2 - (mk)^2\right]|a_m|^2. \tag{2.5}$$

Since $(mk-2)^2\beta^2 - (mk)^2 < 0$ for $m,k \in N$ and $\beta \in (0,1]$, from (2.5) we obtain our result (2.3).

It is easy to verify that for the function defined by formula (2.4) we get

$$|a_{nk-1}| = \frac{2\beta}{nk} .$$

This ends the proof of Theorem 1.

For $k = 1$, $a_0 = 0$ we get the result obtained in [9] and for $k = 1$, $\beta = 1$ we obtain the result given in [8].

Theorem 2. If $F \in \sum^*(\beta,k)$, then for $|z| = r$, $0 < r < 1$, we have

$$\frac{(1 - \beta r^k)^{\frac{2}{k}}}{r} \leqslant |F(z)| \leqslant \frac{(1 + \beta r^k)^{\frac{2}{k}}}{r} . \tag{2.6}$$

The estimates are sharp. The equalities are attained at a point $z = re^{i\varphi}$ respectively by the functions

$$F^*(z) = \frac{(1 - \beta e^{-ik\varphi} z^k)^{\frac{2}{k}}}{z} , \quad F^{**}(z) = \frac{(1 + \beta e^{-ik\varphi} z^k)^{\frac{2}{k}}}{z} . \tag{2.7}$$

Proof. Let $F \in \sum^*(\beta,k)$. From the definition of the family $\sum^*(\beta,k)$ we have

$$\frac{1}{z} + \frac{F'(z)}{F(z)} = \frac{1 - P(z)}{z} .$$

Hence

$$|F(z)| = \frac{1}{|z|} \exp \int_0^1 \mathrm{re} \, \frac{1 - P(zt)}{t} \, dt, \qquad t \in R. \tag{2.8}$$

Using Lemma 3 we get

$$\frac{1 - \beta(tr)^k}{1 + \beta(tr)^k} \leqslant \mathrm{re}\, P(zt) \leqslant \frac{1 + \beta(tr)^k}{1 - \beta(tr)^k} .$$

Considering the inequalities above and (2.8) we obtain the inequalities (2.6). It is not difficult to see that the equalities in (2.6) are attained by the functions (2.7). This completes the proof of the theorem. If $\beta = 1$ then we get the known result given in [4].

We consider the functional

$$(2.9) \qquad -\operatorname{re}\left[1 + \frac{zF''(z)}{F'(z)}\right]$$

defined on $\sum^{*}(\beta,k)$.

For this functional we can prove the following theorems.

T h e o r e m 3. If $F \in \sum^{*}(\beta,k)$, then for each fixed z, $|z| = r$, $0 < r < 1$ the following sharp estimates hold

$$(2.10) \quad -\operatorname{re}\left[1 + \frac{zF''(z)}{F'(z)}\right] \geqslant \begin{cases} m_1(r,\beta) & \text{for } k = 1,\ r \in (0,1) \\ m_2(r,\beta) & \text{for } k \geqslant 2,\ r \in (0,r^*] \\ m_3(r,\beta) & \text{for } k \geqslant 2,\ r \in (r^*,1), \end{cases}$$

where $r^* \in (0,1)$ is the only root of the equation $A(r,\beta) = 0$, and

$$(2.11) \quad A(r,\beta) = (k-1)(1+\beta r^k)^2(1-r^{2k}) - 2kr^k(1+\beta)(1-\beta r^{2k}),$$

$$(2.12) \quad m_1(r,\beta) = \frac{1 - r^2 - 2\beta r^2}{1 - r^2},$$

$$(2.13) \quad m_2(r,\beta) = \frac{1 + 2(1-k)\beta r^k + \beta^2 r^{2k}}{1 - \beta^2 r^{2k}},$$

$$(2.14) \quad m_3(r,\beta) = \sqrt{k(1+R)(2-k+kR)} - \frac{k(1 + \beta^2 r^{2k})}{\beta(1-r^{2k})},$$

while $R = \frac{\varrho^*}{\varrho}$ and ϱ, ϱ^* are defined by (1.8) and (1.13).

