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Ewa Stankiewicz, Wojciech Żakowski

THE (Z, Q)-SYSTEMS

Introduction

The notion of a (Z,Q)-machine has been introduced by Żakowski in [6]. The notion of a (Z,Q)-machine is a generalization of the notion of Pawlak's machine [3], of a k-machine [1], of a simple continuous machine [2] and of a continous simple Z - machine [5]. The (Z,Q)-machine is an abstract model of computing machine which describes the three basic types of that device: digital, analog and hybrid computers. Roughly speaking, a (Z,Q)-machine is a deterministic device which uniquely extends functions of n real variables defined on Z to functions defined on Q (The unique extension of computations is the common attribute of all the computers mentioned above).

In the present paper the idea of the description of the computing machine with the one input is generalised to the multiinput case. The behaviour of the (Z,Q) - system depends on signals acting on each input and the interractions between them. This behaviour is described by - so called - (Z,Q) - processes, whereas a computation and a subcomputation of the (Z,Q) - system are the functions defined on the set of all the moments admissible for all inputs. It seems to the authors that the presented model of computing machine could be the new tool for investigations on the parallel and independent behaviour of multi-input systems.

1. Basic notations and definitions

Let S be an arbitrary but fixed non-empty set, $\mathcal{R}^+$ - the set of all non-negative real numbers and $\mathcal{N}_0$ - the set of all non-negative integers. Let n be an arbitrary but fixed positive integer.

If $a \in \mathcal{R}^+$ and $\emptyset \neq U \subseteq \mathcal{R}^+$ then $U_a = \{t \in \mathcal{R}^+ : t - a \in U\}$.

Let Q_k, $k = 1,\dots,n$ be an arbitrary subset of $\mathcal{R}^+$ such that

$$(1) \qquad \forall_{1 \le k \le n} \; \mathcal{N}_0 \subseteq Q_k \quad \text{and} \quad \forall_{1 \le k \le n} \; \forall_{a \in Q_k} \; (Q_k)_a \subseteq Q_k$$

and let Z_k be an arbitrary subset of Q_k, $k = 1,\dots,n$ fulfilling the following condition

$$(2) \qquad \forall_{1 \le k \le n} \; \forall_{\substack{t_1, t_2 \in Q_k \\ t_1 < t_2}} \left[t_2 \in Z_k \rightarrow t_1 \in Z_k\right] .$$

By Q_0 we shall denote the set $\bigcap_{k=1}^{n} Q_k$. Note that $\mathcal{N}_0 \subseteq Q_0$ and $(Q_0)_a \subseteq Q_0$ for any $a \in Q_0$. We shall also introduce the following notations: $Z_1 = Z_1 \times \dots \times Z_n$, $\mathcal{Q} = Q_1 \times \dots \times Q_n$ and $Z_{\bar{a}} = \{(t_1,\dots,t_n) \in \mathcal{Q} : (t_1 - a_1, \dots, t_n - a_n) \in \mathcal{Q}\}$, where $\bar{a} = (a_1,\dots,a_n) \in \mathcal{Q}$. The set of all mappings $f : Z \to S^n$ such that $f = (f_1,\dots,f_n)$ and $f_i : Z_i \to S$, $i = 1,\dots,n$ is denoted by $\mathcal{F}_Z$; similarly the set of all mappings $f : \mathcal{Q} \to S^n$ such that $f = (f_1,\dots,f_n)$ and $f_i : Q_i \to S$, $i = 1,\dots,n$ is denoted by $\mathcal{F}_{\mathcal{Q}}$. By $\mathcal{A}_{Z,\mathcal{Q}}$ we denote the set of all operators A such that $\emptyset \neq DA \subseteq \mathcal{F}_Z$ and $RA \subseteq \mathcal{F}_{\mathcal{Q}}$. By the $\mathcal{Q}$ - shift operator we mean the operator assigning to every mapping $f : Z_{\bar{a}} \to S^n$, $\bar{a} = (a_1,\dots,a_n)$ the mapping $f^* : Z \to S^n$ such that

$$\forall_{(t_1,\dots,t_n) \in Z} \quad f^*(t_1,\dots,t_n) = f(t_1 + a_1, \dots, t_n + a_n) .$$

by $f_{Z_{\bar{a}}}$ we denote the mapping $(f|Z_{\bar{a}})^*$, where $\bar{a} \in Q$, $f \in \mathcal{F}_Q$ and additionally $f_* = f|Z$.

