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PARTIAL DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Introduction

The partial differential equations that occur in physics and controll the time evolution of a considered physical situation can be treated as a dynamical systems on a Hilbert (or Banach) spaces of functions (e.g. the wave equation, the Klein-Gordon equation) or on an infinite-dimensional manifolds modelled on Banach spaces (e.g. the Euler equation in the hydrodynamics). One obtains however the vector fields which are not continuous and in general are defined only on a dense subspace. Therefore the existencial theorems for such a problems are not the immediate consequences of the theorem about existence and uniqueness of the integral curves for a vector field (satisfying Lipschitz condition) defined on a Banach manifold. In the linear cases the existence theorems arise from the Stone theorem if one proves at first that the linear operator defining the vector field is skew adjoint under the real Hilbert space product (see e.g. [1]).

However if we have to do with the nonlinear densely defined vector field on the Hilbert space we have not any criterium which allows to determine if this field generates (or is a generator) a local one-parameter semigroup (see §3 in [1]).

In this moment the natural question occurs: can we choose the space of functions on which the vector field is constructed from the partial differential equation in such a way that this field is on it suitably regular, i.e. generates on this space a local one-parameter group. It appears that the appropriate choice of the space to reduce the problem to integration of the smooth vector field on a Banach space is fairly easy in the case of partial differential equations with the constant coefficients. However in the nonlinear case an attempt of a realisation of this reduction leads to a very poor spaces of functions, which is motivated in the mentioned below considerations (proposition 3).

1. The spaces $B_r(\Omega)$ and their application for the partial differential equations with constant coefficients

Let $\Omega \subset R^n$ be an open subset. Consider the partial differential equation of the form

$$\frac{\partial^k u}{\partial t^k} = \sum_{|\alpha| \leqslant m} A^\alpha D_\alpha u, \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\{0\} \cup N)^n$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $(t, x_1, \ldots, x_n) \in R \times \Omega$, A^α are the real $p \times p$ matrices, $D_\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$ and u is a function on $R \times \bar{\Omega}$ which takes values in R^p. Assume that the given functions $f_0, f_1, \ldots, f_{k-1}$ on $\bar{\Omega}$ are the initial data for the equation (1), i.e. $\frac{\partial^i u}{\partial t^i}(0,x) = f_i(x)$, $i=0,1,\ldots,k-1$. Define the now unknown functions $u_j := \frac{\partial^j u}{\partial t^j}$, $j=0,1,\ldots,k-1$. Then the equation (1) is equivalent with the following system:

$$\begin{cases} \frac{\partial u_0}{\partial t} = u_1 \\ \frac{\partial u_1}{\partial t} = u_2 \\ \vdots \\ \frac{\partial u_{k-2}}{\partial t} = u_{k-1} \\ \frac{\partial u_{k-1}}{\partial t} = \sum_{|\alpha| < m} A^\alpha D_\alpha u_0 . \end{cases} \quad \text{where } u_j(0,x) = f_j(x) \text{ are given.}$$

We can write this system in the form

$$\frac{\partial \tilde{u}}{\partial t} = \sum_{|\alpha|<m} \tilde{A}^{\alpha} D_{\alpha} \tilde{u}, \tag{2}$$

where $\tilde{u}: R \times \Omega \longrightarrow R^{kp}$, $\tilde{u}(t,x) := (u_0(t,x),\dots,u_{k-1}(t,x))$, and $\tilde{A}^{\alpha}$ are the $kp \times kp$ matrices. The initial condition is then given by the formula: $u(0,x) = (f_0(x), f_1(x),\dots,f_{k-1}(x))$.

The equation (2) we can treat as the equation for an integral curve of a vector field in the following way: if we interpret $t \longmapsto \tilde{u}(t,\cdot)$ as a curve in some space of functions on $\bar{\Omega}$ then we obtain the equation (2) for the (linear) vector field L on this space given by the formula

$$L(f) := \sum_{|\alpha|<m} \tilde{A}^{\alpha} D_{\alpha} f.$$

The equation (2) takes then the form

$$\frac{d\tilde{u}_t}{dt} = L(\tilde{u}_t), \quad \tilde{u}_t(\cdot) := \tilde{u}(t,\cdot) \tag{3}$$

and the given function $\tilde{u}(0,x) = \tilde{u}_0(x)$ we interpret as a point, from which the integral curve $t \longmapsto \tilde{u}_t$ of L starts at the time $t = 0$.

