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NON-CONTINUOUS AND NON-LINEAR HILBERT-HASEMAN PROBLEM FOR A SYSTEM OF ARBITRARY POWER

Let us consider a region S^+ bounded by a smooth Jordan curve L in the complex plane C. A tangent to the curve L with any fixed direction forms some angle for which the Hölder condition holds. Let S^- denote the complement of the set $S^+ \cup L$ to the plane C. The origin is assumed to be in the region S^+.

Let $L_0 = L - \{c_1, c_2, \ldots, c_p\}$, where the points $c_1, c_2, \ldots, c_p \in L$; $p \geq 2$; are labeled in coincidence with the positive orientation of the L. Thus $L_0 = \breve{c_1c_2} \cup \breve{c_2c_3} \cup \ldots \cup \breve{c_pc_1}$.

D e f i n i t i o n (cf. [1]). Let $t' \in \breve{tc}_{n+1} \subset c_nc_{n+1}$ for $n = 1,2,\ldots,p$ and $c_{p+1} = c_1$. Let $W: L_0 \times L_0 \to C$ be some auxiliary function such that

$$(1) \qquad W(t,t') = \left[|t-c_n|\,|t'-c_{n+1}|\right]^{\delta+h},$$

where parameters δ and h are fixed and satisfy the following conditions

$$(2) \qquad 0<\delta<1, \quad 0<h<1, \quad \delta+h<1 .$$

The class $\mathfrak{H}_\delta^h$ is defined to be the set of all continuous functions $\varphi : L_0 \to C$ for which the following inequalities hold

(3)
$$\prod_{n=1}^{p} |t-c_n|^{\delta} |\varphi(t)| \leq \text{const.},$$
$$W(t,t')|\varphi(t)-\varphi(t')| \leq \text{const.}|t-t'|^{h}.$$

The space Λ. Consider a linear space Λ_{L_0}. Its points

(4)
$$U = [f]$$

are all continuous functions $f : L_0 \to C$. For every f the following condition holds

(5)
$$\sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} |f(t)| < \infty .$$

The norm $\|U\|$ of the point (4) is given by the formula

(6)
$$\|U\| = \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} |f(t)|.$$

Let B denote an arbitrary set of indices. The space $\Lambda = \prod_{\beta\in B} \Lambda_\beta$ with the Tichonov topology such that, for every $\beta \in B$, $\Lambda_\beta = \Lambda_{L_0}$ is a locally convex Hausdorff space, [2].

Let us consider the following problem. We want to prove that there exists a system of holomorphic functions $\{\Phi_\beta : S^+ \cup S^- \to C\}_{\beta\in B}$ with the boundary values in class $\mathfrak{H}_\delta^h$ satisfying the condition

$$(7)\quad \Phi_\beta^+\left[\alpha_\beta(t)\right] = G_\beta(t)\Phi_\beta^-(t) +$$

$$+ F_\beta\Big[t,\left\{\Phi_{\gamma_1}^+\left[\alpha_{\gamma_1}(t)\right]\right\}_{\gamma_1\in B},\left\{\Phi_{\gamma_2}^-\left[\alpha_{\gamma_2}(t)\right]\right\}_{\gamma_2\in B},$$

$$\left\{\int_L \frac{\Phi_{\gamma_3}^+\left[\alpha_{\gamma_3}^*(\tau)\right]}{\tau - t}\,d\tau\right\}_{\gamma_3\in B},\left\{\int_L \frac{\Phi_{\gamma_4}^-\left[\alpha_{\gamma_4}^*(\tau)\right]}{\tau - t}\,d\tau\right\}_{\gamma_4\in B}\Big],$$

where α_β, α_β^*, G_β, F_β, for $\beta \in B$, are given functions. Further, for the functions Φ_β the inequalities

$$(8)\qquad \left|\Phi_\beta(z)\right| < \frac{\text{const}}{\left|z-c_n\right|^{\delta_*}}, \qquad n = 1,2,\ldots p, \quad \beta\in B$$

must hold in any sufficiently small neighbourhood of every point $c_1, c_2, \ldots, c_p$, where $\delta_* \in (0;1)$.

The problem stated above is a generalization of some results presented in [9], and in the proof we employ the method which was first used in [3] and [4].

We make the following assumptions:

1. The functions $\alpha_\beta : L \longrightarrow L$, for $\beta \in B$ are one-onto-one. They preserve orientation and do not change the positions of points $c_1, c_2, \ldots, c_p$. These functions have first derivatives $\alpha'_\beta : L \to L$ such that the Hölder condition

$$(9)\qquad \left|\alpha'_\beta(t) - \alpha'_\beta(t')\right| \leqslant k_\alpha \left|t-t'\right|^{h_\alpha}$$

and inequality

$$(10)\qquad 0 < m_\alpha \leqslant \left|\alpha'_\beta(t)\right| \leqslant M_\alpha$$

hold, where $k_\alpha > 0$, $h_\alpha \in (0;1)$, and k_α, m_α, M_α, h_α are some constants.

2. The functions $\alpha^*_\beta : L \to L$, for $\beta \in B$ are one-to-one and onto. They preserve orientation and do not change the positions of points $c_1, c_2, \dots, c_p$. For these functions the inequality

$$(11) \qquad 0 < m_\alpha \leq \left| \frac{\alpha^*_\beta(t) - \alpha^*_\beta(t')}{t - t'} \right| \leq M_\alpha$$

holds, where m_α and M_α are the same constants as in eq. (10).