The equalities in (2.10) are realized by functions of the form

$$(2.15)\quad F_1(z) = F_1(z;\beta) = \frac{1}{z}\exp\int_0^z \frac{2\beta e^{-i\varphi}(r - e^{-i\varphi}\zeta)}{1-(1-\beta)re^{-i\varphi}\zeta-\beta e^{-2i\varphi}\zeta^2}\,d\zeta$$

$$(2.16)\quad F_2(z) = F_2(z;\beta) = \frac{1}{z}\left(1-\beta e^{-ik\varphi}z^k\right)^{\frac{2}{k}},$$

$$(2.17)\quad F_3(z) = F_3(z;\beta) = \frac{1}{z}\exp\int_0^z \frac{2\beta e^{-ik\varphi}(d-e^{-ik\varphi}\zeta^k)\zeta^{k-1}}{1-(1-\beta)de^{-ik\varphi}\zeta^k-\beta e^{-2ik\varphi}\zeta^{2k}}\,d\zeta$$

where

$$(2.18)\quad d(r;\beta) = \frac{1}{r^k}\,\frac{(1+\beta r^{2k}) - (1-\beta r^{2k})\,s_1}{(1+\beta) - (1-\beta)\,s_1},$$

$$(2.19)\quad s_1 = \sqrt{\frac{k(\varrho^*+\varrho)}{2\varrho + k(\varrho^*-\varrho)}}.$$

P r o o f . Let $F \in \sum^*(\beta,k)$. In view of the definition of the family $\sum^*(\beta,k)$ we have

$$-\left[1+\frac{zF''(z)}{F'(z)}\right] = P(z) - z\,\frac{P'(z)}{P(z)}, \qquad P \in \mathcal{P}(\beta,k).$$

Applying the theorem of Zmorowič and lemmas (3-5) we have

$$-\left[1+\frac{zF''(z)}{F'(z)}\right] \geqslant \min_{(s,t)} G(s,t),$$

where the function

$$(2.20)\quad G(s,t) = \left[\frac{(2-k)s^2+k}{2s} - \frac{2ck\varrho^*}{2\varrho}\right]\cos t + \frac{k\varrho^*(s^2+1)}{2s\varrho}$$

is defined in the domain

$$(2.21) \qquad D = \left\{ (s,t):\ c-\rho \leqslant s \leqslant c+\rho,\ -\psi(s) \leqslant t \leqslant \psi(s) \right\}$$

$$\psi(s) = \arc\cos \frac{s^2+1}{2cs}, \qquad 0 \leqslant \psi(s) \leqslant \psi(1)$$

and c, ρ, ρ^* are given by (1.8), (1.12), (1.13), respectively. We verify, by direct calculations, that the function $G(s,t)$ can attain its minimum in the domain D along the diameter $t = 0$ only. Thus the problem of finding the minimum of the function $G(s,t)$ in the domain D under consideration is reduced to one of finding a minimum of the function

$$G_1(s) = G(s,0) \quad \text{in the interval} \quad [c-\rho,\ c+\rho].$$

If $k = 1$, then the function $G_1(s)$ attains its absolute minimum at the point $s = 1$. For $k > 1$ the function $G_1(s)$ attains its absolute minimum at the point s_1 given by formula (2.19) when $s_1 < c+\rho$. If, however, $s_1 \geqslant c+\rho$, then $G_1(s)$ attains its absolute minimum in the closed interval $[c-\rho,\ c+\rho]$ at the end point $c+\rho$. Thus the required result follows from the above reasoning.

It is not difficult to show that for the functions given by (2.15)-(2.17) the equality holds true in the estimate (2.10).

In the same way as above, we can prove the following theorem.

