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# ON THE OPTIMAL CHOICE FROM A SAMPLE DRAWN FROM A KNOWN DISTRIBUTION

## 1. Introduction

Although the general theory of optimal stopping is well developed ([2]) many particular problems interesting in practice are difficult to solve. One of them is so called finite case. In many papers the problem of recognizing the maximum of a stochastic sequence with unknown distribution has been investigated; so called "secretary problem" in [3], [4], and "secretary problem with interview cost" in [1], [5]. In [4] the case of a known distribution is considered too.

In our paper we shall investigate two problems: recognizing the maximum of a sequence of identicaly distributed random variables with a known distribution law and the second problem is maximizing the drawn value. Our results generalize some results of J.Gilbert and F.Mosteller contained in [4].

## 2. Formulation of the problem

Suppose we have a population with a known continuous distribution law. We draw a sample of, at most, N elements out of it. A certain payoff is connected with each drawing to the sample. We want to choose one element out of this sample to maximize the mean payoff. We draw elements to our sample one by one and after each drawing we may either keep the drawn element or cast it off. The payoff depends on the last element but may depends on previous elements too. We have to keep some element by the N-th at the very latest.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $x_1, \dots, x_N$ be a finite sequence of independent identicaly distributed random variables on this space. For $n = 1,2,\dots,N$ let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $x_1,\dots,x_n$ and let $\varphi_n(x_n) = \varphi_n(x_1,\dots,x_n)$ be the n-th payoff. We suppose that $\varphi_n(x_n)$ is $\mathcal{F}_n$-measurable. The problem is to find the optimal stopping rule for the stochastic sequence $[\varphi_n(x_n), \mathcal{F}_n]_{n=1}^{N}$. This is so called "finite case"; the solution always exists and can be obtained by the backward induction.

In our paper we consider the following two cases most interesting in practice:

a) $\varphi_n(x_n) = x_n - c_n, \quad 0 \leq c_1 \leq \dots \leq c_N$

$c_n$ plays the role of the payment for the first $n$ drawings;

b) $\varphi_n(x_n) = E[\psi_n(x_n) | \mathcal{F}_n]$, where

$$\psi_n(x_n) = \begin{cases} 1 - \alpha n & \text{if} \quad x_n \geq x_i \quad \text{for} \quad i = 1,2,\dots,N \\ -\alpha n & \text{otherwise,} \end{cases}$$

in other words we are interested in recognizing the maximum of $\{x_n\}$.

## 3. The case $\varphi_n(x_n) = x_n - c_n$

Let $F$, $f$, $m$ denote the distribution function, the density and the mean of the population respectively. Let $C_n^N$ be the class of these stopping rules $t$ which satisfy $n \leq t \leq N$. Denote by $E^n$ the mean of the payoff for the stopping rule optimal in $C_{N-n+1}^n$ i.e.

$$E^n = \sup\left\{ E\varphi_t(x_t) \;\middle|\; t \in C_{N-n+1}^N \right\}.$$

Definition. The n-th decision number $d_n$ is the smallest number satisfying the inequality $\varphi_n(d_n) \geq E^{N-n}$. The optimal stopping rule will depend on the sequence

$d_i = E^{N-i} + c_i$; if the n-th drawn element is not smaller than $d_n$ we should keep it otherwise we should cast it off and continue the drawing. The above stopping rule is optimal by Theorem 3.2 in [2]. Since we have to keep $x_N$ if we are in the n-th step, $d_N = -\infty$. $E^1 = m - c_N$, so $d_{N-1} = m + c_{N-1} - c_N$. For every $n \geqslant 1$

$$E^{n+1} = P(x_{N-n} \geqslant d_{N-n})E(x_{N-n} - c_{N-n} | x_{N-n} \geqslant d_{N-n}) + E^n \cdot P(x_{N-n} < d_{N-n}),$$

where $d_{N-n} = E^n + c_{N-n}$. After easy computations we get

$$(*) \quad \begin{cases} d_{k-1} = \int_{-\infty}^{+\infty} \max(x, d_k) f(x) dx + c_{k-1} - c_k & k = 2, \dots, N-1 \\ d_{N-1} = m + c_{N-1} - c_N \\ E^N = \int_{-\infty}^{+\infty} \max(x, d_1) f(x) dx - c_1. \end{cases}$$

In the case of a truncated distribution (i.e. if for some $a$ the density $f$ fulfils the conditions $f(x) = 0$ for $x < a$ and $f(x) > 0$ for $a < x < a+\varepsilon$ for same positive $\varepsilon$) we put $d_k = a$ whenever $d_k$ computed as in (*) is smaller than $a$.

