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M. Barros, A. Ramírez

DECOMPOSITION OF QUASI-KÄHLER MANIFOLDS WHICH SATISFY THE FIRST CURVATURE CONDITION

1. Introduction

We consider C^{∞} almost Hermitian manifolds. In this paper we shall be concerned with five classes of almost Hermitian manifolds: K, NK, AK, QK, H. For reference, the defining condition for these classes is as follows

K:	$(\nabla_X J)Y = 0$	Kählerian
NK:	$(\nabla_X J)X = 0$	Nearly-Kählerian
AK:	$dF = 0$	Almost-Kählerian
QK:	$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$	Quasi-Kählerian
H:	$(\nabla_X J)Y - (\nabla_{JX} J)JY = 0$	Hermitian

for $X,Y \in \mathfrak{X}(M)$, where, ∇ is the Riemannian connection adapted to the metric $\langle\ ,\ \rangle$; J-the almost complex structure; $\mathfrak{X}(M)$-the Lie algebra of vector fields on M, and F - the Kähler form given by

$$F(X,Y) = \langle JX,Y\rangle .$$

In this paper, we consider that $\{E_i : 1 \leqslant i \leqslant 2n\}$, with $E_{n+k} = JE_k$, $1 \leqslant k \leqslant n$, is a local frame field on M.

A key to understanding the geometry of these classes of almost Hermitian manifolds are curvature identifies.

In fact, for each of the classes given above there exists a curvature identity [6]. Moreover, in this paper we shall only be concerned with the following curvature identities

(1) $R_{WXYZ} = R_{WXJYJZ}$,

(2) $R_{WXYZ} = R_{WXJYJZ} + R_{WJXYJZ} + R_{WJXJYZ}$,

(3) $R_{WXYZ} = R_{JWJXJYJZ}$,

where R is the Riemannian curvature operator.

For a given class L of almost Hermitian manifolds, let L_i be the subclass of manifolds whose curvature operator satisfies identity (i).

Certain equalities occur among the various classes [6]. In [5], A. Gray give conditions under which a nearly Kähler manifold can be decomposed as a product of Kähler manifold and strict nearly Kähler manifold.

In [1], we study decomposition of AK_2-manifolds. In [3], it is proved that $S^2 \times R^4$ has a structure of QK_1-manifold, non Kählerian, moreover there are not relations among the classes NK, AK_2 and QK_1.

In this article, we study decompositions of quasi-Kähler manifolds which satisfy the first curvature identity and give condition under which a QK_1-manifold can be decomposed as a product of Kähler manifolds and strict quasi-Kähler manifolds which satisfy the first curvature identity.

2. Preliminaries

Let M be an almost Hermitian C^∞ -manifold. Let $\nabla^2 J$, ∇F and $\nabla^2 F$ be tensor fields defined by

$$(\nabla^2 J)(W,X,Y,Z) = \langle (\nabla^2_{WX} J)Y, Z \rangle ,$$

$$\nabla F(X,Y,Z) = XF(Y,Z) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z),$$

$$\nabla^2 F(W,X,Y,Z) = W\nabla F(X,Y,Z) - \nabla F(\nabla_W X, Y, Z) -$$
$$- \nabla F(X, \nabla_W Y, Z) - \nabla F(X,Y,\nabla_W Z)$$

for $W,X,Y,Z \in \mathfrak{X}(M)$.