3. The functions $G_\beta : L \to C$, for every $\beta \in B$ such that

$$(12) \qquad \left| G_\beta(t) - G_\beta(t') \right| \leq g \left| t - t' \right|^{2h},$$

$$(13) \qquad 0 < s \leq \left| G_\beta(t) \right| \leq S$$

on the curve L hold, where g, s and S are some positive constants and $h \in \left(\frac{h_\alpha}{2}; \frac{1}{2} \right)$. Furthermore, it is assumed that there is a natural number n_o for which $|\chi_\beta| \leq n_o$, where $\chi_\beta = \frac{1}{2\pi}\left[\arg G_\beta(t) \right]_L$ for each $\beta \in B$.

4. The functions $F_\beta : \Omega \to C$ for $\beta \in B$ defined on

$$\Omega = \left\{ t \in L_o, \left| u^i_{\gamma_i} \right| < \infty, \ \gamma_i \in B, \ i = 1,2,3,4 \right\}$$

satisfy the following inequalities

$$(14) \quad \left| F_\beta \left[t, \left\{ u^1_{\gamma_1} \right\}_{\gamma_1 \in B}, \left\{ u^2_{\gamma_2} \right\}_{\gamma_2 \in B}, \left\{ u^3_{\gamma_3} \right\}_{\gamma_3 \in B}, \left\{ u^4_{\gamma_4} \right\}_{\gamma_4 \in B} \right] \right| \leq$$

$$\leq \frac{M}{\prod_{n=1}^{p} |t - c_n|^\delta} + M^* \sum_{i=1}^{4} \sup_{\gamma_i \in B} \left| u^i_{\gamma_i} \right|,$$

$$(15)\quad \left|F_\beta\left[t,\left\{u^1_{\gamma_1}\right\}_{\gamma_1\in B},\left\{u^2_{\gamma_2}\right\}_{\gamma_2\in B},\left\{u^3_{\gamma_3}\right\}_{\gamma_3\in B},\left\{u^4_{\gamma_4}\right\}_{\gamma_4\in B}\right]+\right.$$

$$\left.-F_\beta\left[t',\left\{\bar{u}^1_{\gamma_1}\right\}_{\gamma_1\in B},\left\{\bar{u}^2_{\gamma_2}\right\}_{\gamma_2\in B},\left\{\bar{u}^3_{\gamma_3}\right\}_{\gamma_3\in B},\left\{\bar{u}^4_{\gamma_4}\right\}_{\gamma_4\in B}\right]\right| \leq$$

$$\leq \frac{N|t-t'|^h}{W(t,t')} + N^* \sum_{i=1}^{4} \sup_{\gamma_i\in B} \left|u^i_{\gamma_i} - \bar{u}^i_{\gamma_i}\right|$$

on the domain Ω, where M, M^*, N and N^* are some positive constants; h was defined in the assumption 3 and $W(t,t')$ appears in (2).

It is known (cf. e.g. [7]) that the homogeneous Hilbert--Haseman problem with the boundary condition

$$(16)\qquad \Phi^+_\beta\left[\alpha_\beta(t)\right] = G_\beta(t)\Phi^-_\beta(t)$$

has a canonical solution $X_\beta : C \to C$ for every $\beta \in B$ under the assumptions 1 and 3. This solution is nonzero on C.

For the boundary values $X^\pm_\beta : L \to C$ of the solution X_β the following inequalities

$$(17)\qquad \left|X^\pm_\beta(t) - X^\pm_\beta(t')\right| \leq K_x\left|t-t'\right|^h,$$

$$(18)\qquad \left|\frac{1}{X^\pm_\beta(t)} - \frac{1}{X^\pm_\beta(t')}\right| \leq K_x\left|t-t'\right|^h,$$

$$(19)\qquad k_x \leq \left|X^\pm_\beta(t)\right| \leq K_x$$

hold on L, where k_x and K_x are some positive constants, [8].

Let us denote

$$N_\beta(t,\tau) = \frac{1}{2\pi i}\left[\frac{1}{\tau-t} - \frac{\alpha'_\beta(t)}{\alpha_\beta(\tau)-\alpha_\beta(t)}\right]_{\beta\in B} \tag{20}$$

