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ON SOME RELATIONS OF TANGENCY OF ARCS IN METRIC SPACES

Introduction

In this paper we shall use the notions given in the introduction to [2], analogous to [6]. Accordingly, we introduce the relation $T_l(a,b,k,p)$ defined as follows

(1) $T_l(a,b,k,p) = \{(A,B);\ A\cup B\subset E$ and (A,B) is (a,b) - concentrated at the point $p\in E$

and $\frac{1}{r^k} l(A\cap S(p,r)_{a(r)}, B\cap S(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0\}$.

This relation will be called the relation of (a,b) - tengency of sets of order k at the point p.

If $(A,B)\in T_l(a,b,k,p)$, then we shall say that the set A is (a,b) - tangent of order k at the point p to the set B.

There arises the following question: if the sets $A,B\in E_0$ are (a,b) - tangent of order k at the point p of the space (E,l), then when these sets are (a', b') - tangent of order k at the point p of this space?

In the present paper we consider this problem in a metric space (E,ρ) for rectifiable arcs having the Archimedean property at the point p and for real functions defined by the equalities

$$\tag{2}
\begin{aligned}
\rho_0(A,B) &= \sup\left\{\rho(x,B);\ \ x\in A\right\},\\
\rho_1(A,B) &= \max\left\{\rho_0(A,B);\ \rho_0(B,A)\right\},\\
\rho_2(A,B) &= \min\left\{\rho_0(A,B);\ \rho_0(B,A)\right\},\\
\rho_3(A,B) &= \inf\left\{\operatorname{diam}_\rho(\{x\}\cup B);\ x\in A\right\},\\
\rho_4(A,B) &= \max\left\{\rho_3(A,B);\ \rho_3(B,A)\right\},\\
\rho_5(A,B) &= \min\left\{\rho_3(A,B);\ \rho_3(B,A)\right\},\\
\rho_6(A,B) &= \operatorname{diam}_\rho(A\cup B),\\
\rho_7(A,B) &= \inf\left\{\rho(x,B);\ x\in A\right\}
\end{aligned}$$

for $A,\ B\in E_0$. By definition, $\rho(x,B) = \inf\limits_{y\in B}\rho(x,y)$ and $\operatorname{diam}_\rho A$ denotes the diameter of the set A in the metric space (E,ρ). The functions $\rho_i\,(i = 0,1,\ldots,7)$ are special cases of the function l_0 considered in [6] such that

$$\tag{3} \rho_i(\{x\},\{y\}) = l_0(x,y) = \rho(x,y) \quad \text{for} \quad x,y\in E.$$

1. For any sets A,B,C the following inequalities hold

$$\tag{4} \rho(A,C)\leqslant\rho(A,B) + \rho(A\cup B,C) + \operatorname{diam}_\rho B,$$

$$\tag{5} \operatorname{diam}_\rho(A\cup B)\leqslant\operatorname{diam}_\rho A + \operatorname{diam}_\rho B + \rho(A,B)$$

where, by definition, $\rho(A,B) = \rho_7(A,B) = \inf\limits_{x\in A,\ y\in B}\rho(x,y)$.

Now we shall prove the following lemma.

L e m m a 1. If $\rho(A,B) = 0$, then

$$\sup_{x\in(A\cup B)}\rho(x,C) \leqslant \sup_{x\in A}\rho(x,C) + \mathrm{diam}_\rho B. \tag{6}$$

P r o o f. Assume that $\rho(A,B) = 0$. For any sets A,B,C the following equality holds

$$\sup_{x\in(A\cup B)}\rho(x,C) = \max\left\{\sup_{x\in A}\rho(x,C),\ \sup_{x\in B}\rho(x,C)\right\}.$$

If $\max\left\{\sup_{x\in A}\rho(x,C),\ \sup_{x\in B}\rho(x,C)\right\} = \sup_{x\in A}\rho(x,C)$,

then the equality (6) clearly holds. Hence assume that

$$\max\left\{\sup_{x\in A}\rho(x,C),\ \sup_{x\in B}\rho(x,C)\right\} = \sup_{x\in B}\rho(x,C).$$