P r o p o s i t i o n 2.1. Let M be a QK_1-manifold. Then

$$1)\quad \nabla F(X,Y,Z) = \langle (\nabla_X J)Y, Z\rangle ,$$

$$2)\quad \nabla^2 F(W,X,Y,Z) = \langle (\nabla^2_{WX} J)Y, Z\rangle ,$$

$$3)\quad \nabla^2 F(W,X,Y,Y) = 0,$$

$$4)\quad \langle (\nabla^2_{WX} J)Y, Z\rangle - \langle (\nabla^2_{XW} J)Y, Z\rangle = 0,$$

$$(2.1)\quad 5)\quad \langle (\nabla^2_{WX} J)Y, Z\rangle + \langle (\nabla^2_{WJX} J)JY, Z\rangle = - \langle (\nabla_{(\nabla_W J)X} J)JY, Z\rangle - \langle (\nabla_{JX} J)(\nabla_W J)Y, Z\rangle ,$$

$$6)\quad \langle (\nabla^2_{WX} J)Y, JY\rangle = - \langle (\nabla_X J)Y, (\nabla_W J)Y\rangle ,$$

$$7)\quad \langle (\nabla^2_{WX} J)Y, Z\rangle + \langle (\nabla^2_{WX} J)JY, JZ\rangle = - \langle (\nabla_W J)Y, J(\nabla_X J)Z\rangle - \langle (\nabla_X J)Y, J(\nabla_W J)Z\rangle ,$$

$$8)\quad (\nabla_{(\nabla_W J)X} J) = (\nabla_{(\nabla_X J)W} J)$$

for $W,X,Y,Z \in \mathfrak{X}(M)$.

R e m a r k s : a) Expressions 1), 2) and 3) are valid for every almost Hermitian manifold.

b) Expression 4) is valid for every almost Hermitian manifold satisfying the first condition of curvature.

c) Expression 5) is valid for every QK-manifold.

d) Expressions 6), 7) are valid for every QK_2-manifold.

e) Expression 8) is valid for every QK_1-manifold.

P r o p o s i t i o n 2.2. Let M be a QK_1-manifold. Then

$$\langle(\nabla^2_{WX}J)Y,Z\rangle = \frac{1}{2}\left\{\langle(\nabla_W J)Z,J(\nabla_X J)Y\rangle - \langle(\nabla_W J)Y,J(\nabla_X J)Z\rangle - \right.$$

$$\left. - \langle(\nabla_{(\nabla_W J)X}J)JY,Z\rangle\right\}$$

for $W,X,Y,Z \in \mathfrak{X}(M)$.

P r o o f . According to (2.1) 5), 6), 7), 8) $\langle(\nabla^2_{WX}J)Y,Z\rangle$ will be written as a linear combination with constant coefficients of the products $\langle(\nabla_{\cdot}J)\cdot,(\nabla_{\cdot}J)\cdot\rangle$, $\langle J(\nabla_{\cdot}J)\cdot,(\nabla_{\cdot}J)\cdot\rangle$ and $\langle(\nabla_{(\nabla_{\cdot}J)\cdot}J)J\cdot,\cdot\rangle$, that is

(2.2) $\langle(\nabla^2_{WX}J)Y,Z\rangle =$

$$= a_1\langle(\nabla_W J)X,(\nabla_Y J)Z\rangle + a_2\langle(\nabla_W J)Z,(\nabla_X J)Y\rangle +$$

$$+ a_3\langle(\nabla_W J)Y,(\nabla_Z J)X\rangle + a_4\langle(\nabla_W J)X,(\nabla_Z J)Y\rangle +$$

$$+ a_5\langle(\nabla_W J)Z,(\nabla_Y J)X\rangle + a_6\langle(\nabla_W J)Y,(\nabla_X J)Z\rangle +$$

$$+ a_7\langle(\nabla_X J)W,(\nabla_Y J)Z\rangle + a_8\langle(\nabla_Z J)W,(\nabla_X J)Y\rangle +$$

$$+ a_9\langle(\nabla_Y J)W,(\nabla_Z J)X\rangle + a_{10}\langle(\nabla_X J)W,(\nabla_Z J)Y\rangle +$$

$$+ a_{11}\langle(\nabla_Z J)W,(\nabla_Y J)X\rangle + a_{12}\langle(\nabla_Y J)W,(\nabla_X J)Z\rangle +$$