and

$$\left\{\begin{aligned}
x^1_{\gamma_1} &= \frac{1}{2}X^+_{\gamma_1}\left[\alpha_{\gamma_1}(t)\right]\varphi_{\gamma_1}(t) + \\
&\quad + \frac{1}{2\pi i}\; X^+_{\gamma_1}\left[\alpha_{\gamma_1}(t)\right]\int_L \frac{\varphi_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right]}{\tau-\alpha_{\gamma_1}(t)}\,d\tau, \qquad \gamma_1\in B,\\
x^2_{\gamma_2} &= -\frac{1}{2}X^-_{\gamma_2}\left[\alpha_{\gamma_2}(t)\right]\varphi_{\gamma_2}\left[\alpha_{\gamma_2}(t)\right] + X^-_{\gamma_2}\left[\alpha_{\gamma_2}(t)\right]P_{\gamma_2}\left[\alpha_{\gamma_2}(t)\right] + \\
&\quad + \frac{1}{2\pi i}\; X^-_{\gamma_2}\left[\alpha_{\gamma_2}(t)\right]\int_L \frac{\varphi_{\gamma_2}(\tau)}{\tau-\alpha_{\gamma_2}(t)}\,d\tau, \qquad \gamma_2\in B,\\
w^3_{\gamma_3} &= \frac{1}{2}X^+_{\gamma_3}\left[\alpha^*_{\gamma_3}(\tau)\right]\varphi_{\gamma_3}\left\{\alpha^{-1}_{\gamma_3}\left[\alpha^*_{\gamma_3}(\tau)\right]\right\} + \\
&\quad + \frac{1}{2\pi i}\; X^+_{\gamma_3}\left[\alpha^*_{\gamma_3}(\tau)\right]\int_L \frac{\varphi_{\gamma_3}\left[\alpha^{-1}_{\gamma_3}(\eta)\right]}{\eta-\alpha^*_{\gamma_3}(\tau)}\,d\eta, \qquad \gamma_3\in B,\\
x^3_{\gamma_3} &= \int_L \frac{w^3_{\gamma_3}}{\tau-t}\,d\tau, \qquad \gamma_3\in B,\\
w^4_{\gamma_4} &= -\frac{1}{2}X^-_{\gamma_4}\left[\alpha^*_{\gamma_4}(\tau)\right]\varphi_{\gamma_4}\left[\alpha^*_{\gamma_4}(\tau)\right] + X^-_{\gamma_4}\left[\alpha^*_{\gamma_4}(\tau)\right]P_\beta\left[\alpha^*_{\gamma_4}(\tau)\right] + \\
&\quad + \frac{1}{2\pi i}\; X^-_{\gamma_4}\left[\alpha^*_{\gamma_4}(\tau)\right]\int_L \frac{\varphi_{\gamma_4}(\eta)}{\eta-\alpha^*_{\gamma_4}(\tau)}\,d\eta, \qquad \gamma_4\in B,\\
x^4_{\gamma_4} &= \int_L \frac{w^4_{\gamma_4}}{\tau-t}\,d\tau, \qquad \gamma_4\in B.
\end{aligned}\right. \tag{21}$$

Let us consider the system of integral equations

$$(22)\quad \varphi_\beta(t) - \int_L N_\beta(t,\tau)\varphi_\beta(\tau)d\tau = P_\beta(t) +$$

$$+ \left\{X_\beta^+\left[\alpha_\beta(t)\right]\right\}^{-1} F_\beta\left[t, \{x_{\gamma_1}^1\}_{\gamma_1\in B}, \{x_{\gamma_2}^2\}_{\gamma_2\in B}, \{x_{\gamma_3}^3\}_{\gamma_3\in B}, \{x_{\gamma_4}^4\}_{\gamma_4\in B}\right],$$

$$\beta \in B,$$

where $\{P_\beta : C \to C\}_{\beta\in B}$ is a system of entire functions for which

$$(23)\quad \begin{cases} \sup\limits_{\beta\in B}\left[\sup\limits_{t\in L}|P_\beta(t)|\right], \\ \\ \sup\limits_{\beta\in B}\left[\sup\limits_{t,t'\in L}\left|\dfrac{P_\beta(t) - P_\beta(t')}{|t - t'|^h}\right|\right] \end{cases}$$

are not greater than some specified positive number P.

If the system of equations (22) has a solution $\{\varphi_\beta^* : L_0 \to C\}_{\beta\in B}$ in the class $\mathcal{H}_\delta^h$, then (due to the results obtained in [9]) the solution of the problem with conditions (7) and (8) is given by the following formulae

$$(24)\quad \begin{cases} \dfrac{1}{2\pi i} X_\beta(z)\displaystyle\int_L \dfrac{\varphi_\beta^*\left[\alpha_\beta^{-1}(\tau)\right]}{\tau - z}d\tau, \quad \text{for } z \in S^+ \\ \\ \dfrac{1}{2\pi i} X_\beta(z)\displaystyle\int_L \dfrac{\varphi_\beta^*(\tau)}{\tau - z}d\tau + X_\beta(z)P_\beta(z), \quad \text{for } z \in S^- \end{cases} \qquad \beta\in B.$$

Now, basing on the Schauder-Tichonov theorem we shall prove that the system of integral equations (22) has at least one solution in the class $\mathcal{H}_\delta^h$. Assume that for every $\beta \in B$, we have

$$(25)\quad T_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right] = P_\beta(t) + \left\{X_\beta^+\left[\alpha_\beta(t)\right]\right\}^{-1}$$

$$F_\beta\left[t,\left\{x^1_{\gamma_1}\right\}_{\gamma_1\in B},\left\{x^2_{\gamma_2}\right\}_{\gamma_2\in B},\left\{x^3_{\gamma_3}\right\}_{\gamma_3\in B},\left\{x^4_{\gamma_4}\right\}_{\gamma_4\in B}\right],$$

where $x^i_{\gamma_i}$ for $i = 1,2,3,4$ is given by the formulae (21).

In view of (25) the system (22) can be written in the following form

$$(26)\quad \varphi_\beta(t) - \int_L N_\beta(t,\tau)\,\varphi_\beta(\tau)d\tau = T_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right],\quad \beta\in B.$$

In [9] it was shown that the system (26) is equivalent to

$$(27)\quad \varphi_\beta(t) = A_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right],\quad \beta\in B,$$

where

$$(28)\quad A_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right] = T_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right] +$$

$$+\int_L R_\beta(t,\tau)T_\beta\left[\left\{\varphi_\beta(\tau)\right\}_{\beta\in B}\right]d\tau$$

and $R_\beta(t,\tau)$ is the sum of the iterated kernels of $N_\beta(t,\tau)$ and of the resolvent kernel of the bounded kernel (cf. [9]).