T h e o r e m 4. If $F \in \sum^*(\beta,k)$, then for $|z| = r$ $(0 < r < 1)$ we have

$$(2.22) \qquad -\operatorname{re}\left[1 + \frac{zF''(z)}{F'(z)}\right] \leqslant \begin{cases} m_1(r,-\beta) & \text{for } k = 1,\ r \in (0,1) \\ m_2(r,-\beta) & \text{for } k \geqslant 2,\ r \in (0,r^{**}] \\ m_3(r,-\beta) & \text{for } k \geqslant 2,\ r \in (r^{**},1), \end{cases}$$

where $r^{**} \in (0,1)$ is the only root of the equation

$$(2.23) \qquad A(r,-\beta) = 0$$

while $A(r,\beta)$ is defined by (2.11).

The equalities are attained by the functions:

(2.24) $$F_1^*(z) = F_1(z; -\beta)$$

(2.25) $$F_2^*(z) = F_2(z; -\beta)$$

(2.26) $$F_3^*(z) = F_3(z; -\beta)$$

respectively, where $F_1(z;\beta)$, $F_2(z;\beta)$ and $F_3(z;\beta)$ are defined by (2.15) - (2.17).

We put

$$r(F) = \sup\left\{r: -\mathrm{re}\left[1 + \frac{z F''(z)}{F'(z)}\right] > 0, \quad |z| < r\right\} .$$

It is known that the number

$$\text{r.c.} \sum{}^*(\beta,k) = \inf\left\{r(F) : F \in \sum{}^*(\beta,k)\right\}$$

is called the radius of convexity of the family $\sum^*(\beta,k)$.

From Theorem 3 taking into account the compactness of $\sum^*(\beta,k)$ we get

T h e o r e m 5. The radius of convexity for the class $\sum^*(\beta,k)$ is given by

$$\text{r.c.} \sum{}^*(\beta,k) = \begin{cases} \dfrac{1}{\sqrt{1+2\beta}} & \text{for} \quad k = 1 \\ r_0 & \text{for} \quad k > 1, \end{cases}$$

where $r_0 \in (0,1)$ is the only root of the equation

$$(1+\beta)(1-\beta r^{2k})\left[\beta(2-k)(1-r^{2k}) + k(1-\beta^2 r^{2k})\right] = k(1+\beta^2 r^{2k}).$$

T h e o r e m 6. If $F \in \sum^*(\beta,k)$, then for $|z| = r$, $0 < r < 1$ we have

(2.27) $$n(r) \leqslant |F'(z)| \leqslant N(r),$$

where

$$n(r) = \begin{cases} \dfrac{1}{r^2(1-r^2)^\beta} & \text{for } k = 1,\ r \in (0,1) \\[2ex] \dfrac{(1+\beta r^k)^{\frac{2-k}{k}}(1-\beta r^k)}{r^2} & \text{for } k > 1,\ r \in (0,r^{**}] \\[2ex] \dfrac{(1+\beta r^{**k})^{\frac{2-k}{k}}(1-\beta r^{**k})}{r^2} \exp R(r) & \text{for } k > 1,\ r \in (r^{**},1), \end{cases}$$

while

$$R(r) = \int_{r^{**}}^{r} \frac{1 - m_3(r,-\beta)}{r}\, dr$$

and

$$N(r) = \begin{cases} \dfrac{(1-r^2)^\beta}{r^2} & \text{for } k = 1,\ r \in (0,1) \\[2ex] \dfrac{(1-\beta r^k)^{\frac{2-k}{k}}(1+\beta r^k)}{r^2} & \text{for } k > 1,\ r \in (0,r^*] \\[2ex] \dfrac{(1-\beta r^{*k})^{\frac{2-k}{k}}(1+\beta r^{*k})}{r^2} \exp S(r) & \text{for } k > 1,\ r \in (r^*,1), \end{cases}$$

for

$$S(r) = \int_{r^*}^{r} \frac{1 - m_3(r,\beta)}{r}\, dr$$

while $r^*, r^{**} \in (0,1)$ are the only roots of the equations (2.11) and (2.23), respectively.

