DEMONSTRATIO MATHEMATICA
Vol. XI No 2 1978

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THE MAXIMAL k-MACHINES

1. Introduction

The aim of this paper is to give some properties of the k-machines, all computations of which are with the maximal cycle length i.e. 2^k (only the k-machines defined in a two-element alphabet M with total transition function $\varphi : M^k \to M$ will be considered)[1]. Such k-machines will be called maximal.

Problems related to maximal k-machines have been studied by many authors.[2] In Hall's paper [4] the cardinality of the set of all maximal k-machines (k fixed) has been given. Yoeli in [8] shows the possibility of obtaining a maximal k-machine from a given one by modification of its transition function, but he does not give an algorithm for this. Fredricksen in the papers [1] and [2] using Yoeli's method gives an algorithm for constructing a maximal k-machine for arbitrary $k > 1$. Using this algorithm we can not obtain all maximal k-machines. Golomb in [3] has investigated the problem of the existence of maximal cycles for linear k-machines. To every k-machine is assigned a unique polynomial of degree k, the properties of which allow an answer to the question whether the cycle of all its computations is maximal or not. A necessary and sufficient condition for the transition functions of k-machines to be maximal has not yet been given, even for the linear case.

1) The formal definition as well as the fundamental properties of the k-machines in the more general case have been given in [5].

2) Technical applications of the maximal k-machines have been given in Golomb's monograph [3].

This paper consists of some new results relating to maximal k-machines. The set D^k of all k-machines (k is fixed) whose transition functions satisfy the condition $\varphi(t'_1,t_2,\dots t_k) \neq \varphi(t_1,t_2,\dots t_k)$, when $t'_1 \neq t_1$, will be considered here (the transition function of every maximal k-machine satisfy above condition). In the set D^k the distance between two k-machines (having the transition functions φ and ψ, respectively) will be introduced as the cardinality of the set of all sequences $(t_2,\dots t_k) \in M^{k-1}$ for which $\varphi(a,t_2,\dots t_k) \neq \psi(a,t_2,\dots t_k)$ for all $a \in M$.

In the metric space D^k (with the distance taken as the metric) for an arbitrary k-machine, all maximal k-machines belong to some spheres. That k-machine is the center of these spheres (one for all) and their radii are determined uniquely by that center. If as the center of the spheres is taken the k-machine with transition function $\varphi(t_1,\dots t_k) = t_1$, then the above property implies that we will know the number of ones (or zeros) which are the values of transition functions of maximal k-machines at the points $(0,t_2,\dots t_k)$, (or $(1,t_2,\dots t_k)$).

Unfortunately, the results which have been obtained here do not solve completely the problem of maximal k-machines, but in author's opinion these results will lead to the solution in the future.

2. Basic definitions

The notations of [5] will be used here.

Definition 2.1. By a k-machine A_k we mean a pair (M,φ), where $M = \{0,1\}$ is an alphabet and $\varphi : M^k \to M$ is a total function (the transition function of k-machine A_k).

Definition 2.2. By a computation of the k-machine $A_k = (M,\varphi)$ we mean every sequence $T \in M^\infty$ such that

$$(2.1) \qquad \forall_{i \geq 1} \left(T|_{k+i,k+i} = \varphi(T|_{i,i+k-1}) \right) .$$

The set of all computations of the k-machine A_k will be denoted by $C(A_k)$[3].

D e f i n i t i o n 2.3. A sequence $T \in M^{\infty}$ is said to be periodic iff the following condition is satisfied

$$(2.2) \qquad \exists_{p \geq 1} \forall_{i \geq 1} (T|_{p+i} = T|_i) .$$

The least value of p satisfying (2.2) is called the period and $T|_{1,p}$ - the cycle of T.

Let D^k denote the set of all k-machines (k is fixed) all computations of which are periodic. The k-machines of the class D^k will be called periodic. In this paper only periodic k-machines will be considered.

D e f i n i t i o n 2.4. Let us define the relation $Sh \subseteq M^o \times M^o$ (a restriction operation) as follows

$$(2.3) \qquad Sh(U,V) \Leftrightarrow \exists_{i \geq 1} (U = V|_i) ,$$

where M^o denotes the set of all periodic sequences.

R e m a r k 2.1. It follows immediately from the above definition that Sh is equivalence relation in M^o. An equivalence class designated by an element $T \in M^o$ with respect to the relation Sh will be denoted by [T] .

