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ON SOME MEANS OF DOUBLE FOURIER SERIES OF CONTINUOUS FUNCTIONS

1. Let $C_{2\pi}$ be the class of all real functions, 2π-periodic in each variable, continuous everywhere. Let $a_{k,l}$, $b_{k,l}$, $c_{k,l}$, $d_{k,l}$, $S_{k,l}(x,y;f)$ be the coefficients and the (k,l)-th partial sums of the double Fourier series of $f \in C_{2\pi}$, respectively.

Consider the following means of the Fourier series of f:

$$(1)\quad L_{m,n}(x,y;f,\lambda) = \sum_{k=0}^{m}\sum_{l=0}^{n} \lambda_{k,l}^{(m,n)}\, \gamma_{k,l}\, (a_{k,l}\cos kx\cos ly + b_{k,l}\sin kx\cos ly + c_{k,l}\cos kx\sin ly + d_{k,l}\sin kx\sin ly),$$

where

$$\gamma_{k,l} = \begin{cases} 1/4 & \text{if } k = l = 0, \\ 1/2 & \text{if } k=0,\ l>0 \ \text{ or } \ k>0,\ l=0, \\ 1 & \text{if } k>0,\ l>0 \end{cases}$$

$\lambda_{k,l}^{(m,n)}$ are the elements of real matrix $\lambda = \left\| \lambda_{k,l}^{(m,n)} \right\|$ $(0 \le k \le m,\ 0 \le l \le n;\ m,n = 0,1,\dots;\ \lambda_{k,l}^{(m,n)} = 0$ if $k > m$ or $l > n)$, and the means of the Abel-Poisson type

$$P_{r,\varrho}^{(p,q)}(x,y;f) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \varphi_k(r,p)\,\varphi_l(\varrho,q)\,S_{k,l}(x,y;f),$$

where

$$\varphi_k(r,p) = \begin{cases} \dfrac{-1}{\log(1-r)}\,\dfrac{1}{k+1}\,r^{k+1} & \text{if} \quad p=-1, \\ (1-r)^{p+1}\binom{k+p}{p} r^k & \text{if} \quad p=0,1,\ldots, \end{cases}$$

$k=0,1,\ldots;\ p,q=-1,0,1,\ldots;\ r,\varrho \in (0,1)$.

We shall give the sufficient conditions for uniform convergence of the sequence $\{L_{m,n}(x,y;f,\lambda)\}$ to $f(x,y)$ and we shall estimate the quantities

$$A(m,n;\lambda) = \max_{x,y} \left| L_{m,n}(x,y;f,\lambda) - f(x,y)\right|,$$

$$B(r,\varrho;p,q) = \max_{x,y} \left| P_{r,\varrho}^{(p,q)}(x,y;f) - f(x,y)\right|.$$

These expressions will be estimated by $E_{m,n}(f)$ - the best approximation of the function $f \in C_{2\pi}$ by the trigonometric polynomials of two variables of order (m,n) at most.

Below, M_i, $M_i(a,b)$ will denote absolute constants and positive constants depending on a,b only.

2. Let $\Delta\lambda_{k,l}^{(m,n)} = \lambda_{k,l}^{(m,n)} - \lambda_{k+1,l}^{(m,n)} - \lambda_{k,l+1}^{(m,n)} + \lambda_{k+1,l+1}^{(m,n)}$ $(0 \leqslant k \leqslant m,\ 0 \leqslant l \leqslant n;\ m,n=0,1,\ldots)$. Let Λ_α, $\alpha > 1$, denote the class of matrices λ such that

$$(2) \qquad \begin{cases} \lambda_{0,0}^{(m,n)} = 1, & m,n=0,1,\ldots, \\ \lim\limits_{m,n\to\infty} \lambda_{k,l}^{(m,n)} = 1 & \text{for any} \quad (k,l), \end{cases}$$

$$(3)\quad \sum_{\mu=0}^{M}\sum_{\nu=0}^{N}\left\{2^{(\mu+\nu)(\alpha-1)}\sum_{k=\mu'}^{2\mu'}\sum_{l=\nu'}^{2\nu'}\left|\Delta\lambda_{k,l}^{(m,n)}\right|^{\alpha}\right\}^{\frac{1}{\alpha}} = O(1)$$

uniformly in m,n, where $M = \left[\log_2(m+1)\right]$, $N = \left[\log_2(n+1)\right]$, $\mu' = 2^{\mu}-1$, $\nu' = 2^{\nu}-1$. Clearly $\Lambda_{\beta} \subset \Lambda_{\alpha}$ if $1<\alpha<\beta$.

