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REMARKS ON FUNCTOR CATEGORIES

Let $\underline{C}$ be a skeletally small (pre)additive category. We recall that $\underline{C}$-Mod is the category of $\underline{C}$-modules, i.e. the category of covariant additive functors from $\underline{C}$ to the category of abelian groups. We have also a natural morphism:

$$f_M:\Big(\prod_{t\in I} Q_t\Big)\otimes M \longrightarrow \prod_{t\in I}(Q_t\otimes M),$$

where $X\otimes N$ is a tensor product of modules over the category $\underline{C}$ (for the definition see below).

Let R be a (not necessarily commutative) ring with unity. We know that for the category of unitary modules over the ring R the following conditions are equivalent:

1) right projective modules are injective
2) left projective modules are injective
3) right injective modules are projective
4) left injective modules are projective.

Each of these conditions defines a class of quasi-Frobenius rings (see [2]). The situation is different in the case of the category $\underline{C}$-Mod. Harada in [3] gave examples from which we may conclude that no implication between points 1) and 3) holds. We show for modules over $\underline{C}$ that conditions 1) and 2) imply 3) and 4), and that the inverse implication holds, too. In the proof of this fact we use a characterisation of locally Noetherian, coherent and perfect functor categories in terms of the morphism f_M. Some of them where studied in [4], [7], and [10].

We know that $\{h^X\}_{X\in \underline{C}}$, where $h^X = Hom_{\underline{C}}(X,-)$ denotes Yoneda functor, is the family of small projective generators of $\underline{C}$-Mod. Every finitely generated free module is a finite coproduct of Yoneda functors.

Now we recall the definition of tensor product of modules over $\underline{C}$. Let N be a $\underline{C}^{op}$-module and M a $\underline{C}$-module. $N \otimes_{\underline{C}} M = \Big(\bigoplus_{X\in \underline{C}} (NX \otimes_Z MX)\Big)/_U$, where U is the subgroup generated by elements $N(f)y \otimes x - y \otimes M(f)x$ for all $f \in Hom_{\underline{C}}(X,Y), x \in MX, y \in NY$. It is well known and easy to check that $N \otimes_{\underline{C}} h^X \simeq NX$, $h_X \otimes_{\underline{C}} M \simeq MX$ (see [5]).

For a given family $\{Q_t\}_{t\in I}$ of $\underline{C}^{op}$-modules and a $\underline{C}$-module M we define a homomorphism

$$f_M:\Big(\prod_{t\in I} Q_t\Big) \otimes_{\underline{C}} M \longrightarrow \prod_{t\in I} (Q_t \otimes_{\underline{C}} M)$$

to be the unique for which the diagram

$$\begin{array}{ccc} \Big(\prod_{t\in I} Q_t\Big) \otimes M & \xrightarrow{f_M} & \prod_{t\in I} (Q_t \otimes M) \\ \Big\downarrow \Pi_t \otimes 1 & & \Big\downarrow \Pi'_t \\ Q_t \otimes M & = & Q_t \otimes M \end{array}$$

is commutative for each $t \in I$, where Π_t, Π'_t are canonical projections.

We start with the following useful result (see [4]).

P r o p o s i t i o n 1:

a) If M is free (resp. finitely generated projective) $\underline{C}$-module, then f_M is a monomorphism (resp. an isomorphism).

b) A $\underline{C}$-module M is finitely generated iff f_M is an epimorphism for all families $\{Q_t\}_{t\in I}$ (even only for families of Yoneda functors).

c) A $\underline{C}$-module M is finitely presented iff f_M is an isomorphism for all families $\{Q_t\}_{t\in I}$ (even only for families of Yoneda functors).

P r o o f . a) easy.

b) Necessity. If we have an epimorphism $\varphi: F_1 = \bigoplus_{i=1}^{n} h^{X_i} \longrightarrow M$ then from commutativity of the diagram

$$\begin{array}{ccc} \left(\prod_{t\in I} Q_t\right)\otimes F_1 & \xrightarrow{f_{F_1}} & \prod_{t\in I}(Q_t\otimes F_1) \\ \downarrow & & \downarrow \\ \left(\prod_{t\in I} Q_t\right)\otimes M & \xrightarrow{f_M} & \prod_{t\in I}(Q_t\otimes M) \\ & & \downarrow \\ & & 0 \end{array}$$

we conclude that f_M is an epimorphism.