$$+ b_1\langle J(\nabla_W J)X,(\nabla_Y J)Z\rangle + b_2\langle J(\nabla_W J)Z,(\nabla_X J)Y\rangle +$$

$$+ b_3\langle J(\nabla_W J)Y,(\nabla_Z J)X\rangle + b_4\langle J(\nabla_W J)X,(\nabla_Z J)Y\rangle +$$

$$+ b_5\langle J(\nabla_W J)Z,(\nabla_Y J)X\rangle + b_6\langle J(\nabla_W J)Y,(\nabla_X J)Z\rangle +$$

$$+ b_7\langle J(\nabla_X J)W,(\nabla_Y J)Z\rangle + b_8\langle J(\nabla_Z J)W,(\nabla_X J)Y\rangle +$$

$$+ b_9\langle J(\nabla_Y J)W,(\nabla_Z J)X\rangle + b_{10}\langle J(\nabla_X J)W,(\nabla_Z J)Y\rangle +$$

$$+ b_{11}\langle J(\nabla_Z J)W,(\nabla_Y J)X\rangle + b_{12}\langle J(\nabla_Y J)W,(\nabla_X J)Z\rangle +$$

$$+ c_1\langle(\nabla_{(\nabla_W J)X}J)JY,Z\rangle + c_2\langle(\nabla_{(\nabla_W J)Y}J)JX,Z\rangle +$$

$$+ c_3 \langle (\nabla_{(\nabla_W J)Z^J} J)JY, X\rangle + c_4 \langle (\nabla_{(\nabla_X J)Y^J} J)JW, Z\rangle +$$
$$+ c_5 \langle (\nabla_{(\nabla_X J)Z^J} J)JW, Y\rangle + c_6 \langle (\nabla_{(\nabla_Y J)Z^J} J)JW, X\rangle .$$

As (2.2) must satisfy (2.1) 3), 4), 5), 6) it follows that all the coefficients are null except

$$b_2 = -\frac{1}{2},\ b_6 = \frac{1}{2},\ c_1 = -\frac{1}{2}$$

and the proposition is proved.

In [1], we defined for an almost Hermitian manifolds M, the distribution K^* by

$$K^*(p) = \{X \in T_p(M); (\nabla_Y J)X = 0 \quad \text{for} \quad Y \in T_p(M)\}$$

for each $p \in M$.

Moreover, we proved that this distribution is integrable for an AK_2-manifold.

In [4], this is proved for a NK-manifold.

C o r o l l a r y 2.2. Let M be a QK_1-manifold. Then on any open subset of M on which $\dim K^*(p)$ is constant, the distribution $p \mapsto K^*(p)$ is integrable. Furthermore, the integral submanifolds are Kähler submanifolds of M.

P r oo f . Let W and X be vector fields which at each point lie in the distribution $p \mapsto K^*(p)$. Then

$$(\nabla_Y J)W = 0 \quad \text{and} \quad (\nabla_Y J)X = 0 \quad \text{for all } Y.$$

Differentiating this and making use of (2.1), we obtain

$$(\nabla_Y J)[W,X] = 0 \quad \text{for all } Y.$$

Hence $[W,X]$ lies in the distribution $p \mapsto K^*(p)$. Therefore, by the Frobenius theorem, it follows that the distribution is integrable on open sets of M on which $\dim K^*(\rho)$ is constant. The second statement is immediate.

3. Ricci curvature and Ricci $*$ curvature

For any almost Hermitian manifolds, there are two useful contractions of the curvature tensor. Let $\{E_1,\dots,E_n,E_{n+1},\dots,E_{2n}\}$ be with $E_{n+k} = JE_k$, $1 \leq k \leq n$, a local frame field. We define linear transformations ρ (Ricci curvature) and ρ^* (Ricci $*$ curvature) by [5]

$$\langle \rho X,Y\rangle = \sum_{i=1}^{2n} R_{XE_iYE_i},$$

$$\langle \rho^* X,Y\rangle = \frac{1}{2}\sum_{i=1}^{2n} R_{XJYE_iJE_i}.$$

If M is a AH_1-manifold, then $\rho = \rho^*$.