We shall prove the following two theorems.

Theorem 1. If $f \in C_{2\pi}$, $\lambda \in \Lambda_{\alpha}$, $1<\alpha\leqslant 2$, then

$$(4)\quad \lim_{m,n\to\infty} L_{m,n}(x,y;f,\lambda) = f(x,y)$$

uniformly in (x,y).

Proof. The means (1) have the integral form

$$L_{m,n}(x,y;f,\lambda) = \sum_{k=0}^{m}\sum_{l=0}^{n}\Delta\lambda_{k,l}^{(m,n)}\, S_{k,l}(x,y;f) =$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\, K_{m,n}(u,v)\, du\, dv,$$

where

$$K_{m,n}(u,v) = \sum_{k=0}^{m}\sum_{l=0}^{n}\Delta\lambda_{k,l}^{(m,n)}\, D_k(u)\, D_l(v),$$

$$D_k(t) = 1/2 + \sum_{i=1}^{k}\cos it.$$

We shall prove that the condition (3) implies the inequality

$$(5)\quad L_{m,n} = \frac{4}{\pi^2}\int_{0}^{\pi}\int_{9}^{\pi}\left|K_{m,n}(u,v)\right|\, du\, dv \leqslant M_1$$

for $m,n = 0,1,\dots$. Denote M, N, μ', ν' as in (3). It is easily seen, that

$$K_{m,n}(u,v) = \sum_{\mu=0}^{M} \sum_{\nu=0}^{N} U_{\mu,\nu}(u,v),$$

where

$$U_{\mu,\nu}(u,v) = \sum_{k=\mu'}^{2\mu'} \sum_{l=\nu'}^{2\nu'} \Delta\lambda_{k,l}^{(m,n)} D_k(u) D_l(v).$$

We have

$$(6)\quad L_{m,n} \leqslant \sum_{\mu=0}^{M} \sum_{\nu=0}^{N} \left(\int_0^{2^{-\mu}} \int_0^{2^{-\nu}} + \int_0^{2^{-\mu}} \int_{2^{-\nu}}^{\pi} + \int_{2^{-\mu}}^{\pi} \int_0^{2^{-\nu}} + \int_{2^{-\mu}}^{\pi} \int_{2^{-\nu}}^{\pi} \right) \left| U_{\mu,\nu}(u,v) \right| du\, dv =$$

$$= \sum_{\mu=0}^{M} \sum_{\nu=0}^{N} \sum_{i=1}^{4} I_{\mu,\nu}^{(i)} .$$

We estimate $I_{\mu,\nu}^{(1)}$ and $I_{\mu,\nu}^{(3)}$.

$$I_{\mu,\nu}^{(1)} \leqslant \int_0^{2^{-\mu}} \int_0^{2^{-\nu}} \left(\sum_{k=\mu'}^{2\mu'} \sum_{l=\nu'}^{2\nu'} \left| \Delta\lambda_{k,l}^{(m,n)} \right| (k+1)(l+1) \right) du\, dv \leqslant$$

$$\leqslant 4 \left\{ 2^{(\mu+\nu)(\alpha-1)} \sum_{k=\mu'}^{2\mu'} \sum_{l=\nu'}^{2\nu'} \left| \Delta\lambda_{k,l}^{(m,n)} \right|^{\alpha} \right\}^{\frac{1}{\alpha}} ,$$