For any $f \in \mathcal{F}_Q$, φ_f denotes the mapping $\varphi_f : Q_o \to S^n$ defined as follows

$$\forall_{t \in Q_o} \quad \varphi_f(t) \underset{df}{=} f(t,\dots,t) .$$

D e f i n i t i o n 1 . An operator $M \in \mathcal{A}_{Z,Q}$ is said to be a (Z,Q) - system iff

(3) $$\forall_{f \in DM} \left[(Mf|Z) = f\right] ,$$

(4) $$\forall_{f \in RM} \ \forall_{\bar{a}=(a,\dots,a) \in Q_o^n} \left[f_{Q_{\bar{a}}} \in RM\right] .$$

The elements of the set DM are called the initial mappings of M and the elements of the set RM are called the processes of that system. If $f = (f_1,\dots,f_n)$ is a process of the (Z,Q) - system, then the mapping φ_f is called the computation of that system; the function $f_i|Q_o$, $i = 1,\dots,n$ is called the i-th subcomputation of the system.

For $n = 1$ the (Z,Q) - system is the (Z,Q)-machine [6]. In that case processes of the system are computations of the machine.

The (Z,Q) - system can be said to be an abstract model of the computing device with informations coming from n independent sources. By Z_k we mean the set of all the moments such that informations from the k-th source $(k=1,\dots n)$ are supplied . Q_k is the set of all the admissible moments for the k-th source. Conditions (3) and (4) characterize the principle of work of the (Z,Q) - system. Condition (3) requires the unique extension of an n-dimensional information, whereas condition (4) secures the "closure" of RM with respect to "translations in time" at every possible moment for every source. By the computation of the system we mean the "display" of the process in the set of all the moments admissible for all sources.

2. The basic properties of processes and computations of the $(\mathcal{Z},\mathcal{Q})$ - system

Definition 2. A mapping $f \in \mathcal{F}_{\mathcal{Q}}$ is said to be periodic iff there exists $\tau \in Q_0$ such that the following condition is satisfied

$$(5) \qquad \forall_{(t_1,\ldots,t_n)\in\mathcal{Q}} \quad f(t_1+\tau,\ldots,t_n+\tau) = f(t_1,\ldots,t_2) .$$

Lemma 1. If $f = (f_1,\ldots,f_n) \in \mathcal{F}_{\mathcal{Q}}$ is a periodic mapping, then the mapping φ_f is periodic, moreover the functions f_i are the periodic functions of variable t.

Proof. Let $\tau \in Q_0$ be the element satysfying (5). Since condition (1) is true for Q_0, then $t+\tau \in Q_0$ for any $t \in Q_0$ and we get the equality

$$\varphi_f(t+\tau) = f(t+\tau,\ldots,t+\tau) = f(t,\ldots,t) = \varphi_f(t) .$$

The second statement follows from the structure of f.

Definition 3. A mapping $f \in \mathcal{F}_{\mathcal{Q}}$ is said to be $\mathcal{Z}$ - injective iff $f_{\mathcal{Z}_{\bar{a}}} \neq f_{\mathcal{Z}_{\bar{b}}}$ for any $\bar{a},\bar{b} \in Q_0{}^n$, $\bar{a} = (a,\ldots,a)$, $\bar{b} = (b,\ldots,b)$, $a \neq b$.

Definition 4. A mapping $\varphi : Q_0 \to S^n$ is called Z-injective $(\emptyset \neq Z \subseteq Q_0)$ iff $(\varphi|Z_a)^* \neq (\varphi|Z_b)^*$ for any $a,b \in Q_0$, $a \neq b$.

Let us assume that $Q_1 = Q_2 = \ldots = Q_n = Q$ and $Z_1 = Z_2 = \ldots = Z_n = Z$, until Lemma 3. Then we have $Q = Q_0$ and $Z \subseteq Q_0$. We shall investigate relations between $\mathcal{Z}$ - injectivity of the mapping $f = (f_1,\ldots,f_n) \in \mathcal{F}_{\mathcal{Q}}$, Z - injectivity of the mapping φ_f and Z - injectivity of the functions f_i [4].