D e f i n i t i o n 2.5. By a complexity degree of A_k (denoted by $\deg(A_k)$) we mean the cardinality of the set of all equivalence classes designated by relation Sh in the set $C(A_k)$[4].

D e f i n i t i o n 2.6. By a distance between two k-machines $A_k = (M,\varphi)$ and $B_k = (M,\psi)$ (denoted by dist (A_k, B_k)) we mean the cardinality of the set $\{U \in M^{k-1}: \forall_{a \in M} (\varphi(aU) \neq \psi(aU))\}$.

3) A set $E \subset M^{\infty}$ is said to be a k-computation set iff there is a k-machine A_k such that $E = C(A_k)$.

4) For each computation $T \in C(A_k)$ all its restrictions belong to $C(A_k)$ (it has been shown in [5]).

D e f i n i t i o n 2.7. A k-machine A_k is included in a k-machine B_k (denoted by $A_k \subseteq B_k$) iff the following condition is satisfied

$$\text{dist}(A_k, B_k) = \deg(A_k) - \deg(B_k) \text{ .}^{5)} \tag{2.4}$$

D e f i n i t i o n 2.8. By a dimension of a k-machine $A_k = (M, \varphi)$ in the point $a \in M$ (denoted by $\dim_a(A_k)$) we mean the cardinality of the set $\{U \in M^k : U|_{1,1} = a, \varphi(U) \neq a\}$.

3. Basic theorems

Some theorems which are necessary for understanding the further results will be given.

For arbitrary sequence $U \in M^k$ let $^*U$ denote such a sequence $V \in M^k$ that $U|_{1,1} \neq V|_{1,1}$ and $U|_{2,k} = V|_{2,k}$.

T h e o r e m 3.1. A k-machine $A_k = (M, \varphi)$ is periodic iff the following condition is satisfied

$$\forall_{U \in M^k} (\varphi(U) \neq \varphi(^*U)) \text{ .} \tag{3.1}$$

The proof of this theorem has been given in [10].

We remind that only periodic k-machines will be considered.

R e m a r k 3.1. It follows from Theorem 3.1 that the dimension of an arbitrary periodic k-machine A_k defined in two-element alphabet does not depend on the point $a \in M$, and will be denoted by $\dim(A_k)$.

T h e o r e m 3.2. For arbitrary k-machines A_k and B_k - if $\text{dist}(A_k, B_k) = 1$ then $|\deg(A_k) - \deg(B_k)| = 1$.

An idea of the proof of Theorem 3.2 is based on Yoeli's paper [8] and will be not recalled here.

T h e o r e m 3.3. (D^k, dist) is a metric space.

P r o o f . It follows immediately from the definition of dist that for all k-machines A_k and B_k we have

5) The fundamental properties of the relation $\subseteq$ have been studied in [9].

$dist(A_k,B_k) \geq 0$ ($dist(A_k,B_k) = 0$ iff $A_k = B_k$) and $dist(A_k,B_k) = dist(B_k,A_k)$.

Let us consider arbitrary k-machines $A_k = (M,\varphi)$, $B_k = (M,\psi)$ and $C_k = (M,\xi)$. The condition $dist(A_k,B_k) \leq dist(A_k,C_k) + dist(C_k,B_k)$ follows from the condition

$$G \subseteq (E \cup F) , \tag{3.2}$$

where $E = \{U \in M^{k-1} : \varphi(OU) \neq \xi(OU)\}$, $F = \{U \in M^{k-1} : \psi(OU) \neq \xi(OU)\}$ and $G = \{U \in M^{k-1} : \varphi(OU) \neq \psi(OU)\}$.

By a ball and sphere of center A_k and radius $r \geq 0$ will be understood the sets $B(A_k,r) = \{B_k \in D^k : dist(A_k,B_k) < r\}$ and $Sph(A_k,r) = \{B_k \in D^k : dist(A_k,B_k) = r\}$, respectively[6].

L e m m a 3.1. For arbitrary k-machines A_k and B_k, $d = dist(A_k,B_k) - |deg(A_k) - deg(B_k)|$ is an even number.

The proof of this lemma has been given in [9].

L e m m a 3.2. For arbitrary k-machine A_k and the numbers p and q, the difference p-q is an even number iff for arbitrary k-machines $B_k \in Sph(A_k,p)$ and $C_k \in Sph(A_k,q)$ the difference $deg(B_k) - deg(C_k)$ is an even number.