$$I^{(3)}_{\mu,\nu} \leqslant \int_{2^{-\mu}}^{\pi}\left(\sum_{l=\nu'}^{2\nu'}\left|\sum_{k=\mu'}^{2\mu'}\Delta\lambda_{k,l}^{(m,n)}\ D_k(u)\right|\int_0^{2^{-\nu}}\left|D_l(v)\right|dv\right)du \leqslant$$

$$\leqslant \sum_{l=\nu'}^{2\nu'}\left(\int_{2^{-\mu}}^{\pi}\left|\sum_{k=\mu'}^{2\mu'}\Delta\lambda_{k,l}^{(m,n)}\ \sin ku\right|\frac{1}{\operatorname{tg}\frac{u}{2}}\,du + \right.$$

$$\left. + \int_{2^{-\mu}}^{\pi}\left|\sum_{k=\mu'}^{2\mu'}\Delta\lambda_{k,l}^{(m,n)}\ \cos ku\right| du\right).$$

Applying the Hölder and the Hausdorff-Young inequalities, we obtain

$$I^{(3)}_{\mu,\nu} \leqslant M_2(\alpha)\sum_{l=\nu'}^{2\nu'}\left(\left[\int_{2^{-\mu}}^{\pi} t^{-\alpha}\,dt\right]^{\frac{1}{\alpha}}\left\{\int_0^{\pi}\left|\sum_{k=\mu'}^{2\mu'}\Delta\lambda_{k,l}^{(m,n)}\ \sin ku\right|^{\frac{\alpha}{\alpha-1}}du\right\}^{1-\frac{1}{\alpha}} + \right.$$

$$\left. + \left\{\int_0^{\pi}\left|\sum_{k=\mu'}^{2\mu'}\Delta\lambda_{k,l}^{(m,n)}\ \cos ku\right|^{\frac{\alpha}{\alpha-1}}du\right\}^{1-\frac{1}{\alpha}}\right) \leqslant$$

$$\leqslant M_3(\alpha)\sum_{l=\nu'}^{2\nu'} 2^{\mu(1-\frac{1}{\alpha})}\left\{\sum_{k=\mu'}^{2\mu'}\left|\Delta\lambda_{k,l}^{(m,n)}\right|^{\alpha}\right\}^{\frac{1}{\alpha}} \leqslant$$

$$\leqslant M_3(\alpha)\left\{2^{(\mu+\nu)(\alpha-1)}\sum_{k=\mu'}^{2\mu'}\sum_{l=\nu'}^{2\nu'}\left|\Delta\lambda_{k,l}^{(m,n)}\right|^{\alpha}\right\}^{\frac{1}{\alpha}}$$

A similar estimate holds for $I^{(2)}_{\mu,\nu}$ and $I^{(4)}_{\mu,\nu}$. By (6) and (3) we obtain (5). It is known ([4]) that the conditions (2) and (5) imply (4) for any $f \in C_{2\pi}$. Hence the theorem has been proved.

Theorem 2. Suppose that $f \in C_{2\pi}$, the matrix λ satisfies the conditions (2) and

$$(7) \qquad \left|\Delta\lambda^{(m,n)}_{k,l}\right| \leqslant M_3 \frac{(k+1)^{\beta}\ (l+1)^{\delta}}{(m+1)^{1+\beta}\ (n+1)^{1+\delta}},$$

$(0 \leqslant k \leqslant m,\ 0 \leqslant l \leqslant n;\ m,n = 0,1,\ldots)$, where β, δ are some non-negative numbers. Then

$$(8) \quad A(m,n,\lambda) \leqslant \frac{M_4(\beta,\delta)}{(m+1)^{1+\beta}(n+1)^{1+\delta}} \sum_{k=0}^{m}\sum_{l=0}^{n} (k+1)^{\beta}\ (l+1)^{\delta}\ E_{k,l}(f)$$

for $m,n = 0,1,\ldots$.