If there exists i_o , $1 \le i_o \le n$ such that the function f_{i_o} is Z - injective, then the mapping f is $\mathcal{Z}$ - injective. To prove it suppose that $f_{Z_{\bar{a}}} = f_{Z_{\bar{b}}}$ for some $\bar{a},\bar{b} \in Q_o^n$, $\bar{a} = (a,\dots,a)$, $\bar{b} = (b,\dots,b)$. This equality means that

$$\forall_{(t_1,\dots,t_n)\in \mathcal{Z}} \quad f(t_1+a,\dots,t_n+a) = f(t_1+b,\dots,t_n+b)$$

in other words

$$\forall_{1\le i\le n} \ \forall_{t\in Z} \quad f_i(t + a) = f_i(t + b) .$$

Especially $f_{i_o}(t + a) = f_{i_o}(t + b)$ for any $t \in Z$, i.e. $(f_{i_o})_{Z_a} = (f_{i_o})_{Z_b}$. The assumption concerning Z - injectivity of f_{i_o} implies that $a = b$, which proves that f is $\mathcal{Z}$ - - injective.

In the following we shall show the existence of the $\mathcal{Z}$ - injective mapping $f \in \mathcal{F}_{\mathcal{Q}}$ such that φ_f is Z - injective, but the function f_i is even not (Z,Q)-computable [6] for some i , $1 \le i \le n$.

E x a m p l e 1 . Let $Q_1 = Q_2 = Q = \mathcal{R}^+$, $Z = Z_1 = Z_2 = \langle 0;1)$, $S = \mathcal{R}$. We define a mapping $f = (f_1,f_2) : (\mathcal{R}^+)^2 \to \mathcal{R}^2$ as follows

$$f_1(t) = t \quad \text{for any} \quad t \in \mathcal{R}^+$$

$$f_2(t) = \begin{cases} 0 & \text{for} \quad t = 2^k ,\ k \in \mathcal{N}_0 \\ 1 & \text{for} \quad t \in \mathcal{R}^+ - \bigcup_{k\in\mathcal{N}_0} \{2^k\} . \end{cases}$$

This mapping is injective, hence it is $Z_1 x Z_2$ - injective for any $Z_1 x Z_2 \subseteq Q_1 x Q_2$. The mapping $\varphi_f : Q \to \mathcal{R}$ is defined by

$$\varphi_f(t) = \begin{cases} (t,0) & \text{for} \quad t = 2^k , \\ (t,1) & \text{for} \quad t \in \mathcal{R}^+ - \bigcup_{k\in\mathcal{N}_0} \{2^k\}. \end{cases}$$

Since this mapping is injective, it is Z - injective for any $Z \subseteq Q$. The function f_1 is not (Z,Q)-computable for any $Z = \langle 0; \tau)$, $\tau \in \mathcal{R}^+ - \{0\}$

Now we shall give some general properties of (Z,Q) - systems for $Z_1,\ldots,Z_n$, $Q_1,\ldots,Q_n$ satysfying conditions (1) - - (2) only. Note at first that, by Definition 1, every operator M being the (Z,Q) - system is an injective operator. Similarly as for (Z,Q)-machines ([4]) it is easy to verify the following lemmas.

L e m m a 3. If $M \in \mathcal{A}_{Z,Q}$ is a (Z,Q) - system, then $f_{Z_{\bar a}} \in DM$ and $M(f_{Z_{\bar a}}) = f_{Q_{\bar a}}$ for any $f \in RM$ where $\bar a = (a,\ldots,a) \in Q_0^n$.

L e m m a 4. If $M \in \mathcal{A}_{Z,Q}$ is a (Z,Q) - system, then for any $f,h \in RM$ and $\bar a, \bar b \in Q_0^n$, where $\bar a = (a,\ldots,a)$, $\bar b = (b,\ldots,b)$, if $f_{Z_{\bar a}} = h_{Q_{\bar b}}$ then $f_{Q_{\bar a}} = h_{Q_{\bar b}}$.

D e f i n i t i o n 5. A mapping $f \in \mathcal{F}_Q$ is said to be a (Z,Q) - process iff there exists a (Z,Q) - system M such that $f \in RM$.

The following theorem can be proved in the same way as it has been done for (Z,Q) - machines.