Now, we define for all AH-manifold, a new operator B by

$$\langle BX,Y\rangle = 2\pi\gamma_1(X,JY) = \sum_{i=1}^{n}\left\{R_{XJYE_iJE_i} - \frac{1}{2}\langle(\nabla_X J)E_i, J(\nabla_{JY}J)E_i\rangle\right\},$$

where γ_1 is the first Chern class of M.

If M is a Kähler manifold, then $B = \rho$.

Proposition 3.1. Let M be a QK_3-manifold (H_3-manifold); then

$$(3.1)\qquad \langle(\rho^*-B)X,Y\rangle = \pm\frac{1}{2}\sum_{i=1}^{n}\langle(\nabla_X J)E_i,(\nabla_Y J)E_i\rangle$$

for $X,Y \in \mathfrak{X}(M)$. In particular, if M is a QK_1-manifold (H_1-manifold), we have

$$(3.2)\qquad \langle(\rho - B)X,Y\rangle = \pm\frac{1}{2}\sum_{i=1}^{n}\langle(\nabla_X J)E_i,(\nabla_Y J)E_i\rangle .$$

The proof is immediate.

In [5], it is proved that for all AH_3-manifolds we have

$$\rho \circ J = J \circ \rho \quad \text{and} \quad \rho^* \circ J = J \circ \rho^*$$

Proposition 3.2. Let M be a QK_3-manifold. Then $B \circ J = J \circ B$.

Proof. We observe that

$$\langle BJX,Y\rangle = \sum_{i=1}^{n} \left\{ R_{XYE_iJE_i} - \frac{1}{2} \langle (\nabla_X J)E_i, J(\nabla_Y J)E_i\rangle \right\} ,$$

$$\langle JBX,Y\rangle = -\langle BX,JY\rangle = \sum_{i=1}^{n} \left\{ R_{XYE_iJE_i} - \right.$$

$$\left. - \frac{1}{2} \langle (\nabla_X J)E_i, J(\nabla_Y J)E_i\rangle \right\} .$$

Now the proposition is immediate.

Proposition 3.3. Let M be a QK_1-manifold. Then

(3.3)

1) $\langle (\nabla_U(\rho-B))X,X\rangle = \langle (\rho-B)X, (\nabla_U J)JX\rangle ,$

2) $2\langle (\nabla_U(\rho-B))X,Y\rangle = \langle (\rho-B)X, (\nabla_U J)JY\rangle + {} + \langle (\rho-B)Y, (\nabla_U J)JX\rangle ,$

3) $(\nabla_U(\rho-B))X + (\nabla_{JU}(\rho-B)JX = 0$

for $U,X,Y \in \mathfrak{X}(M)$.

Proof. We have

$$4\langle (\rho-B)X,X\rangle = \sum_{i=1}^{2n} \| (\nabla_X J)E_i \|^2 . \tag{3.4}$$

Differentiating (3.4) and using (2.1) we obtain

$$(3.5) \quad \langle(\nabla_U(\rho-B))X,X\rangle = \frac{1}{4}\sum_{i=1}^{2n}\langle(\nabla_X J)E_i,(\nabla_{(\nabla_U J)JX}J)E_i\rangle.$$

Now 1) follows from (3.5). 2) is a simple consequence of 1) and 3) is immediate.

4. Decomposition of quasi-Kähler manifolds

Theorem 4.1. Let M be a QK_3-manifold (H_3-manifold) and $p \in M$. Denote by $P_1(p),\dots,P_r(p)$ the eigenspaces of ρ^*-B corresponding to the nonzero eigenvalues of ρ^*-B and put $H(p) = \{X \in T_p(M):(\rho^*-B)X = 0\}$. Then

a) $H(p) = \{X \in T_p(M):(\nabla_X J) = 0\}$,

b) On any open subset of M on which $\dim H(p)$ is constant, the distribution $H(p)$ is integrable.

c) Furthermore, we can decompose the tangent space as a direct sum

$$T_p(M) = H(p) \oplus P_1(p) \oplus \dots \oplus P_r(p).$$

In particular, if M is a QK_1-manifold (H_1-manifold) we have a similar result with ρ.