Let us consider a set denoted by $Z(\varrho,\varkappa)$ in the space Λ. Its points are systems of functions $\left\{f_\beta : L\to C\right\}_{\beta\in B}$ such that on L_0 the following inequalities

$$(29)\quad \begin{cases} \prod_{n=1}^{p}\left|t-c_n\right|^{\delta}\left|f_\beta(t)\right| \le \varrho, \\ W(t,t')\left|f_\beta(t)-f_\beta(t')\right| \le \varkappa\left|t-t'\right|^{h} \end{cases}$$

hold, where ϱ and $\varkappa$ are some positive constants. The set $Z(\varrho,\varkappa)$ is non-empty, convex and compact (cf. [5] and [6]).

Now, the set $Z(\varrho,\varkappa)$ is subjected to the following transformation

$$(30) \qquad \psi_\beta(t) = A_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right], \qquad \beta\in B.$$

The sufficient conditions for the point $\{\psi_\beta\}_{\beta\in B}$ to belong to the set $Z(\varrho,\varkappa)$ will be set up. Assuming that the point $\{\varphi_\beta\}_{\beta\in B}$ belongs to the set $Z(\varrho,\varkappa)$ we investigate the following integrals

$$(31) \qquad I^1_{\gamma_1} = \int_L \frac{\varphi_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right]}{\tau-\alpha_{\gamma_1}(t)}d\tau, \qquad \gamma_1\in B,$$

$$(32) \qquad I^2_{\gamma_2} = \int_L \frac{\varphi_{\gamma_2}(\tau)}{\tau-\alpha_{\gamma_2}(t)}d\tau, \qquad \gamma_2\in B,$$

$$(33) \qquad I^3_{\gamma_3} = \int_L \frac{\varphi_{\gamma_3}\left[\alpha^{-1}_{\gamma_3}(\tau)\right]}{\tau-\alpha^*_{\gamma_3}(t)}d\tau, \qquad \gamma_3\in B,$$

$$(34) \qquad I^4_{\gamma_4} = \int_L \frac{\varphi_{\gamma_4}(t)}{\tau-\alpha^*_{\gamma_4}(t)}d\tau, \qquad \gamma_4\in B.$$

From (7), assumptions 1 and 2, and Pogorzelski's theorem [10] it follows that there exist positive constants K_1, K_2, K_3, K_4 which are independent of the point $\{\varphi_\beta\}_{\beta\in B}$ such that the following formula holds

$$(35) \quad \left\{I^i_{\gamma_i}\right\}_{\gamma_i\in B} \in Z(K_1\varrho+K_2\varkappa, K_3\varrho+K_4\varkappa) \qquad i = 1,2,3,4.$$

Due to (21) and (17)-(19), (31)-(35) for every $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in B$, respectively, we get the following inequalities on L_0

$$(36)\quad \prod_{n=1}^{p} |t-c_n|^{\delta} \left|x^1_{\gamma_1}(t)\right| \leq \frac{1}{2} K_x \varrho + \frac{1}{2\pi} K_x (K_1 \varrho + K_2 \varkappa),$$

$$(37)\quad W(t,t')\left|x^1_{\gamma_1}(t)-x^1_{\gamma_1}(t')\right| \leq \Big[\frac{1}{2}\,(K_x\varrho+\varkappa K_x K_5 M^h_\alpha) +$$

$$+\frac{1}{2\pi} K_x(K_3\varrho+K_4\varkappa) + \frac{1}{2\pi} K_x K_5 M^h_\alpha (K_1\varrho+K_2\varkappa)\Big] |t-t'|^h, \quad \gamma_1 \in B,$$

where $K_5 = \sup\limits_{t,t' \in L_0} \dfrac{W(t,t\ )}{\prod_{n=1}^{p} |t-c_n|^{\delta}}$.

$$(38)\quad \prod_{n=1}^{p} |t-c_n|^{\delta} \left|x^2_{\gamma_2}(t)\right| \leq \frac{1}{2} K_x \varrho \frac{1}{m^p_\alpha} + \frac{1}{2\pi} K_x (K_1\varrho+K_2\varkappa) + K_6 K_x P,$$

with $K_6 = \sup\limits_{t \in L} \prod_{n=1}^{p} |t-c_n|^{\delta}$;

$$(39)\quad W(t,t')\left|x^2_{\gamma_2}(t) - x^2_{\gamma_2}(t')\right| \leq \Big[\frac{1}{2}\ K_x \frac{1}{m_\alpha^{2(\delta+h)}} M^h_\alpha \varkappa + \frac{1}{m^p_\alpha} K_5 K_x M^h_\alpha \varrho +$$

$$+\frac{1}{2\pi} K_x(K_3\varrho+K_4\varkappa) + \frac{1}{2\pi} K_5 K_x M^h_\alpha (K_1\varrho+K_2\varkappa) +$$

$$+ K_7 K_x P(M^h_\alpha + P)\Big] |t-t'|^h, \qquad \gamma_2 \in B,$$

where $K_7 = \sup\limits_{t,t' \in L} W(t,t')$;

$$(40)\quad \prod_{n=1}^{p} |t-c_n|^{\delta} \left|w^3_{\gamma_3}(t)\right| \leqslant \frac{1}{2} K_x \left(\frac{M_\alpha}{m_\alpha}\right)^p \varrho + \frac{1}{2\pi} K_x(K_1 \varrho + K_2 \varkappa),$$

$$(41)\quad \prod_{n=1}^{p} |t-c_n|^{\delta} \left|w^3_{\gamma_3}(t) - w^3_{\gamma_3}(t')\right| \leqslant \Big\{ \frac{1}{2\pi} K_x K_5 M_\alpha^h (K_1 \varrho + K_2 \varkappa) +$$

$$+ \frac{1}{2}\left[K_x \left(\frac{M_\alpha}{m_\alpha}\right)^{2(\delta+h)+h} + K_x M_\alpha^h \left(\frac{M_\alpha}{m_\alpha}\right)^2 K_5 \varrho\right] +$$

