DEMONSTRATIO MATHEMATICA
Vol. XI No 3 1978

Stanisław Kuś

ON THE COMPLETENESS OF CERTAIN SYSTEMS OF FUNCTIONS IN THE SPACES $L^2(0,\infty)$ AND $L^2(-i\infty, i\infty)$

1. Introduction

Let

(*) $$s_{11},\dots, s_{1k_1}, s_{21},\dots, s_{2k_2},\dots$$

be a sequence of complex numbers such that

$$s_{mp} \neq s_{nq} \quad \text{if} \quad m \neq n$$
$$s_{mp} = s_m \quad \text{if} \quad p = 1,\dots, k_m$$
$$\operatorname{Re} s_m < 0 \quad \text{if} \quad m = 1,2,\dots$$
$$s_m \to \infty .$$

In the sequence (*) each of the numbers s_m occurs successively k_m times.

Introducing the recurrent sequence

$$K_0 = 0$$
$$K_l = K_{l-1} + k_l \quad \text{for} \quad l = 1,2,\dots$$

we may write the sequence (*) in the form

(1.1) $$z_N = s_m,$$

where

$$N = K_{m-1} + \varkappa_{mp}$$

$$\varkappa_{mp} = 1,\dots,k_m .$$

We shall consider the sequence of functions

$$(1.2) \qquad x_N(t) = \begin{cases} t^{\varkappa_{mp}-1} e^{s_m t} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leqslant 0 \end{cases}$$

and the sequence of their Laplace transforms

$$(1.3) \qquad X_N(s) = \frac{(\varkappa_{mp} - 1)!}{(s - s_m)^{\varkappa_{mp}}} .$$

It is well known that the sequence (1.2) is linearly independent [10] . It is easy to prove, by reductio ad absurdum, that the sequence (1.3) is linearly independent. It is also easy to verify that the terms of the sequence (1.2) are elements of the Hilbert space $L^2(0,\infty)$ with scalar product defined by the relation

$$(1.4) \qquad (f,g) = \int_0^\infty f(t)\,\overline{g(t)}\,dt .$$

This space will be denoted by H_t.

Now, from Parseval's formula [6]

$$(1.5) \qquad \int_0^\infty f(t)\,\overline{g(t)}\,dt = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s)\,\overline{G(s)}\,ds$$

it follows that the terms of the sequence (1.3) are elements of the Hilbert space $L^2(-i\infty, i\infty)$ with scalar product defined by the relation

(1.6) $$(F,G) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(s)\,\overline{G(s)}\,ds ,$$

where $F(s) = \mathcal{L}[f(t)]$ i $G(s) = \mathcal{L}[g(t)]$.
This space will be denoted by H_s.

From the point of view of theory and applications the essential problem is that of completeness of the sequences (1.2) and (1.3) in corresponding spaces. For a particular case of the sequence (1.2) it has been solved by P.R.Clement [4]. In this paper we solve the problem in the general case. To prove the completeness conditions we make use of the sequence of functions orthonormal in the space $L^2(-i\infty, i\infty)$, equivalent to the sequence (1.3), constructed by A. Cremonesi [5]. This sequence will be considered in the next section of this paper.

2. Orthonormal exponential functions and their transforms

In the paper [5] A. Cremonesi has constructed the sequence of orthonormal functions in the space H_s

(2.1) $$U_N(s) = A_m \cdot \prod_{\mu=0}^{m-1} W_\mu^{k_\mu}(s) \cdot W_m^{\varkappa_{mp}-1}(s) \cdot V_m(s),$$

where

$$A_m = \sqrt{-2\,\mathrm{Re}\, s_m} ,$$

$$W_o^{k_o}(s) = 1,$$

$$W_\mu^{k_\mu}(s) = \left(\frac{s+\bar{s}_\mu}{s-s_\mu}\right)^{k_\mu} \quad \text{for} \quad \mu = 1,\dots,m,$$

$$V_m(s) = \frac{1}{s-s_m} .$$

This sequence is equivalent to the sequence (1.3), i.e. each function of the first sequence is a linear combination of functions of the second sequence, and conversely.