Then we have

$$\sup_{x\in(A\cup B)}\rho(x,C) = \sup_{x\in B}\rho(x,C). \tag{7}$$

Take an arbitrary $\varepsilon > 0$. By assumption, there exist points $y'\in A$, $x'\in B$ such that $\rho(x',y') < \frac{\varepsilon}{2}$. Let x be an arbitrary point of the set B. Then we have

$$\rho(x,C) \leqslant \rho(y',C) + \rho(x,y') \leqslant \sup_{y\in A}\rho(y,C) + \rho(x,y') \leqslant$$

$$\leqslant \sup_{y\in A}\rho(y,C) + \rho(x,x') + \rho(x',y') \leqslant$$

$$\leqslant \sup_{y\in A}\rho(y,C) + \mathrm{diam}_\rho B + \rho(x',y') <$$

$$< \sup_{y\in A}\rho(y,C) + \mathrm{diam}_\rho B + \frac{\varepsilon}{2}.$$

Hence

$$\sup_{x\in B} \rho(x,C) \leqslant \sup_{y\in A} \rho(y,C) + \operatorname{diam}_\rho B + \frac{\varepsilon}{2} <$$

$$< \sup_{y\in A} \rho(y,C) + \operatorname{diam}_\rho B + \varepsilon .$$

Since ε is arbitrary, we infer that

(8) $$\sup_{x\in B} \rho(x,C) \leqslant \sup_{y\in A} \rho(y,C) + \operatorname{diam}_\rho B.$$

From (7) and (8) we obtain the inequality (6), which concludes the proof of the lemma.

Lemma 2. If

(9) $$A_0 = A_1 \cup A_2 \cup A_3,$$

(9') $$B_0 = B_1 \cup B_2 \cup B_3$$

are sets such that

(10) $$\rho_7(A_1, A_i) = 0$$

$$i = 2,3$$

(10') $$\rho_7(B_1, B_i) = 0$$

then the following inequalities hold

(11) $$\rho_7(A_1,B_1) \leqslant \rho_7(A_0,B_0) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3,$$

(12) $$\rho_6(A_0,B_0) \leqslant \rho_6(A_1,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3,$$

(13) $$\rho_3(A_1,B_0) \leqslant 2\rho_3(A_0,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3,$$

(14) $$\rho_0(A_0,B_1) \leqslant \rho_0(A_1,B_0) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3.$$

P r o o f. From (4), (9) and (10) we obtain

$$\begin{aligned}\rho_7(A_1,B_1) &\leq \rho_7(A_1\cup A_2,B_1) + \rho_7(A_1,A_2) + \operatorname{diam}_\rho A_2 = \\ &= \rho_7(A_1\cup A_2,B_1) + \operatorname{diam}_\rho A_2 \leq \\ &\leq \rho_7(A_1\cup A_2\cup A_3,B_1) + \rho_7(A_1\cup A_2,A_3) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 = \\ &= \rho_7(A_0,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3,\end{aligned}$$

i.e.

$$(11.1)\quad \rho_7(A_1,B_1) \leq \rho_7(A_0,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 .$$

Similarly, from (4), (9') and (10') we obtain

$$(11.2)\quad \rho_7(A_0,B_1) \leq \rho_7(A_0,B_0) + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3.$$

From (11.1) and (11.2) we obtain the inequality (11).
From (5), (9), (9'), (10) and (10') we obtain

$$\begin{aligned}\rho_6(A_0,B_0) &= \operatorname{diam}_\rho(A_1\cup A_2\cup A_3\cup B_1\cup B_2\cup B_3) \leq \\ &\leq \operatorname{diam}_\rho(A_1\cup B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3\end{aligned}$$

which yields the inequality (12).

To prove the inequality (13) let us observe that

$$\operatorname{diam}_\rho B_1 + \operatorname{diam}_\rho(\{x\}\cup B_1) \leq 2\operatorname{diam}_\rho(\{x\}\cup B_1).$$

This implies

$$(13.1)\quad \operatorname{diam}_\rho B_1 + \inf_{x\in A_0} \operatorname{diam}_\rho(\{x\}\cup B_1) \leq 2\inf_{x\in A_0}\operatorname{diam}_\rho(\{x\}\cup B_1).$$