P r o o f . Suppose that F is in $\sum^*(\beta,k)$. We then have

$$\log z^2F'(z) = \log\left|z^2F'(z)\right| + i\arg(z^2 F'(z)).$$

Putting $z = re^{i\varphi}$ we get

$$r\frac{\partial}{\partial r}\log\left|z^2 F'(z)\right| = 1 - \mathrm{re}\left\{-\left[1 + \frac{zF''(z)}{F'(z)}\right]\right\}.$$

Using Theorems 3 and 4 we obtain (2.27). The equality holds in both cases for the functions defined by (2.15) - (2.17) and (2.24) - (2.26), respectively. This completes the proof of the theorem.

3. Let $\sum^{*M}(\beta,k)$ be a family of quasi-β-starlike meromorphic k-symmetric functions f defined by the equation

$$F\left(\frac{1}{f}\right) = M\,F(z), \quad 0<|z|<1, \tag{3.1}$$

where $F \in \sum^{*}(\beta,k)$ and M is a fixed number in the interval $[1,\infty)$. For $\beta = 1$ we obtain the class $\sum^{*M}(k)$ (c.f. [4]), but for $\beta = 1$ and $k = 1$ we get the class $\sum^{*M}$ (c.f. [3]).

Let $M = e^t$, $0\leqslant t<\infty$ and let $f(z,t)$ be a quasi-β-starlike function defined by the equation

$$F\left(\frac{1}{f}\right) = e^tF(z),\ 0\leqslant t<\infty,\ 0<|z|<1,\ F \in \sum^{*}(\beta,k). \tag{3.2}$$

It is easy to see that if F is a fixed function of the class $\sum^{*}(\beta,k)$ and $f(z,t)$ satisfies the equation (3.2), then we have

$$\lim_{t\to\infty} e^{-t} f(z,t) = F(z). \tag{3.3}$$

It follows from the definition of the family $\mathcal{P}(\beta,k)$ that if $P\in\mathcal{P}(\beta,k)$, then $\frac{1}{P}\in\mathcal{P}(\beta,k)$. Making use of this remark, similarly as in [1], [4] we can prove the following result.

T h e o r e m 7. A function f belongs to the class $\sum^{*M}(\beta,k)$ if and only if $f(z) = f(z,T)$, where $f(z,t)$ is a solution of the equation

$$(3.4) \quad \frac{\partial f(z,t)}{\partial t} = f(z,t)\, P\left(\frac{1}{f(z,t)}\right), \quad 0 \leqslant t \leqslant T, \; P \in \mathcal{P}(\beta,k)$$

satisfying the initial condition $f(z,0) = \frac{1}{z}$, where $M = e^T$. It is easy to see that the equation (3.4) is equivalent to the system of equations

$$(3.5) \qquad d\log|f(z,t)| = \mathrm{re}\, P\left(\frac{1}{f(z,t)}\right) dt$$

$$(3.6) \qquad d\arg f(z,t) = \mathrm{im}\, P\left(\frac{1}{f(z,t)}\right) dt.$$

Theorem 7 implies the following theorems.

T h e o r e m 8. If $F \in \sum^{*M}(\beta,k)$, then for $|z| = r < 1$ we have

$$(3.7) \qquad m(r) \leqslant |f(z)| \leqslant M(r),$$

where

$$(3.8) \quad m(r) = \frac{M}{r}\left[\frac{1}{2}(1-\beta r^k)^2 + \frac{r^k}{M^k} + \frac{1}{2}(1-\beta r^k)\sqrt{(1-\beta r^k)^2 + \frac{4\beta r^k}{M^k}}\right]^{\frac{1}{k}}$$

$$(3.9) \quad M(r) = \frac{M}{r}\left[\frac{1}{2}(1+\beta r^k)^2 - \frac{r^k}{M^k} + \frac{1}{2}(1+\beta r^k)\sqrt{(1+\beta r^k)^2 - \frac{4\beta r^k}{M^k}}\right]^{\frac{1}{k}}.$$

These inequalities are sharp.