From Parseval's formula (1.5) it follows that the inverse transforms

$$u_N(t) = \mathcal{L}^{-1}\left[U_N(s)\right] \tag{2.2}$$

form an orthonormal system in the space H_t, equivalent to the sequence (1.2).

It is obvious that equivalent sequences are both either complete or not. Moreover, from a theorem of Paley-Wiener [9] it follows that the sequence of inverse transforms and the sequence of transforms too are both either complete or not. The results of these considerations may be recollected in the following lemma.

L e m m a 2.1. The sequences (1.2), (1.3), (2.1), (2.2) are simultaneously complete, or not, in the corresponding spaces.

3. Particular case

In the sequel we shall investigate a particular case of the considered sequences and present the results concerning completeness in this case.

Assume that

$$k_m = 1 \quad \text{for} \quad m = 1,2,\ldots, \tag{3.1}$$

i.e. let us consider the sequence of unequal complex numbers

$$z_N = s_N, \tag{3.2}$$

the sequence of functions

$$\tilde{x}_N(t) = \begin{cases} e^{s_N t} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leqslant 0, \end{cases} \tag{3.3}$$

the sequence of Laplace-transforms

$$\tilde{X}_N(s) = \frac{1}{s - s_N}, \tag{3.4}$$

the sequence of orthonormal transforms in the space H_s

$$\tilde{U}_N(s) = A_N \cdot \prod_{\mu=0}^{N-1} W_\mu^1(s) \cdot V_N(s), \tag{3.5}$$

equivalent to the sequence (3.4), and the sequence of orthonormal inverse-transforms in the space H_t

$$\tilde{u}_N(t) = \mathcal{L}^{-1}\left[\tilde{U}_N(s)\right] \tag{3.6}$$

equivalent to the sequence (3.3).

The sequences (3.5) and (3.6) have been constructed by W.H. Kautz in the paper [8]. In the paper [4] P.R. Clement has proved that the sequence (3.3) is complete in the space H_t if and only if

$$\sum_{N=1}^{\infty} \frac{-\mathrm{Re}\ s_N}{1 + \left|s_N + \frac{1}{2}\right|^2} = \infty \quad \text{and} \quad \mathrm{Re}\ s_N < 0. \tag{3.7}$$

By virtue of Lemma 2.1 this implies the following lemma.

L e m m a 3.1. The sequences (3.3), (3.6) in the space H_t and the sequences (3.4), (3.5) in the space H_s are complete if and only if the condition (3.7) is fulfilled.

4. General case

We shall now consider a sequence of complex numbers (1.1), a sequence of functions (1.2), the sequence of their Laplace-transforms, and the corresponding orthonormal sequences (2.1) and (2.2). We are going to prove the following theorem.

T h e o r e m 4.1. The sequences (1.2) and (2.1) in the space H_t and the sequences (1.3) and (2.2) in the space H_s are complete if and only if the following condition is fulfilled

$$(4.1) \quad \sum_{m=1}^{\infty} \frac{k_m\,(-\mathrm{Re}\ s_m)}{1+\left|s_m+\frac{1}{2}\right|^2} = \infty\ , \quad \mathrm{Re}\ s_m < 0 \quad \text{and} \quad s_m \longrightarrow \infty .$$

To prove the theorem, we shall construct a sequence, near to the sequence (2.1), of transforms with single poles and we shall prove some lemmas.