It is easy to see that if we want to consider the derivatives in the common sense (not as in the distribution theory) and also assume that L maps the space of functions on $\bar{\Omega}$ into itself, then this domain of L ought to be a subset of $C^{\infty}(\bar{\Omega})$. The sufficient condition for existence of the solutions of (3) on this space is e.g. the existence of some Banach norm on it such that L is the continuous operator.

We will consider some spaces of C^{∞} functions on $\bar{\Omega}$ taking values in R^{kp} with bounded on Ω derivatives. Define

$$a_k(f) := \max_{|\beta|=k} \sup_x |D_{\beta} f(x)|, \quad \text{where } f \in C^{\infty}(\bar{\Omega}, R^{kp}). \tag{4}$$

We introduce the family of spaces of smooth functions indexed by the real positive parameter r:

$$B_r(\Omega) := \left\{ f \in C^\infty(\bar{\Omega}, R^{kp}) : \sup_{0<k\in Z} r^{-k} a_k(f) < \infty \right\}.$$

It is easy to check that $\|\cdot\|_r$ defined by

$$\|f\|_r := \sup_{0<k\in Z} r^{-k} a_k(f) \tag{5}$$

is a norm on $B_r(\Omega)$. Then $(B_r(\Omega), \|\cdot\|_r)$ are the Banach spaces (the completness is evident). Now we sum up the interesting properties of these spaces in the following statement.

T h e o r e m 1. Let $B_r := B_r(\Omega)$. The spaces B_r have the properties:

1^0 For $s<r$ we have $B_s \subset B_r$, and this inclusion is continuous, i.e. $\|f\|_r < \|f\|_s$ for any $f \in B_s$;

2^0 In the case when B_r are the spaces of real (complex) valued functions, they do not form the algebras, but the multiplication of functions continuously maps $B_r \times B_s \longrightarrow B_{r+s}$, moreover

$$\|fg\|_{r+s} < \|f\|_r \|g\|_s .$$

3^0 Each differential operator L with constant coefficients maps any space B_r into itself and is continuous on it.

P r o o f:

ad 1^0 It is evident (see (4) and (5)).

ad 2^0 Consider the functions $f(x_1,\dots,x_n) := \sin rx_1$ and $g(x_1,x_2,\dots,x_n) := \cos rx_1$ on $\bar{\Omega}$. Then

$$a_{2j}(f) = Cr^{2j}, \quad a_{2j+1}(f) = Dr^{2j+1}, \qquad j=0,1,\dots$$

where $C = \sup_{x\in\Omega} |\sin rx_1|$ and $D = \sup_{x\in\Omega} |\cos rx_1|$.

For the function g we obtain

$$a_{2j}(g) = Dr^{2j} \quad \text{and} \quad a_{2j+1}(g) = Cr^{2j+1}.$$

So $f,g \in B_r(\|f\|_r = \max\{C,D\} = \|g\|_r)$. On the other hand, $fg(x) = \frac{1}{2}\sin 2rx_1$ and $a_k(fg) \geq 2^{k-1}r^k \min\{C,D\}$. The above inequality implies that $fg \notin B_r$.

For the proof of the second statement in this point it is clearly sufficient to establish for any functions $f \in B_r$ and $g \in B_s$ the following inequality

$$\|fg\|_{r+s} < \|f\|_r \|g\|_s.$$

Note first that

$$\sup_{x\in\Omega} |D_\alpha f(x)| \leq \|f\|_r r^{|\alpha|}$$

what is straightforward consequence of (4) and (5).
Now

$$\|fg\|_{r+s} = \sup_{0<k} \left((r+s)^{-k} \max_{|\beta|=k} \sup_{x\in\Omega} |D_\beta (fg)(x)|\right) =$$

$$= \sup_k \left((r+s)^{-k} \max_{|\beta|=k} \sup_{x\in\Omega} \left|\sum_{0<\alpha\leq\beta} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n} (D_\alpha f)(D_{\beta-\alpha} g)\right|\right) \leq$$

