DEMONSTRATIO MATHEMATICA
Vol. XI No 3 1978

Zofia Walczak

UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS WITH VALUES IN A TOPOLOGICAL SPACE

E. Kocela [1] proved that a sequence $\{f_n\}$ of common bounded real functions defined on some nonempty set X is uniformly convergent to $f: X \to R$ if and only if for arbitrary: σ-additive family E of subsets of X, increasing sequence $\{n_k\}$ of natural numbers and real numbers y_1, y_2, $y_2 > y_1$, the condition

$$\left\{x: f_{n_k}(x) < a\right\} \in E \quad \text{for any} \quad a \in (y_1, y_2) \quad \text{and} \quad k = 1,2,\ldots$$

implies

$$\left\{x: f(x) < a\right\} \in E \quad \text{for all} \quad a \in (y_1, y_2)$$

and the condition

$$\left\{x: f_{n_k}(x) > a\right\} \in E \quad \text{for any} \quad a \in (y_1, y_2) \quad \text{and} \quad k = 1,2,\ldots$$

implies

$$\left\{x: f(x) > a\right\} \in E \quad \text{for all} \quad a \in (y_1, y_2).$$

E. Kocela and T. Świątkowski [2] generalized this result by considering an arbitrary compact metric space Y instead of closed intervals of R and the sets $\left\{x: f(x) \notin \overline{K(p_0, r)}\right\}$ instead of $\left\{x: f(x) < a\right\}$ and $\left\{x: f(x) > a\right\}$.

It is easy to reformulate the above conditions characterizing uniform convergence of real functions in the case of functions with values in an arbitrary topological space.

D e f i n i t i o n . Let X be a nonempty set and Y - a topological space. A sequence $\{f_n\}$ of functions mapping X into Y is said to be uniformly convergent to $f: X \to Y$ if and only if for arbitrary

a) point $p \in Y$,

b) open sets U_0, U_1 such that $p \in U_0 \subset \overline{U}_0 \subset U_1$,

c) σ-additive family E of subsets of the set X,

d) increasing sequence $\{n_k\}$,

the following condition holds

$$\left(\bigwedge_{\overline{U}_0 \subset U \subset \overline{U} \subset U_1} \bigwedge_k \left\{x: f_{n_k}(x) \notin \overline{U}\right\} \in E \right) \Rightarrow$$

$$\Rightarrow \left(\bigwedge_{\overline{U}_0 \subset U \subset \overline{U} \subset U_1} \left\{x: f(x) \notin \overline{U}\right\} \in E \right).$$

There appears the following problem: Does the uniform convergence in the sense of our definition imply the convergence at any point?

Some results in this direction are formulated in the Theorems 1 and 2.

T h e o r e m 1. If a sequence $\{f_n\}$ of functions mapping a nonempty set X into a normal topological space Y is uniformly convergent to a function $f: X \to Y$, then the sequence $\{f_n\}$ is convergent to f at any point x of X.

P r o o f : Assume that there exist a point $x_0 \in X$, an open set $V \subset Y$ and an increasing sequence $\{n_k\}$ such that $f(x_0) \in V$ and $f_{n_k}(x_0) \notin V$ for any k. Let $p = f(x_0)$ and E be a family of all subsets of the set X including x_0. Put $U_1 = V$ and take an open set U_0 such that $p \in U_0 \subset \overline{U}_0 \subset V$. Then for any open set U satisfying $\overline{U}_0 \subset U \subset$

$\subset \overline{U} \subset V$ the relation $x_0 \in \{x: f_{n_k}(x) \notin \overline{U}\}$ holds. Thus $\{x: f_{n_k}(x) \notin \overline{U}\} \in E$.

On the other hand Y is a normal topological space, so there exists an open neighbourhood U^* of the point p such that $\overline{U}_0 \subset U^* \subset \overline{U}^* \subset V$. Of course $\{x: f(x) \notin \overline{U}^*\} \notin E$. This ends the proof.

C o r o l l a r y . If the space Y is normal then any sequence of functions $f_n: X \to Y$ has at most one limit in the sense of the uniform convergence.

Our theorem fails in the case of an arbitrary topological space.

E x a m p l e . Let $X = \{x\}$ and $Y = \{a, b\}$ be equipped with the topology consisting of the sets $\emptyset$, $\{a\}$, Y. Then every sequence of functions $f_n: X \to Y$ is uniformly convergent to the function $f(x) = a$ in the sense of our definition, but it is convergent to a at x if and only if $f_n(x) = a$ for any sufficiently large n.

T h e o r e m 2. Let Y be a regular topological space. If we have $a_n \to a$, where $a_n \in Y$, $a \in Y$, then the sequence $\{f_n\}$, $f_n(x) = a_n$ for any $x \in X$, is uniformly convergent to the function f, where $f(x) = a$ for any x of X.

P r o o f : Let $a_n \to a$ and take some $p \in Y$, open sets U_0, U_1, σ-additive family E of subsets of the set X and increasing sequence $\{n_k\}$ such that $p \in U_0 \subset \overline{U}_0 \subset U_1$ and

$$\bigwedge_{\overline{U}_0 \subset U \subset \overline{U} \subset U_1} \bigwedge_{k} \{x: f_{n_k}(x) \notin \overline{U}\} \in E.$$

If for every open set U such that $\overline{U}_0 \subset U \subset \overline{U} \subset U_1$ and for every k, we have $a_{n_k} = f_{n_k}(x) \in \overline{U}$, then $\{x: f_{n_k}(x) \notin \overline{U}\} = \emptyset$ and now $\emptyset \in E$. On the other hand, if $a_n \to a$, then $a \in \overline{U}$ and $\{x: f(x) \notin \overline{U}\} = \emptyset \in E$.

Let now for every open set U satisfying $\overline{U}_0 \subset U \subset \overline{U} \subset U_1$ and for every k, a_{n_k} does not belong to $\overline{U}$. Then $\{x: f_{n_k}(x) \notin \overline{U}\} = X \in E$. The equality $U_1 = \bigcup_{\overline{U}_0 \subset U \subset \overline{U} \subset U_1} \overline{U}$ holds because the space Y is regular. So we have $a_{n_k} \notin U_1$, $a \notin U_1$ and $\{x: f(x) \notin \overline{U}\} = X \in E$.

The proof in the case where there exist open sets U, U' and numbers k, k' such that $\overline{U}_0 \subset U \subset \overline{U} \subset U_1$, $\overline{U}_0 \subset U' \subset \overline{U}' \subset U_1$, $a_{n_k} \in \overline{U}$ and $a_{n_k} \in \overline{U}'$, is trivial.

R e m a r k . If in our definition we replace an open set U by a closed set F such that $U_0 \subset F \subset U_1$, then it is easy to obtain that uniform convergence of the sequence $f_n: X \to R$ to the function $f: X \to R$ implies convergence in the sense of our definition and that Theorem 2 holds for any T_1 - space Y.

REFERENCES

[1] E. K o c e l a : On invariants of the uniform convergence. Sc. Bull. of Łódź Technical Univ., Math. 1 (1972), 105 - 118 (in Polish, English summary).

[2] E. K o c e l a , T. Ś w i ą t k o w s k i : On some characterization of uniform convergence. Sc. Bull. of Łódź Technical Univ., Math. 7 (1975) 11 - 15.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, ŁÓDŹ
Received July 21, 1977.