$$+ \frac{1}{2\pi} K_x (K_3 \varrho + K_4 \varkappa)\Big\} \left|t-t'\right|^h, \qquad \gamma_3 \in B,$$

$$(42)\quad \prod_{n=1}^{p} |t-c_n|^{\delta} \left|w^4_{\gamma_4}(t)\right| \leqslant \frac{1}{2} K_x \frac{1}{m_\alpha^p} \varrho + \frac{1}{2\pi} K_x(K_1\varrho + K_2\varkappa) + K_6 K_x P,$$

$$(43)\quad W(t,t')\left|w^4_{\gamma_4}(t) - w^4_{\gamma_4}(t')\right| \leqslant \Big[\frac{1}{2\pi} K_5 K_x M_\alpha^h (K_1 \varrho + K_2 \varkappa) +$$

$$+ \frac{1}{2}\left(K_x \frac{1}{m_\alpha^{2(\delta+h)}} M_\alpha^h \varkappa + \frac{1}{m_\alpha^p} K_5 K_x M_\alpha^h \varrho\right) + K_7 K_x P (M_\alpha^h + P) +$$

$$+ \frac{1}{2\pi} K_x (K_3 \varrho + K_4 \varkappa)\Big] |t-t'|^h, \qquad \gamma_4 \in B.$$

Taking into account (40)-(43) and applying once more Pogorzelski's theorem we infer that there exist positive constants K_8, K_9, K_{10}, K_{11} which are independent of the transformed point and such that

$$(44)\quad \left\{x^i_{\gamma_i}(t)\right\}_{\gamma_i \in B} \in Z(K_8 w_i + K_9 w_i^*, K_{10} w_i + K_{11} w_i^*), \quad i = 1,2,$$

where the following notations are introduced

$$(45)\quad \begin{cases} w_3 = \frac{1}{2} K_x\left(\frac{M_\alpha}{m_\alpha}\right)^p \varrho + \frac{1}{2\pi} K_x(K_1\varrho + K_2\varkappa), \\ w_3^* = \frac{1}{2}\left[K_x\left(\frac{M_\alpha}{m_\alpha}\right)^{2(\delta+h)+h} + K_x M_\alpha^h\left(\frac{M_\alpha}{m_\alpha}\right)^p K_5\varrho\right] + \\ \qquad + \frac{1}{2\pi} K_x(K_3\varrho + K_4\varkappa) + \frac{1}{2\pi} K_5 K_x M_\alpha^h (K_1\varrho + K_2\varkappa), \\ w_4 = \frac{1}{2} K_x \frac{1}{m_\alpha^p}\varrho + \frac{1}{2\pi} K_x(K_1\varrho + K_2\varkappa) + K_6 K_x P, \\ w_4^* = \frac{1}{2}\left(K_x \frac{1}{m_\alpha^{2(\delta+h)}} M_\alpha^h\varkappa + K_5 \frac{1}{m_\alpha^p} K_x M_\alpha^h\varrho\right) + \frac{1}{2\pi} K_x(K_3\varrho + K_4\varkappa) + \\ \qquad + \frac{1}{2\pi} K_5(K_1\varrho + K_2\varkappa) K_x M_\alpha^h + K_7 P K_x(M_\alpha^h + P). \end{cases}$$

Furthermore, according to (14), (19), (30), (36), (38), (44) on L_o we have

$$(46)\quad \prod_{n=1}^{p} |t-c_n|^\delta\, |\psi_\beta(t)| \leqslant M_R \prod_{n=1}^{p} |t-c_n|^\delta \left|T_\beta\left[\{\varphi_\beta(t)\}_{\beta\in B}\right]\right| \leqslant$$

$$\leqslant M_R\left\{\frac{1}{k_x}\left[M + M^*(w_1 + w_2 + K_8 w_3 + K_9 w_3^* + K_8 w_4 + K_9 w_4^*)\right] + K_6 P\right\}$$

for every $\beta \in B$, where

$$(47)\quad \begin{cases} w_1 = \frac{1}{2} K_x\varrho + \frac{1}{2\pi} K_x(K_1\varrho + K_2\varkappa), \\ w_2 = \frac{1}{2} K_x\varrho\frac{1}{m_\alpha^p} + \frac{1}{2\pi}(K_1\varrho + K_2\varkappa) + K_6 K_x P, \\ M_R = 1 + \sup_{\beta\in B}\left[\sup_{t\in L_o} \prod_{n=1}^{p} |t-c_n|^\delta \int_L \frac{|R_\beta(t,\tau)|}{\prod_{n=1}^{p} |\tau-c_n|^\delta}\, dl_\tau\right] \end{cases}$$

Let us notice (cf.[9]) that from (26) it follows that the transformation (30) takes up the form

$$\psi_\beta(t) = \int_L N_\beta(t,\tau)\psi_\beta(\tau)d\tau + T_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right]. \tag{48}$$