P r o o f . Suppose that $p \geq q$. There exists $D_k \in Sph(A_k,q)$ such that $dist(A_k,B_k) = dist(A_k,D_k) + dist(D_k,B_k)$. Then we have

$$dist(D_k,B_k) = p - q . \tag{3.3}$$

On the other hand it follows from Lemma 3.1 that $r = dist(A_k B_k) - (deg(B_k) - deg(A_k))$ and $s = dist(A_k,D_k) - (deg(D_k) - deg(B_k))$ are even numbers. Then we have $deg(D_k) - deg(B_k) = dist(A_k,B_k) - dist(A_k,D_k) - r + s$ and then

$$deg(D_k) - deg(B_k) = p - q - r + s . \tag{3.4}$$

As r and s are even numbers, the difference $deg(D_k) - deg(B_k)$ is even iff p-q is even.

6) If $r = 0$ then $B(A_k,r) = \emptyset$

It follows from Lemma 3.1 that $t = \text{dist}(A_k, D_k) - (\deg(A_k) - \deg(D_k))$ and $u = \text{dist}(A_k, C_k) - (\deg(A_k) - \deg(C_k))$ are even numbers. As $\text{dist}(A_k, D_k) = \text{dist}(A_k, C_k)$, the $\deg(C_k) - \deg(D_k)$ is an even number.

The condition (3.3) implies that $\deg(C_k) - \deg(B_k)$ is an even number iff $p - q$ is an even one.

4. The maximal k-machines set

Some sets consisting of the maximal k-machines as well as the sets which do not consist of maximal k-machines will be shown.

For arbitrary k-machine $A_k = (M, \varphi)$ let $\bar{A}_k$ denote the k-machine $B_k = (M, \psi)$ such that $\varphi(U) \neq \psi(U)$ for all $U \in M^k$.

D e f i n i t i o n 4.1. A k-machine A_k is said to be maximal iff the following condition is satisfied

$$(4.1) \qquad \forall_{B_k \in D^k} \left(\text{dist}(A_k, B_k) = 1 \Rightarrow \deg(B_k) > \deg(A_k)\right) .$$

The set of all maximal k-machines will be denoted by $M(D^k)$.

D e f i n i t i o n 4.2. A k-machine A_k is said to be minimal iff the following condition is satisfied

$$(4.2) \qquad \forall_{B_k \in D^k} \left(\text{dist}(A_k, B_k) = 1 \Rightarrow \deg(B_k) < \deg(A_k)\right) .$$

The set of all minimal k-machines will be denoted by $m(D^k)$.

R e m a r k 4.1. It follows from the conditions (4.1) and (4.2) that the sets $M(D^k)$ and $m(D^k)$, can be understood as the sets of all k-machines for which the function deg attains a local minimum and maximum, respectively. The notions of the maximal k-machine and of the minimal one are used with respect to the period of their computations but not with respect to their complexity degree.

D e f i n i t i o n 4.3. Let A_k be an arbitrary k-machine. Each number $r \geq 0$ satisfying the condition

(4.3) $\exists_{i\geq 0}$ $(r = \deg(A_k) - 1 + 2i \;\&\; r \leq 2^{k-1} - \deg(\bar{A}_k + 1)$

is called the principal radius of A_k [7].

The set of all principal radii of A_k will be denoted by $R(A_k)$.

T h e o r e m 4.1. A k-machine A_k is maximal iff $\deg(A_k) = 1$.

The proof of this theorem has been given in [7].

C o r o l l a r y 4.1. A k-machine A_k is maximal iff period of its arbitrary computation is of 2^k.

T h e o r e m 4.2. For arbitrary k-machine A_k all maximal k-machines B_k such that $A_k \sqsubseteq B_k$ belong to the sphere $Sph(A_k, \deg(A_k)-1)$.

P r o o f . If B_k is a maximal k-machine, then it follows from Theorem 4.1 that $\deg(B_k) = 1$. As $A_k \sqsubseteq B_k$, it follows from Definition 2.7 that $dist(A_k, B_k) = \deg(A_k)-1$.

C o r o l l a r y 4.2. For arbitrary k-machine A_k there are no maximal k-machines in the ball $B(A_k, \deg(A_k)-1)$.