$$\leq \sup_k \left((r+s)^{-k} \max_{|\beta|=k} \sum_{0<\alpha<\beta} \binom{\beta_1}{\alpha_1} \cdot\cdot \binom{\beta_n}{\alpha_n} \sup_{x\in\Omega} |D_\alpha f(x)| \sup_{x\in\Omega} |D_{\beta-\alpha} g(x)|\right) \leq$$

$$\leq \|f\|_r \|g\|_s \sup_k \left((r+s)^{-k} \max_{|\beta|=k} \sum_{0<\alpha<\beta} \binom{\beta_1}{\alpha_1} \cdot\cdot \binom{\beta_n}{\alpha_n} r^{|\alpha|} s^{|\beta|-|\alpha|}\right) =$$

$$= \|f\|_r \|g\|_s,$$

because

$$\sum_{0<\alpha<\beta}\left(\prod_{j=1}^{n}\binom{\beta_j}{\alpha_j}\right) r^{|\alpha|}\, s^{|\beta|-|\alpha|} = \sum_{0<\alpha<\beta}\ \prod_{j=1}^{n}\binom{\beta_j}{\alpha_j} r^{\alpha_j}\, s^{\beta_j-\alpha_j} =$$

$$= (r+s)^{\beta_1}(r+s)^{\beta_2}\ \dots\ (r+s)^{\beta_n} = (r+s)^{|\beta|}.$$

ad 3^o To prove that the operators L of the form $L=\sum_{|\beta|<m} A^{\beta}D_{\beta}$ are continuous, it is sufficient to show the following inequalities:

(i) for a matrix $A : R^{kp} \longrightarrow R^{kp}$, we have

$$\|A\circ f\|_r < C\ \|f\|_r,$$

where the constant C can depend on A, but not on f;

(ii)
$$\|D_{\beta}f\|_r \leqslant M_{\beta}\ \|f\|_r,$$

where the constants M_{β} do not depend on f.

It is easy to see that (i) holds, if as C any number bigger than $\|A\|$ is taken, because

$$\sup_{x\in\Omega}|D_{\alpha}A\circ f(x)| = \sup_{x\in\Omega}|A\ D_{\alpha}f(x)| < \sup_{x\in\Omega}\|A\|\ |D_{\alpha}f(x)| =$$

$$= \|A\|\ \sup_{x\in\Omega}|D_{\alpha}f(x)|.$$

Now

$$\|D_{\beta}f\|_r = \sup_{k}\ (r^{-k}\max_{|\alpha|=k}\ \sup_{x\in\Omega}|D_{\alpha+\beta}f(x)|)\leqslant$$

$$\leqslant\sup_{k}\left(r^{-k}\max_{|\gamma|=k+|\beta|}\ \sup_{x}|D_{\gamma}f(x)|\right) = r^{|\beta|}\sup_{k} r^{-k-|\beta|}a_{k+|\beta|}(f) =$$

$$= r^{|\beta|}\sup_{|\beta|\leqslant j}(r^{-j}a_j(f)) < r^{|\beta|}\sup_{0\leqslant j}(r^{-j}a_j(f)) = r^{|\beta|}\|f\|_r,$$

what proves (ii) - as one can take $M_\beta = r^{|\beta|}$. Thus in the space B_r we have estimated the norm of the operator D_β: $\|D_\beta\|_r < r^{|\beta|}$. Using properly the functions

$$f_y(x) := \prod_{i=1}^{n} \sin r(x_i+y_i), \quad y \in R^n,$$

one can check that $\|D_\beta\|_r = r^{|\beta|}$ indeed. Q.E.D.

As an immediate consequence of Theorem 1, we obtain the following result.