From above and basing on (15), (17), (18), (37), (44) for every $\beta \in B$ we get the following estimations on L_0

$$\begin{aligned}
W(t,t')\,\left|\psi_\beta(t) - \psi_\beta(t')\right| &\leqslant \\
\leqslant W(t,t')&\left| T_\beta\left[\left\{\varphi_\beta(t)\right\}_{\beta\in B}\right] - T_\beta\left[\left\{\varphi_\beta(t')\right\}_{\beta\in B}\right]\right| + \\
+ W(t,t')&\left|\int_L \left[N_\beta(t,\tau) - N_\beta(t',\tau)\right] \psi_\beta(\tau)d\tau\right| \leqslant \\
\leqslant \Big\{ &\frac{1}{k_x}\left[N+N^*(w_1^*+w_2^*+K_{10}w_3+K_{11}w_3^*+K_{10}w_4+K_{11}w_4^*)\right] + \\
+ &N_R\, K_x K_5\left[M+M^*(w_1+w_2+K_8w_3+K_9w_3^*+K_8w_4+K_9w_4^*\right] + \\
+ &K_7P\Big\}\, |t-t'|^h,
\end{aligned} \tag{49}$$

where

$$\begin{cases}
w_1^* = \frac{1}{2}\left(K_x\varkappa+\rho K_xK_5M_\alpha^h\right) + \frac{1}{2\pi}K_x(K_3\rho+K_4\varkappa) + \\
\quad + \frac{1}{2\pi}K_xK_5M_\alpha^h(K_1\rho+K_2\varkappa), \\
w_2^* = \frac{1}{2}\left(K_x\frac{1}{m_\alpha^{2(\delta+h)}}M_\alpha^h\varkappa + \frac{1}{m_\alpha^\rho}K_5K_xM_\alpha^h\rho\right) + \frac{1}{2\pi}K_x(K_3\rho+K_4\varkappa) + \\
\quad + \frac{1}{2\pi}K_5(K_1\rho+K_2\varkappa)K_xM_\alpha^h + K_7PK_x(M_\alpha^h+P),
\end{cases} \tag{50}$$

$$(50)\qquad N_R = \sup_{\beta\in B}\left[\sup_{t,t'\in L_0}\int_L \frac{\left|N_\beta(t,\tau)-N_\beta(t',\tau)\right| W(t,t')}{|t-t'|^h \prod_{n=1}^{p} |t-c_n|^\delta} d\tau\right].$$

The existence of the upper bounds M_R and N_R defined by (47) and (50) is guaranteed by the assumptions 1,2,3 and the properties of the kernel $N_\beta(t,\tau)$ (cf. [9]). From the estimations (46) and (49) we obtain the following sufficient conditions for the transformation (30) to map the set $Z(\varrho,\varkappa)$ into itself

$$(51)\qquad \begin{cases} C_1 + C_2M^* + C_3M^*\varkappa + C_4M^*\varrho \leqslant \varrho, \\ C_5+C_6M^*+C_7N^*+(C_8M+C_9N)\varkappa + (C_{10}M+C_{11}N^*)\varrho \leqslant \varkappa \end{cases},$$

where the constants C_i, $i = 1,2,\dots,11$, are independent of M^*, N^*, of the function F_β and of the point $\{\varphi_\beta\}_{\beta\in B} \in$ $\in Z(\varrho,\varkappa)$.

It is easy to prove that if the constants M^* and N^* are sufficiently small to satisfy the inequalities

$$(52)\qquad \begin{cases} M^* \leqslant \dfrac{1}{2\max(C_3+C_4,\ C_8+C_{10})}, \\ N^* \leqslant \dfrac{1}{2\max(C_9,\ C_{11})}, \end{cases}$$

then it is possible to choose $\varrho = \varrho_0$ and $\varkappa = \varkappa_0$ such that (51) holds. It can be done for example by setting ϱ_0 and $\varkappa_0$ as follows

$$(53)\qquad \begin{cases} \varrho_0 = \dfrac{-C_3M^*(C_5+C_6M^*+C_7N^*)+(C_8M^*+C_9N^*-1)(C_1+C_2M^*)}{C_3M^*(C_{10}M^*+C_{11}N^*)+(1-C_4M^*)(C_8M^*+C_9N^*-1)} \\ \varkappa_0 = \dfrac{-(C_1+C_2M^*)(C_{10}M^*+C_{11}N^*)+(C_5+C_6M^*+C_7N^*)(C_4M^*-1)}{C_3M^*(C_{10}M^*+C_{11}N^*)+(1-C_4M^*)(C_8M^*+C_9N^*-1)} \end{cases}.$$

In this way we have proved the following theorem.

T h e o r e m 1. If for the constants in the assumption 3 the conditions (52) hold, then the transformation (30) maps the set $Z(\rho_0, \varkappa_0)$ into the same set, where ρ_0 and $\varkappa_0$ are given by (53).

Now, we shall prove the following theorem.

T h e o r e m 2. The transformation (30) is continuous on the set $Z(\rho_0, \varkappa_0)$.

P r o o f : Λ_{L_0} is a metric space. Assuming

$$\forall \varepsilon>0 \exists m_0 \in D \forall \beta \in B \forall m \in D \wedge m \succ m_0 \sup_{t\in L_0} \left[\prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| \varphi_\beta^{(m)}(t) - \varphi_\beta(t) \right| \right] < \varepsilon$$

it is sufficient to prove that

$$\forall \varepsilon>0 \exists m_* \in D \forall \beta \in B \forall m \in D \wedge m \succ m_* \sup_{t\in L_0} \left[\prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| \psi_\beta^{(m)}(t) - \psi_\beta(t) \right| \right] < \delta_0 \varepsilon$$

holds, where D is an directed set and δ_0 is a positive constant.