C o r o l l a r y 4.3. For arbitrary k-machine A_k we have

$$Sph(A_k, \deg(A_k)-1) \cap M(D^k) \neq \emptyset .$$

R e m a r k 4.2. It follows from Corollary 4.2 that the equation $\deg(A_k) - 1 + 2p + \deg(\bar{A}_k) - 1 = 2^{k-1}$ has the unique solution $p \geq 0$. Thus $r_1 = \deg(A_k)-1$ is always the principal radius of A_k.
As $dist(A_k, \bar{A}_k) = 2^{k-1}$, it follows from Lemma 3.1 that $\deg(A_k)$ is even iff $\deg(\bar{A}_k)$ is even. Then $r_2 = 2^{k-1}+1 - \deg(\bar{A}_k)$ is the principal radius of A_k.

T h e o r e m 4.3. For arbitrary k-machine A_k,
$M(D^k) \subseteq \bigcup_{r \in R(A_k)} Sph(A_k, r)$

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7) Let us observe that $dist(A_k, \bar{A}_k) = 2^{k-1}$ and the sets of all k-machines which belong to the sphere $Sph(A_k, r)$ and $Sph(\bar{A}_k, 2^{k-1}-r)$ are identical.

P r o o f . Let B_k $Sph(A_k, deg(A_k)-1)$ be an arbitrary maximal k-machine. It follows from Corollary 4.3 that such a k-machine exists. Consider a k-machine $C_k \in Sph(A_k, r+1)$, where $r \in R(A_k)$. As $p = (deg(A_k)-1)-(r+1) = deg(A_k)-1-$ $-deg(A_k)+1-2i-1 = -(2i+1)$ is an odd number, it follows from Lemma 3.2 that $q = deg(B_k) - deg(C_k) \neq 0$, and thus $deg(C_k) \neq 1$. Besides that, Corollary 4.2 implies that there are maximal k-machines neither in the ball $B(A_k, deg(A_k)-1)$ nor in the ball

$$B(\bar{A}_k, deg(\bar{A}_k)-1) = \left\{B_k \in D^k: \ dist(A_k, B_k) > 2^{k-1} - deg(\bar{A}_k)+1\right\} .$$ [8]

R e m a r k 4.3. Theorem 4.3 does not decide whether each sphere $Sph(A_k, r)$, where $r \in R(A_k)$, contains the maximal k-machines.

Let $A_k^o = (M, \varphi_o)$ be a k-machine the transition function of which is defined as follows $\varphi_o(U) = U|_{1,1}$ for all $U \in M^k$.

T h e o r e m 4.4. If A_k is a maximal k-machine, then the following condition is satisfied

$$(4.4) \qquad \exists_{r \in R(A_k^o)} \ (dim(A_k) = r) .$$

Proof of this theorem immediately follows from Theorem 4.3 and remarks that $dim(A_k^o) = 0$ and $dist(A_k^o, A_k) = dim(A_k)$.

T h e o r e m 4.5. For arbitrary number $k > 1$ we have

$$M(D^k) = \bigcap_{A_k \in m(D^k)} \ \bigcup_{r \in R(A_k)} Sph(A_k, r) .$$

P r o o f . Let $F = \bigcap_{A_k \in m(D^k)} \bigcup_{r \in R(A_k)} Sph(A_k, r)$. It follows from Theorem 4.3 that $M(D^k) \subseteq F$. We shall prove the inverse inclusion.

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8) It follows from Remark 4.2 that the number $2^{k-1} - deg(\bar{A}_k)+$ $+1$ is the greatest principal radius of A_k.

Let $B_k \in F$. Then there exists $C_k \in m(D^k)$ such that $C_k \sqsubseteq B_k$. It follows from Theorem 4.2 and the definition of F that the k-machine B_k must belong to the sphere $Sph(C_k, deg(C_k)-1)$. Then we have $dist(B_k, C_k) = deg(C_k) - deg(B_k) = deg(C_k)-1$ and thus $deg(B_k) = 1$.

R e m a r k 4.4. It can be proved that A_k is a minimal k-machine $(A_k \in m(D^k))$ iff each of its computation is a (k-1)-computation, in particular if its computation set is the union of two (k-1)-computation sets. This will be the subject of a separate paper.[9]

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9) The condition when k-computation set is the union of two (k-1)-computation sets has been given in [6].

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Received March 14, 1977.