T h e o r e m 2. The differencial operators with constant coefficients generate on any space $B_r(\Omega)$ the one--parameter groups of bounded, linear operators. In particular, in the space X of functions on $R \times \bar{\Omega}$ taking values in R^{kp} and satisfying the condition:

$(u \in X) \longleftrightarrow$ (exists $r > 0$ such that

(i) $$\|u(t,\cdot)\|_r < \infty \text{ for any } t \in R;$$

(ii) $$\forall t_0 \in R \; \forall \epsilon > 0 \; \exists \delta \; |t-t_0| < \delta \Longrightarrow$$
$$\longrightarrow \|u(t,\cdot) - u(t_0,\cdot)\|_r < \epsilon ;$$

(iii) $$\forall t_0 \in R \; \forall \epsilon > 0 \; \exists \delta \; |t-t_0| < \delta \longrightarrow$$
$$\longrightarrow \left\| \frac{\partial u}{\partial t}(t,\cdot) - \frac{\partial u}{\partial t}(t_0,\cdot) \right\|_r$$

the equation (2) with the initial condition $\tilde{u}(0,\cdot) = f(\cdot) \in \bigcup_{s>0} B_s(\Omega)$ possesses a unique global solution (defined for all $t \in R$ and $x \in \bar{\Omega}$).

P r o o f. The first part of Theorem 2 is a straightforward consequence of Theorem 1. Next notice that the space X consists of all differentiable curves on the spaces $B_r(\Omega)$. If $f(\cdot) \in B_s(\Omega)$, then, accordingly to the first part of the theorem, exists exactly one solution of (3) in the space $B_s(\Omega)$:

$$\tilde{u}(t,\cdot) = e^{tL} f(\cdot).$$

Obviously $\tilde{u} \in X$. It remains to prove the uniqueness of solution in the space X. Suppose that $v \in X$ and v satisfies (2) with the same initial condition $v(0,\cdot) = f(\cdot)$. Then $t \longmapsto v(t,\cdot)$ is a C^1 curve in some space $B_r(\Omega)$ and satisfies (3), i.e. $\frac{dv_t}{dt} = Lv_t$. Suppose now that $r \geqslant s$. If so, then in accordance with the point 1^o of Thm 1 the curve $t \longmapsto \tilde{u}_t$ is also a differentiable curve on $B_r(\Omega)$, and satisfies (3). The uniqueness of integral curves on B_r and the equality $u_o = v_o$ imply that $u = v$. In the case when $r < s$, the argument is the same, but u and v play the reversal parts. Q.E.D.

Notice that any function u of the form $u(t,x) = g(t)f(x)$, where $f \in \bigcup_r B_r(\Omega)$ and g is a real function of class C^1 on R, is an element of the space X introduced in Theorem 2. Though this theorem states existence and uniqueness of solutions in the case of simpler equations than considered in Cauchy-Kovalevska Theorem, it admits the nonanalytic functions (X contains the functions not differentiable twice in t). Still the proper thing to do is to point out that the spaces $B_r(\Omega)$ consist of the analytic functions only which can be widened analytically on the whole R^n. This property of B_r - functions follows from the assumed bounded velocity of the growth of derivatives. On the other hand, there exist analytic functions (even on R^n) with bounded all derivatives which are not in $\bigcup_{r>0} B_r(\Omega)$.

E x a m p l e 1. Let $\Omega := \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\tilde{f}(x) := \frac{1}{1+x^2}$. The function $\tilde{f}$ is analytic on R^1 and on $\bar{\Omega}$ has the uniformly convergent expansion into the power series:

$$\tilde{f}(x) := \tilde{f}|_{\Omega}(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

The derivatives of function $\tilde{f}$ are given be the formulas:

$$\tilde{f}^{(2k)}(x) = \frac{(2k)!}{(1+x^2)^{2k+1}} \sum_{l=0}^{k} \binom{2k+1}{2l} (-1)^{l+k} x^{2l},$$

$$\tilde{f}^{(2k+1)}(x) = \frac{-(2k+1)!}{(1+x^2)^{2k+2}} \sum_{l=0}^{k} \binom{2k+2}{2l+1} (-1)^{k+l} x^{2l+1}$$

and, as is easy to see, are bounded on the whole R^1. Now taking $x = 0$, we obtain the following inequality: $a_{2k}(f) \geqslant (2k)!$. As for each $r>0$ the sequence $\frac{2k!}{r^{2k}}$ is unbounded, we have $f \notin \bigcup_r B_r(\Omega)$.