Proof. By (2) and (7) we have

$$\left|L_{m,n}(x,y;f,\lambda) - f(x,y)\right| = \left|\sum_{k=0}^{m}\sum_{l=0}^{n} \Delta\lambda^{(m,n)}_{k,l} \left\{S_{k,l}(x,y;f) - f(x,y)\right\}\right| \leqslant$$

$$\leqslant \frac{M_5(\beta,\delta)}{(m+1)^{1+\beta}\ (n+1)^{1+\delta}} \sum_{\mu=0}^{M}\sum_{\nu=0}^{N} 2^{\mu\beta+\nu\delta} \sum_{k=\mu'}^{2\mu'}\ \sum_{l=\nu'}^{2\nu'} \left|S_{k,l}(x,y;f) - f(x,y)\right|,$$

where M,N,μ',ν' are as in (3).

Applying (3) of [3] we get

$$A(m,n,\lambda) \leqslant \frac{M_6(\beta,\delta)}{(m+1)^{1+\beta}\ (n+1)^{1+\delta}} \sum_{\mu=0}^{M}\sum_{\nu=0}^{N} 2^{\mu(1+\beta)+\nu(1+\delta)}\ E_{\mu',\nu'}(f)$$

and, by the monotony of $\{E_{m,n}(f)\}$, the estimate (8) follows.

R e m a r k . If the matrix λ satisfies the Fomin conditions (2) and

$$(9) \qquad \left((m+1)(n+1)\right)^{-1+\alpha} \sum_{k=0}^{m} \sum_{l=0}^{n} \left|\Delta\lambda_{k,l}^{(m,n)}\right|^{\alpha} = O(1),$$

$(m,n = 0,1,\dots)$, for some $\alpha > 1$, ([1], [2]), then $\lambda \in \Lambda_{\alpha}$. There exist matrices $\lambda \in \Lambda_{\alpha}$ for which the condition (9) does not hold, e.g.

$$\lambda_{k,l}^{(m,n)} = \left(1 - \frac{h(k)}{h(m+1)}\right)\left(1 - \frac{h(l)}{h(n+1)}\right)$$

$(0 \leqslant k \leqslant m,\ 0 \leqslant l \leqslant n;\ m,n = 0,1,\dots)$, where

$$h(s) = \begin{cases} 0 & \text{if} \quad s = 0, \\ \displaystyle\sum_{\nu=0}^{s-1} \frac{1}{\nu+1}\left(\sum_{\mu=0}^{\nu} \frac{1}{\mu+1}\right)^{\sigma} & \text{if} \quad s = 1,2,\dots,\ \sigma \geqslant -1. \end{cases}$$

3. We shall estimate the quantity $B(r,\rho;p,q)$. We shall need the following lemma.

L e m m a . Let $f \in C_{2\pi}$ and $D = \max\limits_{x,y} |f(x,y)|$. Then

$$(10) \qquad \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varphi_k(r,p)\, \varphi_l(\rho,q)\, \left|S_{k,l}(x,y;f)\right| \leqslant M_7(p,q)\, D,$$

$$(11) \qquad \sum_{k=0}^{m-1} \sum_{l=0}^{\infty} \varphi_l(\rho,q)\, \left|S_{k,l}(x,y;f)\right| \leqslant M_8(q)\, D\, m$$

for all (x,y); $r,\rho \in (0,1)$; $p,q = 0,1,\dots$, $m = 1,2,\dots$.

For $p = q = 0$ these inequalities are given in [3] (lemma 2). The proof of our lemma is analogous.

T h e o r e m 3. If $f \in C_{2\pi}$, then we have

$$(12)\quad B(r,\rho;-1,-1) \leq \frac{M_9}{|\log(1-r)|\,|\log(1-\rho)|}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\frac{1}{(k+1)(l+1)}E_{k,l}(f)$$

for $r \in \langle r_0,1)$, $\rho \in \langle \rho_0,1)$,

$$(13)\quad B(r,\rho;-1,q) \leq M_{10}(q)\frac{(1-\rho)^{1+q}}{|\log(1-r)|}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\frac{(l+1)^q}{k+1}E_{k,l}(f)$$

for $q = 0,1,\ldots$, $r \in \langle r_0,1)$, $\rho \in (0,1)$,

$$(14)\quad B(r,\rho;p,q) \leq M_{11}(p,q)(1-r)^{1+p}(1-\rho)^{1+q}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}(k+1)^p(l+1)^q E_{k,l}(f)$$

for $p,q = 0,1,\ldots$; $r,\rho \in (0,1)$, where $m = [1/(1-r)]$, $n = [1/(1-\rho)]$, $0 < r_0, \rho_0 < 1$.