Due to (10), (11), (17), (18) for every $\gamma_1 \in B$ and for every $m \succ m_0$ we have

$$(54) \quad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| x_{\gamma_1}^{1(m)}(t) - x_{\gamma_1}^{1}(t) \right| \leqslant$$

$$\leqslant \frac{1}{2} K_x \varepsilon + \frac{1}{2\pi} K_x \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| \int_{L_0} \frac{\varphi_{\gamma_1}^{(m)}\left[\alpha_{\gamma_1}^{-1}(\tau)\right] - \varphi_{\gamma_1}\left[\alpha_{\gamma_1}^{-1}(\tau)\right]}{\tau - \alpha_{\gamma_1}(t)} d\tau \right| .$$

We want to estimate the integral in the formula (54). Let us rewrite it in the following form

$$(55)\quad \int_{L_0} \frac{\varphi^{(m)}_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right] - \varphi_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right]}{\tau - \alpha_{\gamma_1}(t)}\, d\tau =$$

$$= \int_{l} \frac{\varphi^{(m)}_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right] - \varphi^{(m)}_{\gamma_1}(t)}{\tau - \alpha_{\gamma_1}(t)}\, d\tau + \left[\varphi^{(m)}_{\gamma_1}(t) - \varphi_{\gamma_1}(t)\right] \int_{l} \frac{d\tau}{\tau - \alpha_{\gamma_1}(t)} +$$

$$+ \int_{l} \frac{\varphi_{\gamma_1}(t) - \varphi_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right]}{\tau - \alpha_{\gamma_1}(t)}\, d\tau + \int_{L_0 - l} \frac{\varphi^{(m)}_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right] - \varphi_{\gamma_1}\left[\alpha^{-1}_{\gamma_1}(\tau)\right]}{\tau - \alpha_{\gamma_1}(t)}\, d\tau ,$$

where the arc L is a part of an arc $c_n \smile c_{n+1}$, $n=1,2,\dots,p$, containing the point $\alpha_{\gamma_1}(t)$. We assume that the length of the arc l is not greater than the half of the length of the shortest arc $c_n \smile c_{n+1}$. In the special case, when any end-point of the arc l coincides with any end-point of the arc $c_n \smile c_{n+1}$, we assume that the length of the arc connecting the point $\alpha_{\gamma_1}(t)$ to the common end-point of arcs I and $c_n \smile c_{n+1}$ is no greater than the length of the remainder of the l.

The set $Z(\rho_0, \varkappa_0)$ is a cartesian product of a collection of closed sets, hence it is also closed. Let the point $\{\varphi_\beta : L_0 \to C\}_{\beta\in B}$ be the limit of a general sequence $\{\varphi^{(m)}_\beta\}_{\beta\in B}$, $m \in D$, of points of the set $Z(\rho_0, \varkappa_0)$. It follows that the coordinates of the point $\{\varphi_\beta : L_0 \to C\}_{\beta\in B}$ satisfy conditions (29) with the constants ρ_0 and $\varkappa_0$ (given by (53)), for every $\beta \in B$. Let us denote the terms on the right-side of the formula (55) by $I^1_{\gamma_1}$, $I^2_{\gamma_1}$, $I^3_{\gamma_1}$, and $I^4_{\gamma_1}$, respectively. Then for every $\gamma_1 \in B$ and for every $m \in D$ we obtain

$$(56)\quad \sup_{t\in L_o} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| I^1_{\gamma_1}(t) + I^2_{\gamma_1}(t) \right| \leqslant$$

$$\leqslant \frac{2\varkappa_o}{m_\alpha^{p(\delta+h)+h}} \cdot M_\alpha^{2(\delta+h)} \sup_{t\in L_o} \prod_{n=1}^{p} \left|\alpha_{\gamma_1}(t)-c_n\right|^{\delta+h} \int_l \frac{dl_\tau}{W\left[\tau,\alpha_{\gamma_1}(t)\right]\left|\tau-\alpha_{\gamma_1}(t)\right|^{1-h}} < \varepsilon,$$

$$(57)\qquad \sup_{t\in L_o} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| I^3_{\gamma_1}(t) \right| \leqslant$$

$$\leqslant 2\varrho_o \frac{1}{m_\alpha^{ph}} \sup_{t\in L_o} \prod_{n=1}^{p} \left|\alpha_{\gamma_1}(t) - c_n\right|^h \left| \int_l \frac{d\tau}{\tau - \alpha_{\gamma_1}(t)} \right| < \varepsilon ,$$

if the length of the arc l is sufficiency small, [1]. Let the length of l be fixed to satisfy inequalities (56) and (57). Then $I^4_{\gamma_1}$ is a regular integral. If $m \succ m_o$ and $\gamma_1 \in B$, then for the arc l lying within the arc $c_n \smile c_{n+1}$, the integral $I^4_{\gamma_1}$ can be estimated as follows

$$(58)\qquad \sup_{t\in L_o} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| I^4_{\gamma_1}(t) \right| \leqslant \frac{C_{14} M_\alpha^{\delta+h}}{|l|} \varepsilon ,$$

but in the case, when the arc l has the common end-point c_n with the arc $c_n \smile c_{n+1}$, then $I^4_{\gamma_1}$ can be estimated as follows

$$\text{(59)}\qquad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| I^4_{\gamma_1}(t)\right| \leqslant$$

$$\leqslant \left[\frac{C_{15}|L| M_\alpha^{\delta+h}}{|l|^{1+\delta+h}} + C_{16} M_\alpha^{\delta+h} \int_{l^*} \frac{dl_\tau}{|\tau - c_n|^{\delta+h}(|\tau - c_n| + m_\alpha)}\right] \varepsilon,$$

where $l^* \in L$ is an arc of the length $|l|$. This arc has only one common point c_n with the arc l. The constants C_{14}, C_{15}, C_{16} are dependent only on L_0. Combining the results (54), (56)-(59) we deduce that there is a constant

$$\delta_1 = \frac{K_x}{2} + \frac{K_x}{2\pi} \left\{ 2+\max\left[\frac{C_{14} M_\alpha^{\delta+h}}{|l|}, \frac{C_{15}|L| M_\alpha^{\delta+h}}{|l|^{1+\delta+h}} + \right.\right.$$

$$\left.\left. + C_{16} M_\alpha^{\delta+h} \int_{l^*} \frac{dl_\tau}{|\tau - c_n|^{\delta+h}(|\tau - c_n| + m_\alpha)}\right]\right\}$$

such that

$$\text{(60)}\qquad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| x^{1(m)}_{\gamma_1}(t) - x^1_{\gamma_1}(t)\right| < \delta_1 \varepsilon$$

for every $m \succ m_0$ and for every $\gamma_1 \in B$.