T h e o r e m 1. A mapping $f \in \mathcal{F}_Q$ is a (Z,Q) - process iff

$$(6) \qquad \forall_{\substack{\bar a, \bar b \in Q_0^n \\ \bar a = (a,\ldots,a) \\ \bar b = (b,\ldots,b)}} \left[f_{Z_{\bar a}} = f_{Z_{\bar b}} \longrightarrow f_{Q_{\bar a}} = f_{Q_{\bar b}} \right].$$

Comparing this theorem and Definition 3 it is clear that every Z - injective mapping $f \in \mathcal{F}_Q$ is a (Z,Q) - process. Consequently, for $Z_1 = Z_2 = \ldots = Z_n = Z$, $Q_1 = Q_2 = \ldots = Q_n = Q$ if $f = (f_1,\ldots,f_n) \in \mathcal{F}_Q$ is such that f_i is Z - injective for some i, $1 \le i \le n$, then f is a (Z,Q) - process.

The class of the (Z,Q) - processes is characterized by the following result.

T h e o r e m 2 . If $f \in \mathcal{F}_Q$ is a (Z,Q) - process, then it satisfies exactly one of the following conditions:

(7) f is Z - injective,

(8) there exist $\bar{a}_0, \bar{b}_0 \in Q_0^n$ where $\bar{a}_0 = (a_0,\dots,a_0)$, $\bar{b}_0 = (b_0,\dots,b_0)$, $a_0 \neq b_0$, such that $f_{Z_{\bar{a}_0}} = f_{Z_{\bar{b}_0}}$ and

for any $\bar{a}, \bar{b} \in Q_0^n$, $\bar{a} = (a,\dots,a)$, $\bar{b} = (b,\dots,b)$, $a < b$ if $b - a \in Q_0$, then the equality $f_{Z_{\bar{a}}} = f_{Z_{\bar{b}}}$ implies that $f_{Q_{\bar{a}}}$ is a periodic mapping of period $b - a$. Moreover, if Q_0 has the property that for any $a, b \in Q_0$, $a < b$ the relation $b - a \in Q_0$ holds, then condition (8) is also a necessary condition for a mapping to be a (Z,Q) - process.

The following theorem proves the existence of mappings $f \in \mathcal{F}_Q$ which are not (Z,Q) - processes.

T h e o r e m 3 . If $\bar{S} \geq 2$ then for any $Z \subsetneq Q$ there exists a mapping $f \in \mathcal{F}_Q$ which is not a (Z,Q) - process.

P r o o f . Let $(t_1^0,\dots,t_n^0) \in Q - Z$. Then there exists i, $1 \leq i \leq n$ such that $t_i \in Q_i - Z_i$. Denote by t_0

$\max_m \left\{t_{i_1}^0,\dots,t_{i_n}^0\right\}$, where $t_{i_m} \in Q_{i_m} - Z_{i_m}$ for any $1 \leq m \leq k_n$.

One can choose $w \in Q_0$ such that $w > t_0$. It follows from condition (2) that $(w,\dots,w) \in Q - Z$. Let $y^{(1)}, y^{(2)} \in S$ and $y^{(1)} \neq y^{(2)}$. We define a mapping $f = (f_1,\dots,f_n)$ as follows

$$f_i(t) = \begin{cases} y^{(1)} & \text{for } t < 4w \\ y^{(2)} & \text{for } t \geq 4w \end{cases} \qquad \text{for } i \in \left\{i_1,\dots,i_{k_n}\right\}$$

and

$$f_i(t) = \begin{cases} y^{(1)} & \text{for } 2kw \leq t < (2k+1)w \\ y^{(2)} & \text{for } (2k+1)w \leq t < (2k+2)w \end{cases} \qquad k \in \mathcal{N}_0$$

$$\text{for } i \in \{1,\dots,n\} - \left\{i_1,\dots,i_{k_n}\right\}.$$

Note that for any $(t_1,\dots,t_n) \in Z$ $t_{i_1} + 2w < 4w, \dots, t_{i_{k_n}} + 2w < 4w$.

Now it is easy to check that $f(t_1,\dots,t_n) = f(t_1+2w,\dots,t_n+2w)$ for any $(t_1,\dots,t_n)\in \mathcal{Z}$, i.e. $f_{\mathcal{Z}} = f_{\mathcal{Z}_{2w}}$. On the other hand we have $f(2w,\dots,2w) = (y^{(1)},\dots,y^{(1)}) \neq f(4w,\dots,4w) = f_{\mathcal{Q}_{2w}}(2w,\dots,2w)$ which proves that f is not a $(\mathcal{Z},\mathcal{Q})$ - process.

Note that there exist $(\mathcal{Z},\mathcal{Q})$ - systems such that the set of i-th subcomputations (for some i, $1<i<n$) of that system cannot be a subset of any (Z,Q)-computable set [6].