P r o o f . We prove the inequality (12). Let $m = [1/(1-r)]$, $n = [1/(1-\rho)]$. Clearly, for any (x,y) we have

$$\left|P_{r,\rho}^{(-1,-1)}(x,y;f)-f(x,y)\right| \leq \frac{1}{|\log(1-r)|\,|\log(1-\rho)|}\left(\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}+\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}+\sum_{k=0}^{m-1}\sum_{l=n}^{\infty}+\right.$$

$$\left.+\sum_{k=m}^{\infty}\sum_{l=0}^{n-1}\right)\frac{r^k\rho^l}{(k+1)(l+1)}\left|S_{k,l}(x,y;f)-f(x,y)\right| = \sum_{i=1}^{4}W_i(r,\rho).$$

Let $T^*_{k,l}(x,y)$ denote the best approximation polynomial of function $f \in C_{2\pi}$ by trigonometric polynomials of order non superior to (k,l). We have

$$W_2(r,\varrho)\leqslant\frac{1}{|\log(1-r)|\,|\log(1-\varrho)|\,(m+1)(n+1)}\left(\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} r^k \varrho^l \left|S_{k,l}\left(x,y;f-T^*_{m,n}\right)\right|+\right.$$

$$\left.+\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} r^k \varrho^l \left|f(x,y) - T^*_{m,n}(x,y)\right|\right).$$

By (10),

$$W_2(r,\varrho) \leqslant \frac{M_{12}\, E_{m,n}(f)}{|\log(1-r)|\;|\log(1-\varrho)|\;(m+1)(n+1)(1-r)(1-\varrho)}$$

and, for $r \in \langle r_0,1)$, $\varrho \in \langle \varrho_0,1)$, we get

$$W_2(r,\varrho) \leqslant M_{13}\; E_{m,n}(f).$$

Using (11), we obtain

$$W_3(r,\varrho)\leqslant\frac{1}{|\log(1-r)|\,|\log(1-\varrho)|\,(n+1)}\sum_{\mu=0}^{M} 2^{-\mu}\sum_{k=\mu'}^{2\mu'}\sum_{l=n}^{\infty}\varrho^l\left(\left|S_{k,l}(x,y;f-T^*_{\mu',n})\right|+\right.$$

$$\left.+\left|T^*_{\mu',n}(x,y) - f(x,y)\right|\right)\leqslant$$

$$\leqslant \frac{M_{14}}{|\log(1-r)|\,|\log(1-\rho)|\,(n+1)(1-\rho)} \sum_{\mu=0}^{M} E_{\mu',n}(f) \leqslant$$

$$\leqslant \frac{M_{15}}{|\log(1-r)|\,|\log(1-\rho)|} \sum_{\mu=0}^{M} E_{\mu',n}(f),$$

where $M = [\log_2 m]$, $\mu' = 2^{\mu} - 1$.

Analogously we obtain the following inequality

$$W_4(r,\rho) \leqslant \frac{M_{16}}{|\log(1-r)|\,|\log(1-\rho)|} \sum_{\nu=0}^{[\log_2 m]} E_{m,2^{\nu}-1}(f).$$

Proceeding as in the proof of Theorem 2 and applying the properties of the best approximation $E_{m,n}(f)$, we obtain

$$W_i(r,\rho) \leqslant \frac{M_{17}}{|\log(1-r)|\,|\log(1-\rho)|} \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \frac{E_{k,l}(f)}{(k+1)(l+1)}$$

$(i = 1,2,3,4)$ and we have (12). The proof of inequalities (13), (14) is similar.

From (12) - (14) we obtain the following corollary.

C o r o l l a r y . If $f \in C_{2\pi}$; $p,q = -1,0,1,\ldots$, then

$$\lim_{r,\rho \uparrow 1} B(r,\rho;p,q) = 0.$$

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