C o r o l l a r y . In the case of a truncated distribution if for every $i$ $c_i - c_{i-1} \geqslant m - a$ then the optimal stopping rule is "keep the first drawn value".

Now let us consider the same problem but with the possibility of keeping more than one, say $k$, values. Our payoff is the sum of the payoff's for the kept values.

Let us introduce the following notations:

$A_n^{N,k}$ - the mean payoff for the stopping rule optimal in the class $C_{N-n+1}^N$ when we still have to choose $k$ elements, $N \geqslant n \geqslant k$;

$a_n^{N,k}$ - the n-th decision number in a k-choice game of the lenght $N$.

It is easy to see that $A_n^{N,1} = E^n$ and $a_n^{N,1} = d_n$.
The optimal stopping rule for this problem depends on k sequences of decision numbers. After the first choice the problem reduces to the problem with k-1 choices. We shall define the solution inductively. For k=1 the solution is given by (*). Suppose we have our problem solved for k-1 choices than

$$\begin{cases} A_k^{N,k} = k\cdot m - c_N - c_{N-1} - \dots - c_{N-k+1}, \\ A_{k+1}^{N,k}(a) = P(x_{N-k} \geqslant a)\left[E(x_{N-k} - c_{N-k} \mid x_{N-k} \geqslant a) + A_k^{N,k-1}\right] + \\ \qquad + \left[1 - P(x_{N-k} \geqslant a)\right] A_k^{N,k}, \\ A_{k+1}^{N,k} = \max\limits_a A_{k+1}^{N,k}(a). \end{cases}$$

Differentiating $A_{k+1}^{N,k}(a)$ with respect to a and equating the result to zero we get

$$\begin{cases} a_{N-k}^{N,k} = A_k^{N,k} - A_k^{N,k-1} + c_{N-k} \\ A_{k+1}^{N,k} = A_{k+1}^{N,k}\left(a_{N-k}^{N,k}\right) . \end{cases}$$

Repeating the above procedure for $n = k,\dots,N-1$ we get

$$\begin{cases} a_{N-n}^{N,k} = A_n^{N,k} - A_n^{N,k-1} + c_{N-n} \\ A_{n+1}^{N,k} = A_n^{N,k} + \int\limits_{a_{N-n}^{N,k}}^{+\infty} (x - c_{N-n}) f(x)dx - (A_n^{N,k} - A_n^{N,k-1}) \int\limits_{a_{N-n}^{N,k}}^{+\infty} f(x)dx. \end{cases}$$

$A_N^{N,k}$ is the mean payoff for the whole game.

Proposition. If a distribution of a population is truncated at $a$ and $c_i - c_{i-1} \geqslant m - a$ for every $i$ than the optimal stopping rule in the $k$-choice game is: "keep the first $k$ elements".

## 4. The case $\varphi_n(x_n) = E[\psi_n(x_n)|\mathcal{F}_n]$

In this case we want to know where the maximum actually is, and not how big it is. So, if we have a population with some fixed distribution law ($F$ - distribution function) our problem can be reduced to the problem of finding the maximum of a sequence of random variables having the uniform distribution on the interval $\langle 0,1\rangle$. Indeed, the distribution function $F$, mapping the maximum on the maximum, transforms our random variable to a random variable with uniform distribution on the interval $\langle 0,1\rangle$. The advantage of this transformation is that in the case of the uniform distribution on $\langle 0,1\rangle$ the drawn value is the value of the distribution function. From now on we shall assume that $x_1, x_2, \ldots, x_N$ is a sequence of random variables uniformly distributed on the interval $\langle 0,1\rangle$. Notice that it is enough to consider the case $\alpha < 1$; for $\alpha \geqslant 1$ the optimal stopping rule is to stop after the first drawing. Indeed, if we did not stop in the first step our payoff would be at most $1-2\alpha$, otherwise it would be at least $-\alpha$.

