DEMONSTRATIO MATHEMATICA
Vol. XI No 2 1978

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ON A RANDOM LINEAR PARABOLIC EQUATION

In this paper we consider the linear parabolic equation

$$(0.1)\quad Lu = \sum_{i,j=1}^{n} a_{ij}(x,t,\omega)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t,\omega)u_{x_i} + c(x,t,\omega)u - u_t = 0$$

with real random coefficients defined in a strip $G = \{(x,t):\ x \in R^n,\ 0 < t < T\}$ 1). Under suitable assumptions, applying the same method as in [4] (see also [1], chapter 1), we prove the existence of a fundamental solution $Z(x,t,\xi,\tau,\omega)$ $(x,\xi \in R^n,\ 0 \leq \tau < t \leq T)$ of equation (0.1). Next this fundamental solution is used in proving of the existence of a solution $u(x,t,\omega)$ of the Cauchy problem

$$(0.2)\qquad Lu = f(x,t,\omega),\quad (x,t) \in G_0 = R^n \times (0,T\rangle ,$$

$$(0.3)\qquad u(x,0,\omega) = g(x,\omega),\quad x \in R^n.$$

1. Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probabilistic space. By $L_p(\Omega)$ $(1 \leq p \leq \infty)$ we denote the Banach space of all real random variables $f(\omega)$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with finite norm

$$\|f\|_p = \left[\int_\Omega |f(\omega)|^p\, P(d\omega)\right]^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,\quad \|f\|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} |f(\omega)|.$$

1) Throughout this paper we shall use only real random functions. Therefore the adjective "real" will be omitted.

Let $u(x,\omega)$, $x \in D \subset R^k$ be a real random function defined on $(\Omega, \mathcal{F}, \mathcal{P})$ (see [3], p.59)[2]. If $u: D \to L_p(\Omega)$, then the strong limit, strong continuity and strong derivatives of u are called respectively the L_p-limit, L_p-continuity and L_p-derivatives of u.

We shall consider the equation (0.1) under the following assumptions, denoted collectively by (H) (cf. [4]).

(H) For some $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 1$ the following conditions are satisfied

$$a_{ij}(x,t,\omega) = a_{ji}(x,t,\omega), \quad (x,t,\omega) \in G \times \Omega_0;$$

there are positive constants $\lambda_0, \lambda_1 (\lambda_1 \geq \lambda_0)$ such that

$$(1.1) \quad \lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t,\omega)\xi_i\xi_j \leq \lambda_1 |\xi|^2, \quad (x,t,\omega) \in G \times \Omega_0, \xi \in R^n,$$

where $|\xi|^2 = \sum_{i=1}^{n} \xi_i^2$;

there exist constants $A_0 > 0$, $\alpha \in (0,1)$ such that for any (x,t,ω), $(x',t',\omega) \in G \times \Omega_0$ holds the inequality

$$(1.2) \quad \left|a_{ij}(x,t,\omega) - a_{ij}(x',t',\omega)\right| \leq A_0(|x-x'|^{\alpha} + |t-t'|^{\alpha/2});$$

there is a constant $A_1 > 0$ such that for any $(x,t,\omega) \in G \times \Omega$

$$\left|b_i(x,t,\omega)\right|, \quad \left|c(x,t,\omega)\right| \leq A_1.$$

Moreover we assume that the coefficients $b_i(x,t,\omega)$, $c(x,t,\omega)$ are measurable random functions (see [3], p.211), L_p-continuous ($p \in \langle 1, \infty \rangle$ being some constant) for $t \in \langle 0,T \rangle$ and there are satisfied the inequalities

2) I.e. that $u: D \times \Omega \to R$ and $u(x,\omega)$ is measurable for each $x \in D$. In the sequel we shall use the concept of random function also for function $u(x,\omega)$ defined for $x \in D$, $\omega \in \Omega_x \subset \Omega$ (where $P(\Omega_x) = 1$) and, of course, measurable with respect to ω (see [3], p.59 or [2], p.140).

(1.3) $\|b_i(x,t,\cdot) - b_i(x',t,\cdot)\|_p$, $\|c(x,t,\cdot) - c(x',t,\cdot)\|_p < A_2|x-x'|^{\sigma}$,

$x, x' \in R^n$, $t \in \langle 0,T\rangle$, $A_2 > 0$ being a constant.

R e m a r k. It follows from (1.2) that $a_{ij}(x,t,\omega)$ are measurable random functions. Moreover the condition (1.1) implies that $|a_{ij}(x,t,\omega)| < \lambda_1$, $(x,t) \in G$, $\omega \in \Omega_0$.

For $(x,t) \in G$, $\omega \in \Omega_0$ there exists a matrix $[a^{ij}(x,t,\omega)]$ inverse to the matrix $[a_{ij}(x,t,\omega)]$. Obviously $a^{ij}(x,t,\omega)$ are random functions satisfying, by (1.1), (1.2), the following conditions:

(1.4) $$\mu_0|\xi|^2 < \sum_{i,j=1}^{n} a^{ij}(x,t,\omega)\xi_i\xi_j < \mu_1|\xi|^2, \quad (x,t) \in G, \ \omega \in \Omega_0, \ \xi \in R^n$$

($0 < \mu_0 < \mu_1$ being constants depending only on λ_0 and λ_1) and

(1.5) $$\left|a^{ij}(x,t,\omega) - a^{ij}(x',t',\omega)\right| \leq A_3\left(|x-x'|^{\alpha} + |t-t'|^{\alpha/2}\right), \quad (x,t),(x',t') \in G,$$

$\omega \in \Omega_0$ (A_3 being a constant depending only on A_0, λ_0 and λ_1).