Sufficiency. Take $I = \dot{\bigcup}_{X\in \underline{C}} MX$ (disjoint union) and $Q_t = h_X$ for $t \in MX$. For each finitely generated submodule $N \overset{j}{\hookrightarrow} M$ we have a commutative diagram

$$\begin{array}{ccccc} \left(\prod_{t\in I} Q_t\right)\otimes M & \xrightarrow{f_M} & \prod_{t\in I}(Q_t\otimes M) & \xrightarrow{g} & \prod_{X\in\underline{C}}\prod_{t\in MX}(MX)_t \\ \uparrow & & \uparrow \prod_t (1\otimes j) & & \uparrow \\ \left(\prod_{t\in I} Q_t\right)\otimes N & \xrightarrow{f_N} & \prod_{t\in I}(Q_t\otimes N) & \xrightarrow{g'} & \prod_{X\in\underline{C}}\prod_{t\in MX}(NX)_t, \end{array}$$

where $(MX)_t = MX$, $(NX)_t = NX$ for $t \in MX$, g and g' are isomorphisms induced by isomorphisms $Q_t\otimes M \simeq MX$, $Q_t\otimes N \simeq NX$ for $t \in MX$ and the right hand morphism is induced by natural inclusions $NX \hookrightarrow MX$. Let $a = \langle\langle t\rangle_{t\in MX}\rangle_{X\in\underline{C}} \in \prod_{X\in\underline{C}}\prod_{t\in MX}(MX)_t$. By the assumption f_M is an epimorphism;

then $a = gf_M(b)$ for same $b \in (\Pi Q_t) \otimes M$. Then one can find a finitely generated submodule $N \xrightarrow{j} M$ and an element $c \in \left(\prod_{t\in I} Q_t\right) \otimes N$ such that $b = (1 \otimes j)c$. The commutativity of the above diagram with such an N implies that $MX \subseteq NX$ for each $X \in \underline{C}$ and therefore $M = N$ is finitely generated.

c) We assume that f_M is an isomorphism. From a) we know that M is finitely generated. Hence we have an exact sequence $0 \to K \to F \to M \to 0$, where F is a finitely generated free $\underline{C}$-module. This sequence induces the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\left(\prod_{t\in I} h_{X_t}\right) \otimes K & \longrightarrow & \left(\prod_{t\in I} h_{X_t}\right) \otimes F & \longrightarrow & \left(\prod_{t\in I} h_{X_t}\right) \otimes M & \longrightarrow & 0 \\
\downarrow f_K & & \simeq\downarrow f_F & & \simeq\downarrow f_M & & \\
0 \longrightarrow \prod_{t\in I} (h_{X_t} \otimes K) & \longrightarrow & \prod_{t\in I} (h_{X_t} \otimes F) & \longrightarrow & \prod_{t\in I} (h_{X_t} \otimes M) & \longrightarrow & 0.
\end{array}$$

Since f_K is an epimorphism then from a) we have that K is a finitely generated $\underline{C}$-module. The inverse implication can be proved similarly.

We call a category $\underline{C}$-Mod locally coherent if every finitely generated submodule of a free module is finitely presented, or equivalently, every finitely generated submodule of any h^X is finitely presented (see [6]).

If $\underline{C}$-Mod is locally Noetherian then, of course, it is locally coherent.

P r o p o s i t i o n 2. $\underline{C}$-Mod is locally Noetherian iff f_M is a monomorphism for all M and all families $\{Q_t\}_{t\in I}$ of flat objects in $\underline{C}^{op}$-Mod (see [10] Prop.2.4).

P r o o f . Let $\{Q_t\}_{t\in I}$ be a family of flat $\underline{C}^{op}$-modules and let M be a $\underline{C}$-module. By the assumption, M is a union of Noetherian submodules $M_k \subseteq M$. Then for each k we have a commutative diagram

$$\begin{array}{ccc} \left(\prod Q_t\right) \otimes M_k & \xrightarrow{f_{M_k}} & \prod(Q_t \otimes M_k), \\ \downarrow & & \downarrow \varphi_k \\ \left(\prod Q_t\right) \otimes M & \xrightarrow{f_M} & \prod(Q_t \otimes M), \end{array}$$

where f_{M_k} is an isomorphism and φ_k is a monomorphism since Q_t are flat. Then it follows that f_M is a monomorphism.