We modify the sequence (1.1) so as to obtain a sequence of unequal complex numbers

$$(4.2) \qquad \tilde{z}_N = s_m - \delta_{m\varkappa_{mp}},$$

where the numbers $\delta_{m\varkappa_{mp}}$ satisfy the inequalities

$$(4.3) \qquad \delta_{m\varkappa_{mp}} > 0 \quad \text{and} \quad -\mathrm{Re}\ (s_m - \delta_{m\varkappa_{mp}}) > 0 .$$

Using the sequence $\tilde{z}_N$ we shall build the orthonormal sequence of Kautz-functions

$$(4.4) \qquad \tilde{U}_N(s) = \tilde{A}_N \cdot \prod_{\mu=0}^{m-1} \tilde{W}_{\mu}^{k_\mu}(s) \cdot \tilde{W}_m^{\varkappa_{mp}-1}(s) \cdot \tilde{V}_{m\varkappa_{mp}}(s),$$

where $\tilde{A}_N = \sqrt{-2\ \mathrm{Re}\ (s_m - \delta_{m\varkappa_{mp}})}$,

$\tilde{W}_o^{k_o}(s) = 1$,

$$\tilde{W}_{\mu}^{k_{\mu}}(s) = \frac{s + \bar{s}_{\mu} - \delta_{\mu 1}}{s - s_{\mu} + \delta_{\mu 1}} \cdots \frac{s + \bar{s}_{\mu} - \delta_{\mu k_{\mu}}}{s - s_{\mu} + \delta_{\mu k_{\mu}}} \text{ for } \mu = 1, \ldots, m,$$

$$\tilde{V}_{m\varkappa_{mp}}(s) = \frac{1}{s - s_m + \delta_{m\varkappa_{mp}}} .$$

In view of (3.7) the necessary and sufficient condition for the sequence (4.4) to be complete may be written in the form

$$(4.5) \quad \sum_{m=1}^{\infty} \left(\sum_{\varkappa_{mp}=1}^{k_m} \frac{-\mathrm{Re}(s_m - \delta_{m\varkappa_{mp}})}{1 + \left| s_m - \delta_{m\varkappa_{mp}} + \frac{1}{2} \right|^2} \right) = \infty \quad \text{and} \quad \mathrm{Re}\, s_m < 0.$$

L e m m a 4.2. The sequences (2.1) and (4.4) are both either complete in the space H_s or not.

P r o o f . I. A direct calculation (making use of residua) shows that

$$\lim_{\delta_{m\varkappa_{mp}} \to 0^+} (\tilde{U}_K U_K) = \begin{cases} 1 & \text{when } n = K \\ 0 & \text{when } n \neq K \end{cases} \qquad K = 1, 2, \ldots$$

Then it follows from the equality

$$\|U_K - \tilde{U}_K\|^2 = \|U_K\|^2 + \|\tilde{U}_K\|^2 - (U_K \tilde{U}_K) - (\tilde{U}_K U_K)$$

that

$$\lim_{\delta_{m\varkappa_{mp}} \to 0^+} \|U_K - \tilde{U}_K\| = 0, \qquad K = 1, 2, \ldots$$

Similarly, the inequality

$$\|\tilde{U}_K - \sum_{n=1}^{N} (\tilde{U}_K U_n)\, U_n\| \leq \|\tilde{U}_K - U_K\| + |1-(\tilde{U}_K U_K)\, U_K| \cdot \|U_K\| +$$

$$+ \sum_{\substack{n=1\\ n\neq K}}^{N} |(\tilde{U}_K U_n)| \cdot \|U_n\|$$

implies

$$\lim_{\delta_{m\varkappa_{mp}} \to 0+} \|\tilde{U}_K - \sum_{n=1}^{N} (\tilde{U}_K U_n)\, U_n\| = 0, \quad K = 1,2,\ldots; \quad N > K.$$

This means that for every number $\varepsilon > 0$, there exists a number $\delta(\varepsilon) > 0$ such that if

$$(4.6) \qquad 0 < \delta_{m\varkappa_{mp}} < \delta(\varepsilon)$$

then

$$(4.7) \qquad \|U_K - \tilde{U}_K\| < \varepsilon, \quad K = 1,2,\ldots$$

and

$$(4.8) \qquad \|\tilde{U}_K - \sum_{n=1}^{N} (\tilde{U}_K U_n) U_n\| < \varepsilon \quad K = 1,2,\ldots; \quad N > K.$$

In what follows we shall consider only functions $\tilde{U}_K$ such that the numbers $\delta_{m\varkappa_{mp}}$ satisfy the condition (4.6).