Now we shall try to answer the question when it is reasonable to stop the stochastic sequence $\{\varphi_n(x_n), \mathcal{F}_n\}$ for $\alpha \in \langle 0,1)$. Let $E^n(x)$ denote the mean payoff for the stopping rule optimal in the class $C^N_{N-n+1}$ with the additional assumption that $x$ is the maximum of values drawn in the first $N-n$ steps. $E^n(x)$ is a decreasing function of $x$. If we have already drawn values $x_1, \ldots, x_n$ then it is reasonable to keep $x_n$ only in two cases:

a) $x_n = \max\limits_{i \leqslant n} x_i$ and $E^{N-n}(x) \leqslant \varphi_n(x)$, $x = \max(x_1, \ldots, x_n)$

b) $x_n < x = \max\limits_{i \leqslant n} x_i$, but $-\alpha n \geqslant E^{N-n}(x)$

(the cost is so big that even the optimal strategy will not help).
For $n = 1,2,\dots,N-1$ let $d_n$ be the smallest $x \in \langle 0,1\rangle$ such that $E^{N-n}(x) \leqslant \varphi_n(x)$ and $D_n$ be the smallest $x \in \langle 0,1\rangle$ such that $E^{N-n}(x) \leqslant -\alpha n$. For $n=N$ we put $d_N = D_N = 0$. Observe that $D_n \geqslant d_n$ for each $n$. By [2,th.3.2] the optimal stopping rule $t$ is of the form

$$t = \inf\{n \leqslant N: \max(x_1,\dots,x_n) = x_n \ \& \ x_n \geqslant d_n \text{ or } \max(x_1,\dots,x_n) \geqslant D_n\}.$$

We shall improve this result, namely we shall prove that $t$ is of the form

$$t = \inf\{n \leqslant N: \max(x_1,\dots,x_n) = x_n \ \& \ x_n \geqslant d_n\}.$$

Since $D_n \geqslant d_n$ it is enough to prove that for every $\alpha$ decision numbers $D_n(\alpha)$ form a nondecreasing sequence. Indeed, it is easy to see that conditions $\max(x_1,\dots,x_i) \geqslant D_i \geqslant D_{i-1}$ and $\max(x_1,\dots,x_{i-1}) < D_{i-1}$ imply $\max(x_1,\dots,x_i) = x_i$ and $x_i \geqslant d_i$.

P r o p o s i t i o n . For every $\alpha$ decision numbers $D_i(\alpha)$ $i = 1,\dots,N-1$ form a non-decreasing sequence.

P r o o f :

1) Notice that $D_{N-i} \leqslant D_{N-i+1}$ if and only if the following implication holds: if $\max(x_1,\dots,x_{N-i}) = D_{N-i+1}$ then stop in the $N-i$ step. Thus for

$$\alpha \geqslant \frac{1-D^{i}_{N-i+1}(\alpha)}{i} \quad \text{we have} \quad D_{N-i}(\alpha) \leqslant D_{N-i+1}(\alpha) \qquad i=2,\dots,N-1.$$

Suppose that $\max(x_1,\dots,x_{N-i}) = D_{N-i+1}$. If we do not stop in the $N-i$ step then we shall stop in $N-i+1$ step with the probability 1. So to have $D_{N-i} \leqslant D_{N-i+1}$ it is enough to have

$$-\alpha(N-i) \geqslant \frac{1-D^i_{N-i+1}(\alpha)}{i} - \alpha(N-i+1) = E^i(D_{N-i+1}(\alpha)).$$

The last equality holds because

$$P(x_{N-i+1} > x_1,\ldots,x_{N-i+1} > x_N \mid \max(x_1,\ldots,x_{N-i}) = D_{N-i+1}(\alpha)) =$$

$$= \int_{D_{N-i+1}}^{1} x^{i-1}dx = \frac{1-D^i_{N-i+1}(\alpha)}{i}.$$

2) If $d_{N-i+1} \leqslant D_{N-i}$, then

$$\left[D_{N-i}(\alpha) \leqslant D_{N-i+1}(\alpha) \Longrightarrow \alpha \geqslant \frac{1-D^i_{N-i}(\alpha)}{i}\right].$$

3) If $\alpha \leqslant \frac{1}{i}$ then $d_{N-i+1}(\alpha) \leqslant D_{N-i}(\alpha)$ .