Conversely, assume that $\underline{C}$-Mod is locally Noetherian. We shall show that every $\underline{C}$-submodule $\mathfrak{a}$ of any h^X, $X \in \underline{C}$, is finitely generated. The exact sequence $0 \to \mathfrak{a} \to h^X \to h^X/\mathfrak{a} \to 0$ induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & (\prod h_{X_t}) \otimes \mathfrak{a} & \longrightarrow & (\prod h_{X_t}) \otimes h^X & \longrightarrow & (\prod h_{X_t}) \otimes h^X/\mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow f_{\mathfrak{a}} & & \downarrow \simeq & & \downarrow f_{h^X/\mathfrak{a}} & & \\ 0 & \longrightarrow & \prod(h_{X_t} \otimes \mathfrak{a}) & \longrightarrow & \prod(h_{X_t} \otimes h^X) & \longrightarrow & \prod(h_{X_t} \otimes h^X/\mathfrak{a}) & \longrightarrow & 0, \end{array}$$

where $f_{h^X/\mathfrak{a}}$ is an isomorphism by the assumption. Hence $f_{\mathfrak{a}}$ is an epimorphism and by Proposition 1. $\mathfrak{a}$ is finitely generated.

Recall that a category is said to be perfect if each of its objects has a projective cover.

P r o p o s i t i o n 3. The following three conditions are equivalent:

i) $\underline{C}$-Mod is perfect,

ii) every flat $\underline{C}$-module is projective,

iii) f_M is a monomorphism for all flat $\underline{C}$-modules M.

This is a part of Theorem 5.4 in [7].

P r o p o s i t i o n 4. If in the category $\underline{C}^{op}$-Mod every injective object is flat and every flat $\underline{C}$-module is a directed union of finitely presented modules then the category $\underline{C}$-Mod is perfect.

P r o o f . Let M be a flat $\underline{C}$-module and suppose $M = \bigcup_{k\in K} M_k$ is a directed union of finitely presented modules M_k. By proposition 3 it is sufficient to show that f_M is a monomorphism. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & (\prod Q_t)\otimes M & \longrightarrow & (\prod \hat{Q}_t)\otimes M \\ & & \downarrow f_M & & \downarrow g_M \\ 0 & \longrightarrow & \prod(Q_t\otimes M) & \longrightarrow & \prod(\hat{Q}_t\otimes M), \end{array}$$

where $\hat{Q}_t$ is an injective envelope of Q_t. Now it is enough to show that g_M is a monomorphism. Suppose $\operatorname{Ker} g_M \neq 0$. Then there exists k such that the composed map $(\prod \hat{Q}_t)\otimes M_k \longrightarrow (\prod \hat{Q}_t)\otimes M \longrightarrow \prod(\hat{Q}_t\otimes M)$ has a non-trivial kernel. Since each of the modules $\hat{Q}_t$ is flat, then from commutativity of the diagram

$$\begin{array}{ccccc} & & (\prod \hat{Q}_t)\otimes M_k & \longrightarrow & (\prod \hat{Q}_t)\otimes M \\ & & \simeq\downarrow g_{M_k} & & \downarrow \\ 0 & \longrightarrow & \prod(\hat{Q}_t\otimes M_k) & \longrightarrow & \prod(\hat{Q}_t\otimes M) \end{array}$$

where g_{M_k} is an isomorphism, we get contradiction.

C o r o l l a r y 5. If all injective $\underline{C}^{op}$-modules are flat and the category $\underline{C}$-Mod is locally Noetherian then $\underline{C}$-Mod is perfect.

A $\underline{C}$-module M is FP-injective if $\operatorname{Ext}^1_{\underline{C}}(P,M) = 0$ for all finitely presented $\underline{C}$-module P (see [8], [11]).

Let G be a $\underline{C}$-module. We define the character $\underline{C}^{op}$-module G^+ by formula $G^+(X) = \operatorname{Hom}_{Ab}(G(X), {}^{Q}/_{Z})$ (see [8]). From the Yoneda equivalence it is easy to check that the morphism

$$\chi_{h^X}\colon G^+\otimes h^X \longrightarrow (h^X, G)^+$$

defined on generators by

$$\chi(g^+ \otimes f)(F) = g^+(F(f)),$$

where $g^+ : G(Y) \longrightarrow Q/_Z$, $f: X \longrightarrow Y$, $F : h^X \longrightarrow G$, is an isomorphism. We remark that the homomorphism χ_M is an epimorphism if M is a finitely generated $\underline{C}$-module, and it is an isomorphism if M is a finitely presented $\underline{C}$-module.