II. Let the sequence of functions $\{U_K\}$ be complete in the space H_s. Suppose the sequence of functions $\{\tilde{U}_K\}$ to be not complete in this space. Then there exists a function $F_o(s)$ satisfying simultaneously both conditions

1^o $\|F_o\| > 0$

2^o $(\tilde{U}_K F_o) = 0.$

We shall now estimate the scalar product. To this aim we can use condition 2^o, Schwarz's inequality and condition (4.7).

$$|(U_K, F_o)| = |(U_K - \tilde{U}_K, F_o)| \leq \|U_K - \tilde{U}_K\| \cdot \|F_o\| \leq \varepsilon \cdot \|F_o\| .$$

The number $\varepsilon > 0$ being arbitrary we have

$$(U_K F_o) = 0$$

whence, since the sequence $\{U_K\}$ is complete, we get

$$F_o(s) = 0, \qquad s \in (-i\infty , i\infty).$$

This being contrary to condition 1^o the completeness of the sequence $\{\tilde{U}_K\}$ has been established.

III. Suppose the sequence of functions $\{\tilde{U}_K\}$ to be complete in the space H_s. To prove the completeness of the sequence $\{U_K\}$ we have to show that it is closed, i.e. that for every function $F \in H_s$, there exists a sequence of coefficients $\{c_n\}$ such that

$$\lim_{N \to \infty} \|F - \sum_{n=1}^{N} c_n U_n\| = 0 . \tag{4.9}$$

Let the coefficients assume the following values

$$c_n = \sum_{k=1}^{K} (F\tilde{U}_k)(\tilde{U}_k U_n), \quad K = 1,2,\ldots; \ n = 1,2,\ldots$$

and let

$$R_{kN} = \tilde{U}_K - \sum_{n=1}^{N} (\tilde{U}_K\ U_n)\ U_n.$$

From the equality

$$\left\| F - \sum_{n=1}^{N} c_k U_n \right\| = \left\| F - \sum_{k=1}^{K} (F\ \tilde{U}_k)(U_k - R_{kN}) \right\|$$

we deduce the estimation

$$(4.10) \quad \left\| F - \sum_{n=1}^{N} c_k U_n \right\| \leq \left\| F - \sum_{k=1}^{K} (F\tilde{U}_k)\ \tilde{U}_k \right\| + \sum_{k=1}^{K} \left|(F\tilde{U}_k)\right| \cdot \left\| R_{kN} \right\|.$$

The completeness of the sequence $\{\tilde{U}_K\}$ implies the closure condition

$$\lim_{K\to\infty} \left\| F - \sum_{k=1}^{K} (F\tilde{U}_k)\ \tilde{U}_k \right\| = 0$$

and the convergence of the series $\sum_{k=1}^{\infty} |(F\tilde{U}_k)|^2$, whence the condition

$$\lim_{k\to\infty} \left|(F\tilde{U}_k)\right| = 0.$$

This means that for every number $\varepsilon > 0$, there exists a number $K(\varepsilon)$ such that if

$$K \geqslant K(\varepsilon)$$

then

$$\left\| F - \sum_{k=1}^{K} (F\ \tilde{U}_k)\ \tilde{U}_K \right\| < \varepsilon$$

and

$$\left|(F\ \tilde{U}_k)\right| < \varepsilon \ .$$

Moreover, (4.8) implies that

$$\|R_{kN}\| < \varepsilon \ , \quad k = 1,2,\dots; \quad N > k.$$

Let $K = K(\varepsilon)$ be fixed. From inequality (4.10) we get

$$\| F - \sum_{k=1}^{N} c_n U_n \| \leqslant \varepsilon + K(\varepsilon)\cdot\varepsilon^2, \quad N > K(\varepsilon)$$

whence it finally results that condition (4.9) is satisfied and that the sequence $\{U_n\}$ is complete.