From (5), (9), (9'), (10), (10'), (11.1) and (13.1) we obtain

$$\rho_3(A_1,B_0) = \inf_{x\in A_1} \operatorname{diam}_\rho(\{x\} \cup B_0) \leqslant$$

$$\leqslant \inf_{x\in A_1} \operatorname{diam}_\rho(\{x\} \cup B_1) + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3 \leqslant$$

$$\leqslant \inf_{x\in A_1} (\operatorname{diam}_\rho B_1 + \rho(x,B_1)) + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3 =$$

$$= \rho(A_1,B_1) + \operatorname{diam}_\rho B_1 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3 \leqslant$$

$$\leqslant \rho(A_0,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_1 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3 \leqslant$$

$$\leqslant \inf_{x\in A_0} \operatorname{diam}_\rho(\{x\} \cup B_1) + \operatorname{diam}_\rho B_1 + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3$$

$$\leqslant 2 \inf_{x\in A_0} \operatorname{diam}_\rho(\{x\} \cup B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3 =$$

$$= 2\rho_3(A_0,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3,$$

that is

$$\rho_3(A_1,B_0) \leqslant 2\rho_3(A_0,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 + \operatorname{diam}_\rho B_2 + \operatorname{diam}_\rho B_3$$

which gives the inequality (13).

From Lemma 1 and from (9) we have

$$\rho_0(A_0,B_1) = \sup_{x\in A_0} \rho(x,B_1) \leqslant \sup_{x\in A_1} \rho(x,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3 =$$

$$= \rho_0(A_1,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3.$$

Consequently

$$(14.1) \qquad \rho_0(A_0,B_1) \leqslant \rho_0(A_1,B_1) + \operatorname{diam}_\rho A_2 + \operatorname{diam}_\rho A_3.$$

Taking into account (11.2) we obtain

(14.2) $\varrho_0(A_1,B_1) \leqslant \varrho_0(A_1,B_0) + \operatorname{diam}_\varrho B_2 + \operatorname{diam}_\varrho B_3.$

From (14.1) and (14.2) we obtain the inequality (14). Taking into account the definitions of the functions $\varrho_0, \varrho_3, \varrho_6, \varrho_7$ and the inequalities (11) - (14) we obtain the following inequalities

(15) $\varrho_3(A_i,B_j) \leqslant 2\varrho_3(A_k,B_l) + \operatorname{diam}_\varrho A_2 + \operatorname{diam}_\varrho A_3 + \operatorname{diam}_\varrho B_2 + \operatorname{diam}_\varrho B_3,$

(15') $\varrho_s(A_i,B_j) \leqslant \varrho_s(A_k,B_l) + \operatorname{diam}_\varrho A_2 + \operatorname{diam}_\varrho A_3 + \operatorname{diam}_\varrho B_2 + \operatorname{diam}_\varrho B_3,$

where $i,j,k,l = 0,1$; $s = 0,6,7$.

2. Assume that in the metric space (E,ϱ) we are given real, non-negative functions $a_i, i = 1,2,$ defined in a right-hand neighbourhood of 0, which satisfy the following condition

(16) $$a_i(r) \xrightarrow[r \to 0+]{} 0.$$

Let us put

(17) $$\check{a}(r) = \max\{a_1(r),\ a_2(r)\},$$

(17) $$\hat{a}(r) = \min\{a_1(r),\ a_2(r)\},$$

(18) $$A(r) = A \cap S(p,r)_{\check{a}(r)} - A \cap S(p,r)_{\hat{a}(r)} \quad \text{for} \quad A \subset E.$$

We shall prove the following lemma.

L e m m a 3. If there exists $\eta > 0$ such that for every $r,u \in (0,\eta)$

$$S(p,r)_u = \left\{ x \in M; \quad r - u < \varrho(p,x) < r + u \right\}, \tag{19}$$

then we have

$$A(r) = A_1(r) \cup A_2(r), \tag{20}$$

where

$$A_1(r) = \left\{ x \in A; \quad r - \check{a}(r) < \varrho(p,x) \leqslant r - \hat{a}(r) \right\}, \tag{21}$$

$$A_2(r) = \left\{ x \in A; \quad r + \hat{a}(r) \leqslant \varrho(p,x) < r + \check{a}(r) \right\}. \tag{21'}$$