Proof. The first statement follows from (3.1). It is proved in [6] that $p \mapsto H(p)$ is integrable. The last statement is a well-known fact in linear algebra.

Definition 4.2. Let M be a QK_1-manifold. We call M strict provided that for all $p \in M$ and $X \in T_p(M)$ with $X \neq 0$ we have $(\nabla_X J) \neq 0$ ([5]).

Let M be a simply connected QK_1-manifold and $M = M^0 \times M^1 \times \dots \times M^q$ de Rham's decomposition of M. Here M^0 denotes a Euclidean space, and $M^1,\dots,M^q$ are ireducible Riemannian manifolds.

We form a new decomposition $M = M^k \times M^s$ from de Rham's decomposition. Here M^k denotes the product of all factors

M^i which have the following property: $(\nabla_X J) = 0$ for all $X \in \mathfrak{X}(M^i)$. Then M^s is the product of the remaining factors.

Similarly as in [5], we shall make the following assumption: $\rho - B$ is parallel on M. By (3.3), 3) this is quite reasonable.

L e m m a 4.3. On each factor M^i of de Rham's decomposition of M there exists a constant λ_i such that $(\rho-B)X = \lambda_i X$ for $X \in \mathfrak{X}(M^i)$.

For the proof we use a method similar to that in [5].

L e m m a 4.4. $M^i \subset M^k$ if and only if $\lambda_i = 0$.

P r o o f . If $M^i \subset M^k$, $(\nabla_X J) = 0$ for all $X \in \mathfrak{X}(M^i)$ then $(\rho-B)X = 0$ and $\lambda_i = 0$.

L e m m a 4.5. If $p \in M^k$, then $T_p(M^k) = H(p)$.

P r o o f . If $X \in T_p(M)$ then $X = \sum_{i=1}^{q} X_i$, where X_i is the component of X in $T_p(M^i)$. Suppose $X \in T_p(M^k)$. Then $(\nabla_X J) = \sum_{M^i \subset M^k} (\nabla_{X_i} J) = 0$ so that $X \in H(p)$. Conversely, if $H \in H(p)$, then

$$0 = (\rho-B)X = \sum_{M^i \subset M^s} \lambda_i X_i .$$

Thus $X_i = 0$ for $M^i \subset M^s$ and $X \in T_p(M^k)$.

L e m m a 4.6. M^k is a Kähler submanifold of M and M^s is a strict QK_1-submanifold of M.

P r o o f . That M^k and M^s are almost complex submanifolds of M follows from the fact that $(\rho-B)\circ J = J \circ (\rho-B)$. From [2], M^k is Kählerian and M^s is strict QK_1.

T h e o r e m 4.7. Let M be a QK_1-manifold, that is not Kählerian. Then $\rho - B$ is parallel if and olny if for any $U \in \mathfrak{X}(M)$, $(\nabla_U J)$ is closed for the spectrum of $\rho - B$. In particular if $\rho - B$ has only one eigenvalue then $\rho - B$ is parallel.

P r o o f . By (3.3), 2) we get

$$2\langle(\nabla_U(\rho-B))X,Y\rangle = \langle(\rho-B)(\nabla_U J)X,JX\rangle - \langle(\nabla_U J)(\rho-B)X,JY\rangle .$$

Thus

$$2(\nabla_U(\rho-B))X = (\rho-B)(\nabla_{JU}J)X - (\nabla_{JU}J)(\rho-B)X.$$

Now, the theorem is immediate.

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UNIVERSITY OF GRANADA AND UNIVERSITY OF VALENCIA, SPAIN
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