L e m m a 4.3. The conditions (4.1) and (4.5) are equivalent.

P r o o f . Let us consider two following sequences of numbers a_m and $\tilde{a}_m$:

$$a_m = \frac{k_m \cdot (-\mathrm{Re}\ s_m)}{1 + \left| s_m + \frac{1}{2}\right|^2} ,$$

$$\tilde{a}_m = \sum_{\varkappa_{mp}=1}^{k_m} \frac{-\mathrm{Re}\ (s_m - \delta_{m\varkappa_{mp}})}{1 + \left| s_m - \delta_{m\varkappa_{mp}}\right|^2} \ .$$

The following relation is evident:

$$\lim_{\delta_{m\varkappa_{mp}} \to 0+} \frac{\tilde{a}_m}{a_m} = 1$$

i.e. for each number $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that

$$0 < \delta_{m\varkappa_{mp}} < \delta(\varepsilon)$$

implies

$$\left|\frac{\tilde{a}_m}{a_m} - 1\right| < \varepsilon$$

or

$$(1 - \varepsilon)a_m < \tilde{a}_m < (1 + \varepsilon)a_m.$$

Thence it follows that the series (4.1) and (4.5) are both either convergent or divergent.

Lemmas 2.1, 3.1, 4.2 and 4.3 proved above imply Theorem 4.1.

5. Final remarks

The completeness of a system of functions is a property which conditions the possibility of constructing a base in the corresponding space and, at the same time, the possibility of arbitrarily near approximation in the sense of the metric of this space. In this paper we have considered certain systems of exponential functions and of their Laplace transforms (with single or multiple poles). Theorem 4.1 gives necessary and sufficient conditions under which these systems are bases in the spaces $L^2(0,\infty)$ and $L^2(-i\infty, i\infty)$.

Another important property of these systems, in view of the simplicity of computations, is their orthonormality. One of the most important of them seems to be the system functions introduced by W.H.Kautz, of which a particular case is the system of Laguerre functions. Another particular case is the system of exponential functions with real exponents forming an arithmetic sequence - they are nearly related to the Jacobi polynomials ([1], [2]).

The functions of A. Cremonesi are generalizations of both Laguerre and W.H. Kautz functions. These functions may be applied e.g. in the approximation of solutions of linear differential equations with retarded argument. Exact solutions of such equations in form of series of functions (1.2) may be found in the books [3], [7]. To construct such a solution, it is necessary to know all the roots of the characteristic equation. When only some of these roots are known, with greatest real parts (the so-called dominant roots), it is possible to find an approximate solution. The computations may then be simplified by use of an orthonormalized system.

REFERENCES

[1] H.L. Armstrong: On finding an orthonormal basis for representing transients. IRE Trans., vol. CT - 4, No. 3. 1957.

[2] H.L. Armstrong: On the representation of transients by series of orthogonal functions, IRE Trans., vol. CT - 6, No. 4, 1959.

[3] R. Bellman, K.L. Cooke: Differential--difference equations, New York 1963.

[4] P.R. Clement: On completeness of basis functions used for signal analysis, SIAM Review, vol. 5, No. 2, 1963.

[5] A. Cremonesi: Sull'ortonormalizaccione di un particolare sistema di funzioni, Atti Accad. Lincei, vol. 20. No. 2, 1956.

[6] G. Doetsch: Praktyka przekształcenia Laplace'a, Warszawa 1964.

[7] Ł.E. Elsgolc: Równania różniczkowe z odchylonym argumentem, Warszawa 1966.

[8] W.H. Kautz: Transient synthesis in the time domain, IRE Trans., vol. CT - 1, No 3, 1954.

[9] R. Paley, N. Wiener: Fourier transform in the complex domain, Amer. Math. Soc. Coll. Publ. Vol. XIX, N. Y. 1934.

[10] W.W. Stepanow: Równania różniczkowe. Warszawa 1956.

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY, LUBLIN
Received August 22, 1977.