P r o o f. Let $r, a_i(r) \in (0,\eta)$ for $\eta > 0$, $i = 1,2$. Let $x \in A(r)$. Hence $x \in A$ and $x \in S(p,r)_{\check{a}(r)}$ and $x \notin S(p,r)_{\hat{a}(r)}$. From (19) we infer that

$$r - \check{a}(r) < \varrho(p,x) < r + \check{a}(r)$$

and

$$\varrho(p,x) \leqslant r - \hat{a}(r) \quad \text{or} \quad \varrho(p,x) \geqslant r + \hat{a}(r).$$

This implies

$$r - \check{a}(r) < \varrho(p,x) \leqslant r - \hat{a}(r) \quad \text{or} \quad r + \hat{a}(r) \leqslant \varrho(p,x) < r + \check{a}(r).$$

Hence we have

$$x \in A_1(r) \quad \text{or} \quad x \in A_2(r), \quad \text{i.e.}$$

$$x \in (A_1(r) \cup A_2(r)).$$

Similarly we can show that if $x \in (A_1(r) \cup A_2(r))$, then $x \in A(r)$.

Hence the equality (20) holds.

From (18) and (20) we obtain

$$A \cap S(p,r)_{\check{a}(r)} = A \cap S(p,r)_{\hat{a}(r)} \cup A_1(r) \cup A_2(r). \tag{22}$$

From (21), (21') taking into account the formula

$$A \cap S(p,r)_{\hat{a}(r)} = \left\{ x \in A; \quad r - \hat{a}(r) < \varrho(p,x) < r + \hat{a}(r) \right\}$$

we obtain the following equalities

$$(23) \qquad \varrho(A \cap S(p,r)_{\hat{a}(r)}, \; A_i(r)) = 0 \quad \text{for} \quad i = 1,2.$$

Let A be a rectifiable arc with the Archimedean property at a point $p \in A$, defined in the interval $\langle 0,1 \rangle$ by means of a homeomorphism φ.

Let us recall that an arc A has the Archimedean property at a point $p \in A$ whenever

$$\frac{s(p,x)}{\varrho(p,x)} \xrightarrow[A \ni x \to p]{} 1,$$

where $s(p,x)$ denotes the length of the arc with ends at p and x.

Let us consider the sets defined by the equalities (21) and (21').

L e m m a 4. If the functions a_i, $i = 1,2$, satisfy the condition

$$(24) \qquad a_i(r) < r \quad \text{for} \quad r \in (0,\delta), \; \delta > 0,$$

$$(25) \qquad \frac{|a_2(r) - a_1(r)|}{r} \xrightarrow[r \to 0+]{} 0$$

then

$$\frac{1}{r} \operatorname{diam}_\varrho A_i(r) \xrightarrow[r \to 0+]{} 0, \qquad i = 1,2.$$

P r o o f. Consider the set

$$(26) \qquad \overline{A_1(r)} = \left\{ x \in A; \quad r - \check{a}(r) \leqslant \varrho(p,x) \leqslant r - \hat{a}(r) \right\}.$$

Since $A_1(r) \subset \overline{A_1(r)}$ we have

$$(27) \qquad \frac{1}{r}\,\mathrm{diam}_\rho A_1(r) \leqslant \frac{1}{r}\,\mathrm{diam}_\rho \overline{A_1(r)}.$$

Now we shall prove that

$$(28) \qquad \frac{1}{r}\,\mathrm{diam}_\rho \overline{A_1(r)} \xrightarrow[r \to 0+]{} 0.$$

Let us put

$$(29) \qquad \begin{aligned} \varphi(t'_r) &= x'_r, \\ \varphi(t''_r) &= x''_r \end{aligned}$$

where

$$(29') \qquad \begin{aligned} t'_r &= \inf\left\{ t;\quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad \varphi(t) \in (A \cap S(p, r-\check{a}(r)))\right\}, \\ t''_r &= \sup\left\{ t;\quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad \varphi(t) \in (A \cap S(p, r-\hat{a}(r)))\right\}. \end{aligned}$$

For the points p, x'_r, x''_r the following inequality hold

$$(30) \qquad \rho(p,x''_r) \leqslant \rho(p,x'_r) + s(x'_r, x''_r) \leqslant s(p,x''_r),$$

where $s(p,x''_r)$, $s(x'_r,x''_r)$ denote the length of arcs with ends at p,x''_r and x'_r,x''_r, respectively. The inequality (30) is equivalent to the following one:

$$(30') \qquad 1 \leqslant \frac{\rho(p,x'_r) + s(x'_r,x''_r)}{\rho(p,x''_r)} \leqslant \frac{s(p,x''_r)}{\rho(p,x''_r)}.$$