E x a m p l e 2. It occurs that the problem of solving the heat equation with the given initial data of class B_r is in the space X (described in Thm 2) well posed "in both time directions" (usually in the theory of parabolic equations, we have to do with the semigroups, i. the solutions are defined in future: $t>0$). For example, the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ has in this space the unique solution satisfying the initial condition $u(0,x) = f(x) = \sin x$ ($f \in B_1(R^1)$), given by the formula:

$$u(t,x) = \left(\exp t \frac{d^2}{dx^2}\right) \sin x = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \sin x = e^{-t} \sin x.$$

E x a m p l e 3. Consider the hyperbolic equation $\frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial x^2}$, $(t,x) \in R^2$ with the initial data: $g_o(x) = g(0,x)$ and $h_o(x) = \frac{\partial g}{\partial t}(0,x)$, and let $g_o, h_o \in \bigcup_r B_r(R^1)$. Defining $h := \frac{\partial g}{\partial t}$ to be the new unknown function, we obtain the following system:

$$\begin{cases} \dfrac{\partial g}{\partial t} = h \\ \dfrac{\partial h}{\partial t} = \dfrac{\partial^2 g}{\partial x^2} \end{cases} \quad \text{with the initial data:} \quad \begin{array}{l} g(0,x) = g_o(x) \\ h(0,x) = h_o(x). \end{array}$$

Now we obtain by Thm 2 that the above problem has a unique solution in an appropriate space X of some functions on R^2 taking values in R^2:

$$\begin{pmatrix} g(t,\cdot) \\ h(t,\cdot) \end{pmatrix} = \left(\exp t \begin{bmatrix} 0 & 1 \\ D^2 & 0 \end{bmatrix}\right)\begin{pmatrix} g_o \\ h_o \end{pmatrix} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\begin{bmatrix} D^{2k} & 0 \\ 0 & D^{2k} \end{bmatrix}\begin{pmatrix} g_o \\ h_o \end{pmatrix} +$$

$$+ \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!}\begin{bmatrix} 0 & D^{2l} \\ D^{2l+2} & 0 \end{bmatrix}\begin{pmatrix} g_o \\ h_o \end{pmatrix} =$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} D^{2k} g_o + \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} D^{2l} h_o \\ \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} D^{2k} h_o + \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} D^{2l+2} g_o \end{bmatrix}.$$

Here D denotes differentiation of the real valued functions on R^1: $D = \frac{d}{dx}$. Thus the wanted function g is expressed by the functions g_o, h_o, and their derivatives in the form of the uniformly convergent series:

$$g(t,\cdot) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \frac{d^{2k} g_o}{dx^{2k}} + \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} \frac{d^{2l} h_o}{dx^{2l}}.$$

From the example 1, the following doubt arises naturally: are the spaces B_r rich enough for any applications? The answer to this question is positive. In the case, when the set Ω is bounded, any space $B_r(\Omega)$ contains all polynomials, because for the polynomial of degree m the coefficients a_k defined by (4) vanish for $k>m$. Hence the spaces $B_r(\Omega)$ are dense in $C(\bar{\Omega})$ equipped with the standard norm ($\bar{\Omega}$ is compact). Now we consider the case $\Omega = R^n$.

P r o p o s i t i o n 1. Let $K(0,r)$ denote the closed ball in R^n with center 0 and radius r. If $\varphi \in C_0^\infty(R^n)$ and $\operatorname{supp} \varphi \subset K(0,r)$, then the Fourier transform $\mathcal{F}(\varphi)$ of the function φ is an element of $B_r(R^n)$ (this space consists of complex valued functions). In particular, $\bigcup_r B_r(R^n) \supset \mathcal{F}(C_0^\infty(R^n))$.