E x a m p l e 2. The mapping f defined in Example 1 is $\mathcal{Z}$ - injective for $\mathcal{Z} = Z_1 \times Z_2$ where $Z_1 = Z_2 = \langle 0;\frac{1}{2})$, hence it is a $(\mathcal{Z},\mathcal{Q})$ - process. Consequently, an operator $M\in \mathcal{A}_{\mathcal{Z},\mathcal{Q}}$ defined by the formula

$$DM = \left\{f_{\mathcal{Z}_{\bar{a}}} : \bar{a} = (a, \ )\in Q_0^{\,2}\right\}, \quad M(f_{\mathcal{Z}_{\bar{a}}}) \underset{df}{=} f_{\mathcal{Q}_{\bar{a}}}$$

is the $(\mathcal{Z},\mathcal{Q})$ - system. The function f_2 defined in Example 1 belongs to the set of 1-st subcomputations of that system. Since this function is not (Z_2,Q)- computable, the set of these subcomputations cannot be a subset of any (Z_2,Q) - computable set.

Finally we shall show that for any finite sequence of one dimensional (Z_i,Q_i),- machines M_i, $i = 1,\dots,n$ such that

$\mathcal{N}_0\subseteq Q_i$, $Z_i\subseteq \bigcap_{i=1}^{n} Q_i$ one can construct the $(\mathcal{Z},\mathcal{Q})$ - system.

This system has the property that its set of i-th subcomputations, $i = 1,\dots,n$ is (Z,Q) - computable for some Z,Q (not necessarly equal to Z_i and Q_i).

Let us denote, as previously, by Q_0 the set $\bigcap_{i=1}^{n} Q_i$

and by RM_i, $i = 1,\dots,n$ the set of computations of the machine M_i. For any i, $i = 1,\dots,n$, let us distinguish the subset of such $f\in RM_i$ for which $h\in RM_i$ and $a\in Q_i - Q_0$ such that $f = h_{Q_a}$ does not exist. Denote these sets by R_1M_i, $i = 1,\dots,n$. They have the following property

$$R_1M_i = \left\{f_{Q_a} : f\in R_1M_1 \wedge a\in Q_0\right\} \text{ for } i = 1,\dots,n \tag{9}$$

We define an operator $M : \mathcal{F}_Z \longrightarrow \mathcal{F}_Q$ as follows

$$DM = \left\{ f_Z \ : \ f = (f_1,\dots,f_n) \in R_1M_1 x \ \dots \ xR_1M_n \right\}$$

$$\forall_{h = (h_1,\dots,h_n) \in DM} \left[M(h) \underset{df}{=} (M_1(h_1),\dots,M_n(h_n)) \right]$$

We have to prove that M is the (Z,Q) - system. Let $h \in DM$. Then there exists $f = (f_1,\dots,f_n) \in R_1M_1 x \ \dots \ xR_1M_n$ such that $h = f_Z$. As the immediate consequence of the construction of the sets R_1M_i we obtain that $f_i | Z_i \in DM_i$, $i = 1,\dots,n$. Hence we get

$$M(h) = M(f|Z) = (M_1(f_1|Z_1),\dots,M_n(f_n|Z_n)) = (f_1,\dots,f_n) = f$$

and $M(f|Z)|Z = f|Z = h$ which proves that condition (3) of Definition 1 holds. By equality (9) we get that M satisfies condition (4) as well. The set of i-th subcomputations $(i = 1,\dots,n)$ is (Z_i,Q_o)-computable.

REFERENCES

[1] Z. G r o d z k i : The theory of shift-registers, Inform. Contr. 21 (1972) 196-205.

[2] B. K o n i k o w s k a : Continuous machines, Inform. Contr. 22 (1973) 353-372.

[3] Z. P a w l a k : Stored program computers, Algorytmy 10 (1969) 5-19.

[4] E. S t a n k i e w i c z : On some properties of (Z,Q)--machine and its computations, Bull.Acad.Polon.Sci., Ser.Sci. Math.Astronom.Phys. 25 (1977) 205-208.

[5] W. Ż a k o w s k i : Continuous simple Z-machines, Z-computations and Z-computable sets, Bull.Acad.Polon. Sci., Ser. Sci. Math. Astronom. Phys. 22 (1974) 1063-1067

[6] W. Ż a k o w s k i : A Generalization of the notions of a machine and computability, Inform.Contr. 33 (1977) 166-176.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW

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