From the Archimedean property for the arc A at the point p it follows that

$$(31) \qquad \frac{s(p,x''_r)}{\rho(p,x''_r)} \xrightarrow[r \to 0+]{} 1.$$

From (30) and (31) we obtain

$$\frac{\rho(p,x'_r) + s(x'_r, x''_r)}{\rho(p,x''_r)} \xrightarrow[r\to 0+]{} 1.$$

Hence by definition we have

$$\frac{r - \check{a}(r) + s(x'_r,x''_r)}{r - \hat{a}(r)} \xrightarrow[r\to 0+]{} 1.$$

Consequently,

$$\frac{1 - \frac{\check{a}(r)}{r} + \frac{s(x'_r, x''_r)}{r}}{1 - \frac{\hat{a}(r)}{r}} \xrightarrow[r\to 0+]{} 1.$$

This implies

$$(32)\qquad \frac{\frac{s(x'_r, x''_r)}{r} - \left(\frac{\check{a}(r)}{r} - \frac{\hat{a}(r)}{r}\right)}{1 - \frac{\hat{a}(r)}{r}} \xrightarrow[r\to 0+]{} 0.$$

By (24) we have

$$0 < 1 - \frac{\hat{a}(r)}{r} \leqslant 1.$$

Hence by (32) we obtain

$$(33)\qquad \frac{s(x'_r,x''_r) - (\check{a}(r) - \hat{a}(r))}{r} \xrightarrow[r\to 0+]{} 1.$$

From (25) and (33) it follows that

$$(34)\qquad \frac{s(x'_r,x''_r)}{r} \xrightarrow[r\to 0+]{} 0.$$

From (29), (29') and from the equality (26) we obtain

$$\overline{A_1(r)} \subset (x'_r, x''_r),$$

where (x'_r, x''_r) is the arc with ends x'_r, x''_r.

Hence we have

$$\sup_{x,y\in\overline{A_1(r)}} \rho(x,y) \leqslant \sup_{x,y\in(x'_r,x''_r)} \rho(x,y) \leqslant s(x'_r,x''_r).$$

Consequently, we infer that

$$\frac{1}{r}\operatorname{diam}_\rho \overline{A_1(r)} \leqslant \frac{1}{r}\, s(x'_r, x''_r).$$

Hence by (34) we have

(35) $$\frac{1}{r}\operatorname{diam}_\rho \overline{A_1(r)} \xrightarrow[r\to 0+]{} 0.$$

From (27) and (35) we obtain

(36) $$\frac{1}{r}\operatorname{diam}_\rho A_1(r) \xrightarrow[r\to 0+]{} 0.$$

Similarly we can prove that

(37) $$\frac{1}{r}\operatorname{diam}_\rho A_2(r) \xrightarrow[r\to 0+]{} 0.$$

3. In the metric space (E,ρ) let us consider rectifiable arcs A, B with the origin at a point p, having the Archimedean property. Making use of Lemma 3 and (22) we obtain the following equalities

(38) $$A\cap S(p,r)_{\check{a}(r)} = A\cap S(p,r)_{\hat{a}(r)} \cup A_1(r) \cup A_2(r),$$

$$(38') \qquad B\cap S(p,r)_{\check{b}(r)} = B\cap S(p,r)_{\hat{b}(r)} \cup B_1(r)\cup B_2(r),$$

where $A_1(r)$ and $A_2(r)$ are defined by (21) and (21'), and

$$B_1(r) = \left\{ x\in B;\ \ r - \check{b}(r) < \varrho(p,x) \leqslant r - \hat{b}(r) \right\},$$

$$B_2(r) = \left\{ x\in B;\ \ r + \hat{b}(r) \leqslant \varrho(p,x) < r + \check{b}(r) \right\},$$

where

$$\check{b}(r) = \max\left\{ b_1(r),\ b_2(r) \right\},$$

$$\hat{b}(r) = \min\left\{ b_1(r),\ b_2(r) \right\}.$$

The functions b_1, b_2 satisfy the same assumptions as the functins a_1 and a_2.