P r o o f. It is sufficient to find the constant C, such that for any multiindex α the following inequality holds: $\sup_\xi |D_\alpha \mathcal{F}(\varphi)(\xi)| \leqslant C\, r^{|\alpha|}$. We have

$$\sup_\xi |D_\alpha \mathcal{F}(\varphi)(\xi)| = \sup_\xi \left| \int_{R^n} e^{-ix\cdot\xi}\, x^\alpha \varphi(x)dx \right| =$$

$$= \sup_\xi \left| \int_{K(0,r)} e^{-ix\cdot\xi} x^\alpha \varphi(x)dx \right| \leqslant \sup_\xi \int_{K(0,r)} \left| x^\alpha \varphi(x)\right| dx \leqslant$$

$$\leqslant r^{|\alpha|} \int_{K(0,r)} |\varphi(x)|\, dx \qquad (\text{where } x^\alpha := x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot \ldots \cdot x_n^{\alpha_n}).$$

So we obtain $\|\mathcal{F}(\varphi)\|_r \leqslant C := \int_{R^n} |\varphi(x)|\, dx.$ Q.E.D.

As an immediate consequence of the above proposition and Plancherel's theorem, we obtain the following result:

C o r o l l a r y. If $B_r(R^n)$ and $L^2(R^n)$ denote the spaces of R^p values functions, then the subspace $\bigcup_r B_r(R^n) \cap L^2(R^n)$ is dense in $L^2(R^n)$.

There is some connection between the spaces $B_r(\Omega)$ and $B_r(\Omega')$:

R e m a r k. As is easily seen, if $\Omega' \subset \Omega$, then $B_r(\Omega') \supset B_r(\Omega)\|_{\bar{\Omega}'} = \{f|_{\bar{\Omega}'} : f \in B_r(\Omega)\}$. But the functions exist of class B_r on $\bar{\Omega}'$ which are not restrictions of the functions from $B_r(\Omega)$ (if $\bar{\Omega}' \neq \bar{\Omega}$).

2. The nonlinear partial differential equations and Lipschitz dynamical systems

Now we consider some special cases of the nonlinear equations with unknown function $u(t,x_1,\ldots,x_n)$. The linear operator L in (3) will be replaced by the nonlinear differential operator P (acting in n variables $x_1,\ldots,x_n$):

$$P(g)(x_1,\ldots,x_n) := \tilde{P}(x_1,\ldots,x_n,\ldots,(D_\alpha g)(x_1,\ldots,x_n),\ldots),$$

where $|\alpha| \leq m$ and $\tilde{P}$ is the R^p valued function of $n+p\cdot \sum_{j=0}^{m} \binom{n+j-1}{j}$ variables.

In the simplest (quasilinear) case, in the equations like this

$$\frac{\partial u}{\partial t}(t,x) = P(u_t)(x) \tag{6}$$

already the products of unknown function u and its derivatives occur (or powers of unknown function - e.g. in the nonlinear Klein-Gordon equation: $\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi + m^2\varphi^s$).

Suppose moreover that u (and P) are real valued, i.e. $p=1$. Then if we would like to use the analogous scheme as before, we have to assume additionally that a space of functions (on which the vectorfield P is constructed) forms an algebra. We also need some metric on this space to formulate the Lipschitz condition for the vector field P. At first, we will show that in general the vector fields, satisfying the Lipschitz condition on the algebras equipped with the Fréchet metric, have no unique integral curves.

Let an open set $\Omega \subset R^n$ be bounded. Then the space $Y :=$ $= C^\infty(\bar\Omega)$ with the standard metric $\tilde d$ (invariant under translations) defined by the formula

$$\tilde d(f,g) = d(f-g) := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\sum_{|\alpha|=n} \sup|D_\alpha(f-g)|}{1 + \sum_{|\alpha|=n} \sup|D_\alpha(f-g)|}$$

is the Fréchet space and forms the algebra under addition and multiplication of functions.

P r o p o s i t i o n 2. Let $L : Y \to Y$ be the differential operator with constant coefficients, of order m, i.e. $L = \sum_{|\beta|\leqslant m} a^\beta D_\beta$. Then L is bounded.