P r o o f . Let $f \in \sum^{*M}(\beta,k)$. Then from the equation (3.4) we obtain

(3.10) $$d \log f(z,t) = P\left(\frac{1}{f(z,t)}\right) dt, \quad P \in \mathcal{P}(\beta,k).$$

Hence, by (3.5) and Lemma 3 we have

(3.11) $$\frac{|f(z,t)|^k - \beta}{|f(z,t)|^k + \beta} \leqslant d \log |f(z,t)| \leqslant \frac{|f(z,t)|^k + \beta}{|f(z,t)|^k - \beta} .$$

Integrating (3.1) in the interval from 0 to $T = \log M$ and taking into account the initial condition we obtain

$$\frac{(|f(z,T)|^k + \beta)^2}{|f(z,T)|^k} \leqslant M^k \frac{(1 + \beta r^k)^2}{r^k}$$

and

$$\frac{(|f(z,T)|^k - \beta)^2}{|f(z,T)|^k} \geqslant M^k \frac{(1 - \beta r^k)^2}{r^k}.$$

From the above inequalities we obtain (3.7) - (3.9). It is easy to see that the function realizing the equalities in (3.7) are defined by the equations

$$\frac{(f^k - \beta)^{\frac{2}{k}}}{f} = \frac{M}{z} (1 - \beta z^k)^{\frac{2}{k}} ,$$

$$\frac{(f^k + \beta)^{\frac{2}{k}}}{f} = \frac{M}{z} (1 + \beta z^k)^{\frac{2}{k}}$$

respectively. This ends the proof of Theorem 8.

R e m a r k s . When $k=1$, $\beta=1$ we obtain the known result given in [3]. When $\beta=1$ and $k=1,2,\dots$ we obtain the result given in [4].

T h e o r e m 9. If $f \in \sum^{*M}(\beta,k)$, then we have

$$(3.12)\quad |\arg zf(z)| \leqslant \frac{1}{k} \log \frac{|f(z)|^k - \beta}{|f(z)|^k + \beta} \cdot \frac{1+\beta r^k}{1-\beta r^k}, \quad 0 < |z| = r < 1$$

and the equality in (3.12) holds for the functions f defined by the equations

$$(3.13)\quad \frac{1}{f}\left(f^k + i\beta\right)^{\frac{2}{k}} = \frac{M}{z}\left(1 + \beta i z^k\right)^{\frac{2}{k}},$$

$$(3.14)\quad \frac{1}{f}\left(f^k - i\beta\right)^{\frac{2}{k}} = \frac{M}{z}\left(1 - \beta i z^k\right)^{\frac{2}{k}}.$$

P r o o f . Let $f \in \sum^{*M}(\beta,k)$. Then from (3.5) and (3.6) we obtain

$$(3.15)\quad d \arg f(z,t) = \frac{\operatorname{im} P\left(\frac{1}{f(z,t)}\right)}{\operatorname{re} P\left(\frac{1}{f(z,t)}\right)} \cdot d \log |f(z,t)|.$$

By using Lemmas 1 - 5 it is not difficult to see that

$$(3.16)\quad \frac{-2\beta r^k}{1-\beta^2 r^{2k}} \leqslant \frac{\operatorname{im} P(z)}{\operatorname{re} P(z)} \leqslant \frac{2\beta r^k}{1-\beta^2 r^{2k}}.$$

The estimates (3.16) are sharp. We have equalities respectively for the functions

$$(3.17)\quad P_1(z) = \frac{1-\beta i z^k}{1+\beta i z^k},$$

$$(3.18)\quad P_2(z) = \frac{1+\beta i z^k}{1-\beta i z^k} \quad \text{and} \quad |z| = r.$$

From (3.15) and (3.16) we obtain

$$|d \arg f(z,t)| \leqslant \frac{2\beta |f(z,t)|^k}{|f(z,t)|^{2k} - \beta^2} \cdot d \log |f(z,t)|.$$

Integrating the above inequality in the interval from 0 to $T = \log M$ and taking into account the condition $f(z,0) = \frac{1}{z}$ we obtain (3.12). From (3.17), (3.18) in view of the definition of the family $\sum^{*M}(\beta,k)$ we infer that the functions realizing the equalities (3.12) are of the form (3.13) and (3.14), respectively. This ends the proof of Theorem 9. For $\beta = 1$ we obtain the result given in [4].

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