L e m m a 6. Suppose that $\underline{C}^{op}$-Mod is locally coherent. The following conditions are equivalent:

i) Every injective $\underline{C}$-module is flat.

ii) Every flat $\underline{C}^{op}$-module is FP-injective.

iii) Every finitely generated free $\underline{C}^{op}$-modules is FP-injective, or equivalently, each of the Yoneda functors h_X is FP-injective.

For modules over a ring this lemma is due to Colby [1], theorem 1 and Würfel [11], satz 4.3..

P r o o f . i) $\longrightarrow$ ii). Consider a monomorphism $u : M \longrightarrow F_1 = \bigoplus_{k=1}^{n} h_{Y_k}$, where M is finitely generated. By our assumption M is finitely presented. Further let F be a flat $\underline{C}^{op}$-module. We want to have that the natural morphism $h_F(u):(F_1,F) \longrightarrow (M,F)$ is epimorphic, or equivalently, that $h_F(u)^+:(M,F)^+ \longrightarrow (F_1,F)^+$ is a monomorphism, where $h_F(u)^+$ is the natural homomorphism of groups of characters (see [8], Prop.2.6). Since, by the assumption, F^+ is flat we have the commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow & M \otimes F^+ & \longrightarrow & F_1 \otimes F^+ \\
 & \chi_M \big\downarrow \simeq & & \big\downarrow \simeq \\
h_F(u)^+ : & (M,F)^+ & \longrightarrow & (F_1,F)^+
\end{array}$$

and ii) follows. The implication

ii) $\longrightarrow$ iii) is obvious.

iii) $\longrightarrow$ i). Let Q be an injective $\underline{C}$-module and consider an epimorphism $F \xrightarrow{p} Q^{+} \longrightarrow 0$, where $F = \bigoplus_{t \in I} h_{X_t}$ is a free $\underline{C}^{op}$-module. We derive the monomorphism

$$Q^{++} \xrightarrow{p^{+}} F^{+} \simeq \prod_{t \in I} h^{+}_{X_t}.$$

Hence, by the assumption, Q is a direct summand of F^{+} because p^{+} is a monomorphism and there is a natural embedding $Q \hookrightarrow Q^{++}$. Now, to show that Q is flat, it is enough (from the coherentness of $\underline{C}^{op}$-Mod) to show that h^{+}_{X} is a flat $\underline{C}$-module for every object X (see [4], Satz.4.).

Let $\mathfrak{a}$ be a finitely generated (i.e. finitely presented) submodule of the free $\underline{C}^{op}$-module h_Y. Then the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathfrak{a} \otimes h^{+}_{X} & \longrightarrow & h_Y \otimes h^{+}_{X} \\ & & {\scriptstyle \chi_{\mathfrak{a}}} \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & (\mathfrak{a}, h_X)^{+} & \longrightarrow & (h_Y, h_X)^{+} \end{array}$$

implies condition i).

Now we are in a position to prove the following theorem related to Harada's results in [3].

T h e o r e m 7. Let $\underline{C}$ be a skeletally small (pre)additive category and let $\underline{C}$-Mod (resp. $\underline{C}^{op}$ Mod) denote the category of additive functors from the category $\underline{C}$ (resp. $\underline{C}^{op}$) to the category of abelian groups. Then the following conditions are equivalent:

i) Every injective $\underline{C}$-module and every injective $\underline{C}^{op}$-module is projective.

ii) Every projective $\underline{C}$-module and every projective $\underline{C}^{op}$-module is injective.

P r o o f . If we assume i) then from the well-known Faith theorem ([2] Theorem 1.1) $\underline{C}$-Mod and $\underline{C}^{op}$-Mod are locally Noetherian. From Lemma 6 we have that h_X, h^X are FP-injective and therefore they are injective. Now since these two categories are locally Noetherian we obtain the statement ii).

ii) ⟶ i). By [3], Theorem 3 we know that $\underline{C}$-Mod and $\underline{C}^{op}$-Mod are locally finite and hence perfect and locally coherent. Therefore i) is a consequence of Lemma 6.

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