P r o o f. Let $k \in N$ be such integer that for any β we have $|a^\beta| < k$. Then

$$d(Lf) \leqslant \sum_{|\beta|\leqslant m} d(a^\beta D_\beta f) \leqslant k \sum_{|\beta|\leqslant m} d(D_\beta f).$$

The second inequality follows by the properties of metric:

$$d\left(\frac{a^\beta}{k}\, h\right) \leqslant d(h) \ \left(\text{as } \left|\frac{a^\beta}{k}\right| < 1\right) \quad \text{and} \quad d(kh) \leqslant k\, d(h).$$

Furthermore

$$d(D_\beta f) = \sum_{j=0}^{\infty} 2^{-j} \frac{\sum_{|\alpha|=j} \sup|D_{\alpha+\beta} f|}{1 + \sum_{|\alpha|=j} \sup|D_{\alpha+\beta} f|} \leqslant$$

$$\leqslant \sum_{j=0}^{\infty} 2^{-j} \frac{\sum_{|\gamma|=j+|\beta|} \sup|D_\gamma f|}{1 + \sum_{|\gamma|=j+|\beta|} \sup|D_\gamma f|} = 2^{|\beta|} \sum_{l=|\beta|}^{\infty} 2^{-l} \frac{\sum_{|\gamma|=l} \sup|D_\gamma f|}{1+\sum_{|\gamma|=l} \sup|D_\gamma f|} \leqslant$$

$$\leqslant 2^{|\beta|} d(f).$$

Thus

$$d(Lf) \leq k \cdot \sum_{|\beta| \leq m} 2^{|\beta|} d(f).$$

Taking

$$M := k \sum_{|\beta| \leq m} 2^{|\beta|} = k \sum_{j=0}^{m} 2^j \binom{m+j-1}{j}$$

we obtain for any $f \in Y$: $d(Lf) \leq M\, d(f)$. Q.E.D.

It occurs that even the linear, bounded (and so Lipschitz) operators L do not generate the one - parameter groups on the space Y.

Example 4. Let $L := \frac{d}{dx}$ and $Y := C^{\infty}([0, 1])$. We construct two different integral curves φ_1, φ_0 of L passing by $0 \in Y$. Let $f_0, f_1 \in C^{\infty}(R^1)$ be given by the formulas:

$$f_0(x) = 0; \qquad f_1(x) = \begin{cases} 0 & \text{for } x \in [0,1] \\ \exp -x^{-2} & \text{for } x > 0 \\ \exp -(x-1)^{-2} & \text{for } x < 1 \end{cases}$$

It is easy to see that $\varphi_0, \varphi_1 : R^1 \longrightarrow Y$, given by

$$[\varphi_i(t)](x) := f_i(t+x), \quad i = 0,1,$$

are differentiable curves in Y, $\frac{d}{dt}\varphi_i(t) = L\,\varphi_i(t)$, and $\varphi_0(t) \neq \varphi_1(t)$ for each $t \neq 0$, but $\varphi_0(0) = \varphi_1(0) = 0$.

So if we want to integrate the Lipschitz vector fields on a Fréchet space, we need some additional conditions for a metric d. The study of the construction of integral curves leads in the case of Fréchet spaces of functions to the following notion:

D e f i n i t i o n. By the Fréchet algebra we mean such Fréchet space (Y,d) that Y is algebra and for any $(f,g) \in Y \times Y$

$$(7) \qquad d(fg) \leq d(f)d(g).$$

However the condition (7) is strongly restrictive:

T h e o r e m 3. Let (Y,d) be the Fréchet algebra with unity. Then exists on Y such norm $\| \ \|$ that $(Y, \| \ \|)$ is Banach algebra, and the topology on Y given by this norm is the same as the original topology.

P r o o f. We use Kolmogorov's theorem (see e.g. [2]) which states that the Hausdorff topological vector space is normable iff exists in it a bounded and convex neighborhood of 0. We first prove that balls in Y are bounded. It follows by the inequality $d(rf) < d(r)d(f)$ $(r \in R)$ and $\lim_{r \to 0} d(r) = 0$ (that is $\forall \varepsilon > 0 \ \exists r \neq 0 \ d(r) < \varepsilon$). Denoting by $K(0,r)$ the open ball in Y with center 0 and radius r, we obtain from the above

$$\forall \varepsilon > 0 \ \ \forall r_0 > 0 \ \ \exists r \neq 0 \ \ r\,K(0,r_0) \subset K(0, \varepsilon r_0).$$