Proofs of 2) and 3) are similar to that of 1).

4) $D_{N-i}(\alpha) \leqslant D_{N-i+1}(\alpha) \Longrightarrow \alpha \geqslant \frac{1-D^i_{N-i}(\alpha)}{i}$ .

For $\alpha \leqslant \frac{1}{i}$ 4) results from 2) and 3), and for $\alpha > \frac{1}{i}$ it is trivial.

5) For $x \in \langle 0,1\rangle$ $\frac{1-x^{i+1}}{i+1} \leqslant \frac{1-x^i}{i}$ .

6) $D_{N-i}(\alpha) \leqslant D_{N-i+1}(\alpha) \Longrightarrow D_{N-i-1}(\alpha) \leqslant D_{N-i}(\alpha)$,
results from 4), 5) and 1).

7) $D_{N-2}(\alpha) \leqslant D_{N-1}(\alpha) = 1-\alpha$ , results from 1).

The thesis follows easily from 7) and 6) (induction).

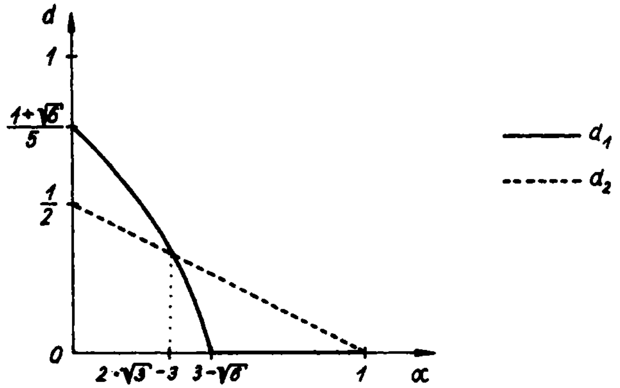
C o r o l l a r y . The optimal stopping rule is of the form: "keep the first $x_i$ such that $x_i = \max(x_1,\ldots,x_i)$ and $x_i \geqslant d_i$".

E x a m p l e . For N=3 we get

$$d_3 = 0$$

$$d_2 = \frac{1}{2}(1-\alpha)$$

$$d_1 = \begin{cases} 0 & 3-\sqrt{6} \leqslant \alpha < 1 \\ \sqrt{\frac{\alpha^2}{6} - \alpha + \frac{1}{2}} & 2\sqrt{3}-3 \leqslant \alpha < 3-\sqrt{6} \\ \frac{1}{5}(1-\alpha) + \frac{1}{5}\sqrt{\alpha^2 - 12\alpha + 6} & 0 \leqslant \alpha < 2\sqrt{3}-3 \end{cases}$$




The three decision numbers computed above are the last three in any game of the lenght at least three. This results from the fact that $d_n$ in the game of N drawings equals $d_{n+1}$ in the game of N+1 drawings. Even in the case N=3 it is rather arduous to compute $d_n$'s and requires solving out some quadratic equations. The quantity and degrees of equations increase with N. We shall show now some facts which enable us to find the optimal stopping rule in some cases.