The same arguments lead to the proof of the existence of positive constants δ_2 and δ_3 such that

$$\text{(61)}\qquad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| x^{2(m)}_{\gamma_2}(t) - x^2_{\gamma_2}(t)\right| < \delta_2 \varepsilon,$$

$$\text{(62)}\qquad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left| w^{i(m)}_{\gamma_i}(t) - w^i_{\gamma_i}(t)\right| < \delta_3 \varepsilon,\ i=3,4,$$

for every $m \succ m_0$ and for every $\gamma_s \in B$, $s=2,3,4$.

Due to results (40) - (45) we see that there exist constants $\rho_1 = \max[w_3(\rho_0,\varkappa_0), w_4(\rho_0,\varkappa_0)]$ and $\varkappa_1 = [w_3^*(\rho_0,\varkappa_0), w_4^*(\rho_0,\varkappa_0)]$ for which $\{w_{\Upsilon_i}^{i(m)}\}_{\Upsilon_i \in B} \in Z(\rho_1,\varkappa_1)$, $i = 3, 4$ holds for every $m \in D$. Since the set $Z(\rho_1,\varkappa_1)$ is closed, it contains the limits $\{w_{\Upsilon_i}^{i}\}_{\Upsilon_i \in B}$ of the generalized sequences $\{w_{\Upsilon_i}^{i(m)}\}_{\Upsilon_i \in B}$, where $i = 3,4$, $m \in D$. From (21) we have

$$(63)\qquad x_{\Upsilon_3}^{3(m)}(t) - x_{\Upsilon_3}^{3}(t) = \int_{L_0} \frac{w_{\Upsilon_3}^{3(m)}(\tau) - w_{\Upsilon_3}^{3}(\tau)}{\tau - t}\, d\tau =$$

$$= \int_{l_1} \frac{w_{\Upsilon_3}^{3(m)}(\tau) - w_{\Upsilon_3}^{3(m)}(t)}{\tau - t}\, d\tau + \int_{l_1} \frac{w_{\Upsilon_3}^{3}(t) - w_{\Upsilon_3}^{3}(\tau)}{\tau - t}\, d\tau +$$

$$+ \left[w_{\Upsilon_3}^{3(m)}(t) - w_{\Upsilon_3}^{3}(t)\right] \int_{l_1} \frac{d\tau}{\tau - t} + \int_{L_0 - l_1} \frac{w_{\Upsilon_3}^{3(m)}(\tau) - w_{\Upsilon_3}^{3}(\tau)}{\tau - t}\, d\tau,$$

where l_1 is an arc containing the point t and of analogous properties as the arc l considered above. Now, for the integrals (63) we can find the estimations similar to (57)-(60). Hence there exists a positive constant δ_3^* such that

$$(64)\qquad \sup_{t \in L_0} \prod_{n=1}^{p} |t - c_n|^{\delta + h} \left| x_{\Upsilon_3}^{3(m)}(t) - x_{\Upsilon_3}^{3}(t) \right| < \delta_3^* \varepsilon$$

holds for every $m \succ m_0$ and for every $\Upsilon_3 \in B$. Analogously, we can state that there exists a positive constant δ_4^* such that

$$(65) \quad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left|x_{\gamma_4}^{4(m)}(t) - x_{\gamma_4}^{4}(t)\right| < \delta_4^* \varepsilon$$

holds for every $m \succ m_0$ and for every $\gamma_4 \in B$.

Due to (15), (30), (59), (60), (64), (65) for every $\beta \in B$ and for every $m \succ m_0$ we obtain

$$(66) \quad \sup_{t\in L_0} \prod_{n=1}^{p} |t-c_n|^{\delta+h} \left|\psi_\beta^{(m)}(t) - \psi_\beta(t)\right| <$$

$$< \frac{1}{k_x} M_0 N^*(\delta_1 + \delta_2 + \delta_3^* + \delta_4^*)\varepsilon ,$$

where

$$(67) \quad M_0 = \sup_{\beta\in B}\left\{1+\sup_{t\in L_0}\left[\prod_{n=1}^{p} |t-c_n|^{\delta+h} \int_{L_0} \frac{|R_\beta(t,\tau)|}{\prod_{n=1}^{p} |\tau-c_n|^{\delta+h}}\, dl_\tau\right]\right\}$$

which was to be proved.

Hence all the assumptions of Schauder-Tichonov theorem are fulfilled. Now we can formulate the following theorem.

T h e o r e m 3. If the assumptions 1-4 are fulfilled and for the constants appearing in the assumption 3 the conditions (52) hold, then the system of integral equations (22) has at least one solution $\left\{\varphi_\beta^* : L_0 \longrightarrow C\right\}_{\beta\in B}$ in the class of functions $\mathfrak{H}_\delta^h$.

Theorem 3 and earlier considerations show that there is at least one solution which can be expressed by the formulae (24) and which satisfies the conditions (8) and (9).

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