From (23), (15) and (15') we obtain the following inequalities

$$(39) \quad \left|\varrho_i(A\cap S(p,r)_{a_2(r)}, B\cap S(p,r)_{b_2(r)}) - \varrho_i(A\cap S(p,r)_{a_1(r)}, B\cap S(p,r)_{b_1(r)})\right| \leqslant$$

$$\leqslant \operatorname{diam}_\varrho A_1(r) + \operatorname{diam}_\varrho A_2(r) + \operatorname{diam}_\varrho B_1(r) + \operatorname{diam}_\varrho B_2(r),$$

$$(40) \quad \varrho_3(A\cap S(p,r)_{a_2(r)}, B\cap S(p,r)_{b_2(r)}) \leqslant 2\varrho_3(A\cap S(p,r)_{a_1(r)}, B\cap S(p,r)_{b_1(r)}) +$$

$$+ \operatorname{diam}_\varrho A_1(r) + \operatorname{diam}_\varrho A_2(r) + \operatorname{diam}_\varrho B_1(r) + \operatorname{diam}_\varrho B_2(r),$$

$$(40') \quad \varrho_3(A\cap S(p,r)_{a_1(r)}, B\cap S(p,r)_{b_1(r)}) \leqslant 2\varrho_3(A\cap S(p,r)_{a_2(r)}, B\cap S(p,r)_{b_2(r)}) +$$

$$+ \operatorname{diam}_\varrho A_1(r) + \operatorname{diam}_\varrho A_2(r) + \operatorname{diam}_\varrho B_1(r) + \operatorname{diam}_\varrho B_2(r).$$

T h e o r e m. If the functions a_i, b_i, $i = 1,2$, satisfy the conditions

$$a_i(r)<r, \quad b_i(r)<r \quad \text{for} \quad r \in (0,\delta), \quad \delta>0,$$

$$\frac{|a_2(r) - a_1(r)|}{r} \xrightarrow[r \to 0+]{} 0, \qquad \frac{|b_2(r) - b_1(r)|}{r} \xrightarrow[r \to 0+]{} 0$$

and there exists $\eta \geq \delta > 0$ such that for any $r,u \in (0,\eta)$

$$S(p,r)_u = \left\{ x \in E;\ \ r - u < \rho(p,x) < r + u \right\}$$

then for any rectifiable arcs A, B with the origin at $p \in E$, having the Archimedean property at p we have

$$(A,B) \in T\rho_i(a_1,b_1,1,p) \Leftrightarrow (A,B) \in T\rho_i(a_2,b_2,1,p), \qquad i = 0,1,\dots,7.$$

P r o o f. For the functions ρ_0, ρ_3, ρ_6, ρ_7 this theorem follows directly from (39), (40), (40') and from Lemma 4. Hence let us assume that $(A,B) \in T\rho_1(a_1,b_1,1,p)$. We shall show $(A,B) \in T\rho_1(a_2,b_2,1,p)$.
If

$$\max\left\{ \rho_0(A,B), \quad \rho_0(B,A) \right\} = \rho_0(A,B)$$

then we have

$$\rho_1(A,B) = \rho_0(A,B). \tag{41}$$

This implies

$$(A,B) \in T\rho_1(a_1,b_1,1,p) \Leftrightarrow (A,B) \in T_0(a_1,b_1,1,p). \tag{42}$$

Since the theorem holds for the function ρ_0, from (41) and (42) we obtain

$$(A,B) \in T\rho_1(a_1,b_1,1,p) \Leftrightarrow (A,B) \in T\rho_1(a_2,b_2,1,p). \tag{43}$$

Now let us assume that

$$\max\left\{\rho_0(A,B),\ \rho_0(B,A)\right\} = \rho_0(B,A)$$

then we have

(44) $$\rho_1(A,B) = \rho_0(B,A).$$

Consequently, we infer that

(45) $$(A,B)\in T\rho_1(a_1,b_1,1,p) \Leftrightarrow (B,A)\in T\rho_0(b_1,a_1,1,p).$$

From (44) and (45), taking into account that the theorem holds for ρ_0, we obtain (43).

Similarly we prove the theorem for the functions ρ_2, ρ_4, ρ_5. Hence the theorem holds for all the functions ρ_i, $i = 0,1,2,\dots,7$.

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