Rewriting we have

$$\forall r_0 > 0 \ \ \forall \varepsilon > 0 \ \ \exists \lambda \neq 0 \ \ K(0,r_0) \subset \lambda K(0, \varepsilon r_0).$$

The last inclusion means that $K(0,r_0)$ is the bounded neighborhood of 0 in Y (as the balls $K(0, \varepsilon r_0)$, $\varepsilon > 0$, make a basis of neighborhoods of 0). On the other hand, Y is locally convex (as Y is the Fréchet space), and thus Y has a basis of convex neighborhoods of 0, so in particular exists a convex neighborhood contained in $K(0,1)$. Because any subset of a bounded set in topological vector space is bounded, the convex neighborhood contained in $K(0,1)$ is also simultaneously bounded. Hence by the mentioned Kolmogorov theorem the space Y is normable.

Now let $\|\cdot\|_1$ be a norm on Y determining the original metric topology. The continuity of multiplication in the algebra Y implies that a constant C exists such that for any $f,g \in Y$ $\|fg\|_1 < C \|f\|_1 \|g\|_1$. Taking the equivalent norm $\|\ \| := C\|\ \|_1$, we obtain from the above that $\|fg\| = C\|fg\|_1 < C^2 \|f\|_1 \|g\|_1 = \|f\| \|g\|$. Thus $(Y, \|\ \|)$ is such Banach algebra that Y has the same topology as defined by d. Q.E.D.

R e m a r k. The condition (7) and the local convexity of Fréchet algebra imply (for algebra with unity!) the result formulated in Thm 3. The following problem remains open: does there exist a nonnormable metric algebra (with unity) fullfilling the condition (7) ? This condition would then mean that Y is locally bounded. However it is well known that there exist the locally bounded (and thus metrizable - see [2]) nonnormable topological vector spaces; e.g. the spaces $L^p(R)$ for $p<1$ are of that type.

Farther on we will consider the Banach algebras of functions such that differentiations are continuous operators. Unfortunately it occurs that this algebras are very poor (the trivial example of such algebra is the space of constant functions). The theorem formulated below for the functions of one variable sufficiently motivates the above statement in the case of several variables too.

P r o p o s i t i o n 3. Let B with the norm $\|\ \|$ be a Banach algebra of functions such that $B \subset C^\infty(R)$ and the differentiation $D : B \ni f \longmapsto f' \in B$ is continuous. Then the functions $g_t(x) := \sin tx$, $h_t(x) := e^{tx}$, $(t \neq 0)$ and $h(x) := x$ are not in B $(g_t, h_t, h \notin B)$.

P r o o f. Suppose, that for some $t \in R$ we have $g_t \in B$. Then $g_t' \in B$, and for any $n \in N$ the function g_{nt} belongs to B, as it is the polynomial of g_t' and g_t. Notice also that $D^2 g_{nt} = -(nt)^2 g_{nt}$, what implies that $\|D^2 g_{nt}\| = n^2 t^2 \|g_{nt}\|$. On the other hand $\|D^2 g_{nt}\| \leq M^2 \|g_{nt}\|$, where the constant M

is the norm of D. Thus we obtain inequality (valid for any $n \in N$):

$$n^2t^2 \leqslant M^2.$$

This inequality holds iff $t = 0$. So $g_t \notin B$ for $t \neq 0$.

Suppose now that $h_t \in B$. Then $h_{nt} = (h_t)^n \in B$, and $\|Dh_{nt}\| = n|t|\,\|h_{nt}\|$. By the same argument as before, these equalities (for any $n \in N$) are compatible with the existence of a bound M iff $t = 0$. Hence $h_t \notin B$ for $t \neq 0$.

Now if $h \in B$, then $h^n \in B$. But $Dh^n = nh^{n-1}$, while $\|Dh^n\| < M\,\|h^n\| \leqslant M\,\|h^{n-1}\|\,\|h\|$, and thus for any $n \in N$ one obtains inequality

$$n < M\,\|h\|.$$

This is impossible, which proves that $h \notin B$. Q.E.D.

One can replace the norm in the above proposition by the metric d sytisfying (7). Then if D is bounded, we obtain the same result. However the proof does not work in the case when D is continuos but simultaneously unbounded.

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