P r o p o s i t i o n .

$$d_{N-i}(\alpha) \leqslant d_{N-i+1}(\alpha) \text{ if and only if } \alpha \geqslant \frac{1-d^{i}_{N-i+1}(\alpha)}{i} .$$

P r o o f . By the definition of $d_n$'s

$$d^i_{N-i}(\alpha) - \alpha(N-i) = E^i(d_{N-i}(\alpha))$$

$(\Longrightarrow)$

$$d^i_{N-i+1}(\alpha) - \alpha(N-i) \geqslant \frac{1-d^i_{N-i+1}(\alpha)}{i} - \left(1-d_{N-i+1}(\alpha)\right)\alpha(N-i+1) +$$

$$+ d_{N-i+1}(\alpha)\cdot E^{i-1}(d_{N-i+1}(\alpha))$$

$(\Longleftarrow)$ Suppose $d_{N-i} > d_{N-i+1}$, then

$$d^i_{N-i+1}(\alpha)-\alpha(N-i) < \frac{1-d^i_{N-i+1}(\alpha)}{i} - \left(1-d_{N-i+1}(\alpha)\right) \alpha(N-i+1) +$$

$+ d_{N-i+1}(\alpha)\, E^{i-1}\left(d_{N-i+1}(\alpha)\right)$ thus by the definition of $d_n$'s we get $\alpha < \frac{1-d^i_{N-i+1}(\alpha)}{i}$ .

C o r o l l a r y . If $d_{N-i}(\alpha) = 0$ then $d_{N-i-1}(\alpha) = 0$ for $i = 1,2,\dots,N-2$.

P r o o f . If the maximum for $i = 1,\dots,N-i$ is zero then $\frac{1}{i} - \alpha(N-i+1)$ is the mean payoff in the $N-i+1$ step. Thus $d_{N-i} = 0$ implies that $-\alpha(N-i) \geqslant E^i(0) \geqslant \frac{1}{i} - \alpha(N-i+1)$. Hence $\alpha \geqslant \frac{1}{i}$ . Suppose that $d_{N-i-1} > 0 = d_{N-i}$; then $\alpha < \frac{1-d^{i+1}_{N-i}}{i+1} = \frac{1}{i+1}$ . The contradiction proves that $d_{N-i-1} = 0$. Let us introduce the following notation:

$$\alpha_i = \left\{\inf\alpha \ : \ \alpha \in (0,1) \ \& \ \alpha = \frac{1-d^i_{N-i+1}(\alpha)}{i}\right\}.$$

By the continuity of the distribution law and by the facts $d_{N-i+1}(0) < 1$, $d_{N-i+1}(1) = 0$ the set in brackets is non-empty. So for $\alpha < \alpha_i$ we have

$$\alpha < \frac{1-d^{i}_{N-i+1}(\alpha)}{i} \quad \text{and} \quad d_{N-i}(\alpha) > d_{N-i+1}(\alpha).$$

The definition of $\alpha_i$ implies:

Proposition. The sequence $(\alpha_i)$ decreases i.e. $1 = \alpha_1 > \ldots > \alpha_N$.

Corollary. For $\alpha < \alpha_i$ $d_{N-i}(\alpha) \geqslant d_{N-i+1}(\alpha) \geqslant \ldots$ $\ldots \geqslant d_N$. In particular for $0 \leqslant \alpha \leqslant \alpha_{N-1}$ the decision numbers $d_i(\alpha)$ form a nonincreasing sequence.

Our last two propositions will generalize results obtained in [4] for $\alpha = 0$ to the case $\alpha > 0$. In proofs we shall limit ourselves to the part with a cost.

Proposition. Let $y_0 = \inf\left\{y \in (0,1) : y^{N-2} = \sum_{k=1}^{N-2} \frac{1}{k}\binom{N-2}{k} y^{N-k-2}(1-y)^k + \frac{1-y^{N-1}}{N-1} \cdot \frac{1-y^{N-2}}{1-y}\right\}$.

For $0 \leqslant \alpha \leqslant \alpha_{N-1} = \frac{1-y_0^N}{N-1}$ the numbers $d_{N-i}$ $i = 1,\ldots,N-1$ fulfill the equations

$$x^i = \sum_{k=1}^{i} \frac{1}{k}\binom{i}{k} x^{i-k}(1-x)^k - \alpha\,\frac{1-x^i}{1-x}.$$

Proof. Suppose we are in the N-i-th step and $x = x_{N-i} = \max(x_1,\ldots,x_{N-i}) = d_{N-i}$. We still have $i$ drawings at most. We shall stop in $N-i+k$ - th step with probability $x^{k-1}(1-x)$ for $k \leqslant i-1$ and in N-th step with probability $x^{i-1}$. This results from the fact that decision numbers do not increase, and so our loss equals

$$-\alpha\left[\sum_{k=1}^{i-1} x^{k-1}(1-x)(N-i+k) + Nx^{i-1}\right] = -\alpha\left[N-i + \frac{1-x^i}{1-x}\right].$$

Enclosing after [4] the part without the cost we get

$$E^i(x) = \sum_{k=1}^{i} \frac{1}{k}\binom{i}{k} x^{i-k}(1-x)^k - \alpha(N-i) - \alpha\left(\frac{1-x^i}{1-x}\right).$$

The mean payoff for the stopping in the N-i-th step is $x^i - \alpha(N-i)$. We get the thesis by equating obtained values.

R e m a r k . By [4] $d^i_{N-i+1}(0) \leq \frac{1}{2}$ for $N \leq 51$. Thus

$$\alpha_{N-1} = \frac{1-d_1^{N-1}(\alpha_{N-1})}{N-1} \geq \frac{1-d_1^{N-1}(0)}{N-1} \geq \frac{1}{2(N-1)}, \quad \text{for} \quad N \leq 51.$$

Hence for $\alpha \leq \frac{1}{2(N-1)}$, $N \leq 51$ we have the possibility of computting the decision numbers.

P r o p o s i t i o n . If the decision numbers form a nonincreasing sequence then the mean payoff for the stopping rule based on these numbers is given by:

$$E^N = \frac{1-d_1^N}{N} + \sum_{r=1}^{N-1}\left[\sum_{i=1}^{r}\left(\frac{d_i^r}{r} - \frac{d_i^N}{N}\right)\frac{1}{N-r} - \frac{d_{r+1}^N}{N}\right] - \alpha\left[1 + \sum_{k=1}^{N-1}\left(\sum_{i=1}^{k}\frac{d_i^k}{k}\right)\right].$$

P r o o f . (for the part with a cost).

Let $A_k$ denotes the event $\{$we stop in the k-th step$\}$. $A_k$ are disjoint, $P\left(\bigcup_{k=1}^{N} A_k\right) = 1$ and the mean cost equals $-\alpha\sum_{k=1}^{N} k\cdot P(A_k)$. Let $B_k = \bigcup_{i=k+1}^{N} A_i$, $k=0,1,\dots,N-1$. We have $-\alpha\sum_{k=1}^{N} k\cdot P(A_k) = -\alpha\left\{\sum_{k=1}^{N-1} k\left[P(B_{k-1})-P(B_k)\right] + N\cdot P(B_{N-1})\right\} = -\alpha\sum_{k=0}^{N-1} P(B_k)$. Let $C_i^k = \{$we stop after k-th step and the i-th number is the biggest among the first k numbers$\}$, $k=1,\dots,N-1$, $i=1,\dots,k$, i.e. $C_i^k = \{x_1 < x_i, \dots, x_{i-1} < x_i, x_{i+1} < x_i, \dots, x_k < x_i, x_i < d_i\}$. For each k $C_i^k$ are disjoint and $\bigcup_i C_i^k = B_k$. It is easy to see that $P(C_i^k) = \int_0^{d_i} x_i^{k-1}dx_i = \frac{d_i^k}{k}$. Now $-\alpha\sum_{k=0}^{N-1} P(B_k) = -\alpha\left[1 + \sum_{k=1}^{N-1}\left(\sum_{i=1}^{k}\frac{d_i^k}{k}\right)\right]$ Q.e.d.

R e m a r k . The above result is usefull not only in the case of a non-increasing sequence of decision numbers. Looking at the mean payoff as at a function of $d_1,\dots,d_N$ defined on the set $\{(d_1,\dots,d_N) : 1 \geqslant d_1 \geqslant d_2 \geqslant \dots \geqslant d_N = 0\}$. We can fix the optimal stopping rule in this set. Namely, the coordinates of the point $(d_1,\dots,d_N)$ in which the function takes its maximal value are the decision numbers in the optimal stopping rule.

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