Let us introduce the function

$$W_{\varrho,\theta}(x,t,\xi,\tau,\omega) =$$

$$= \left[4\pi(t-\tau)\right]^{-\frac{n}{2}} \left\{\det\left[a_{ij}(\varrho,\theta,\omega)\right]\right\}^{-\frac{1}{2}} \exp\left[-\frac{\sum_{i,j=1}^{n} a^{ij}(\varrho,\theta,\omega)(x_i-\xi_i)(x_j-\xi_j)}{4(t-\tau)}\right],$$

$0 \leq \tau < t < T$, $x, \xi, \varrho \in R^n$, $\theta \in \langle 0,T\rangle$, $\omega \in \Omega_0$.

$W_{\varrho,\theta}(x,t,\xi,\tau,\omega)$ is a random function, continuous in $x, t, \xi, \tau, \varrho, \theta$ uniformly with respect to $\omega \in \Omega_0$ and it possesses derivatives (in the usual sense) of any order with respect to the variables x, t, ξ, τ, uniform with respect to $\omega \in \Omega_0$ (i.e. that difference quotients tend to derivatives uniformly with respect to $\omega \in \Omega_0$). These derivatives are random functions with the same property of continuity as $W_{\varrho,\theta}$

and they are also L_q-derivatives for each $q \in \langle 1, \infty \rangle$. Using (1.4) we obtain the following estimates

$$(1.6)\qquad \left|W_{\varrho,\theta}(x,t,\xi,\tau,\omega)\right| < C_1(t-\tau)^{-\frac{n}{2}} \exp\left[-\frac{\mu'|x-\xi|^2}{t-\tau}\right],$$

$$(1.7)\qquad \left|\frac{\partial W_{\varrho,\theta}(x,t,\xi,\tau,\omega)}{\partial x_i}\right| \leq C_1(t-\tau)^{-(n+1)/2} \exp\left[-\frac{\mu|x-\xi|^2}{t-\tau}\right],$$

$$(1.8)\qquad \left|\frac{\partial^2 W_{\varrho,\theta}(x,t,\xi,\tau,\omega)}{\partial x_i \partial x_j}\right|, \left|\frac{\partial W_{\varrho,\theta}(x,t,\xi,\tau,\omega)}{\partial t}\right| \leq$$

$$\leq C_1(t-\tau)^{-(n+2)/2} \exp\left[-\frac{\mu|x-\xi|^2}{t-\tau}\right]$$

for $x,\xi,\varrho \in R^n$, $\theta \in \langle 0,T \rangle$, $0 < \tau < t \leq T$, $\omega \in \Omega_0$, where $C_1 > 0$ is some constant (depending only on λ_0 and λ_1), $\mu' = \frac{\mu_0}{4}$, $0 < \mu < \mu'$.

2. The fundamental solution

We shall prove the following theorem.

T h e o r e m 2.1. If assumptions (H) are satisfied, then there exists a random function $Z(x,t,\xi,\tau,\omega)$, defined for $x,\xi \in R^n$, $0 < \tau < t < T$, $\omega \in \Omega_1$ ($\Omega_1 \in \Omega_0$ being some set such that $P(\Omega_1) = 1$) and possessing the following properties:

1^0 $Z(x,t,\xi,\tau,\omega)$ is continuous in x,t,ξ,τ uniformly with respect to $\omega \in \Omega_1$ and we have

$$(2.1)\qquad \left|Z(x,t,\xi,\tau,\omega)\right| \leq C(t-\tau)^{-\frac{n}{2}} \exp\left[-\frac{\bar{\mu}|x-\xi|^2}{t-\tau}\right],$$

$C, \bar{\mu}$ being some positive constants;

2^o there exist derivatives $Z_{x_i}(x,t,\xi,\tau,\omega)$, uniform with respect to $\omega\in\Omega_1$ and we have the estimate

(2.2) $$\left|Z_{x_i}(x,t,\xi,\tau,\omega)\right| \leqslant C(t-\tau)^{-(n+1)/2}\exp\left|-\frac{\bar{\mu}|x-\xi|^2}{t-\tau}\right| ;$$

3^o there exist L_p-derivatives $Z_{x_ix_j}(x,t,\xi,\tau,\omega)$, $Z_t(x,t,\xi,\tau,\omega)$ which are L_p-continuous in x,t,ξ,τ and fulfil the inequalities

(2.3) $$\left\|Z_{x_ix_j}(x,t,\xi,\tau,\cdot)\right\|_p,\ \left\|Z_t(x,t,\xi,\tau,\cdot)\right\|_p \leqslant$$
$$\leqslant C(t-\tau)^{-(n+2)/2}\exp\left[-\frac{\bar{\mu}|x-\xi|^2}{t-\tau}\right] ;$$

4^o for fixed $\xi\in R^n$, $\tau\in\langle 0,T)$ the function $Z(x,t,\xi,\tau,\omega)$ satisfies, with respect to $x\in R^n$, $t\in(\tau,T\rangle$, $\omega\in\Omega_{x,t,\xi,\tau}$ (where $P(\Omega_{x,t,\xi,\tau}) = 1$), the equation

$$L_{x,t}\left[Z(x,t,\xi,\tau,\omega)\right] \equiv \sum_{i,j=1}^{n} a_{ij}(x,t,\omega)Z_{x_ix_j}(x,t,\xi,\tau,\omega) +$$
$$+ \sum_{i=1}^{n} b_i(x,t,\omega)Z_{x_i}(x,t,\xi,\tau,\omega)+c(x,t,\omega)Z(x,t,\xi,\tau,\omega)-Z_t(x,t,\xi,\tau,\omega) = 0;$$

5^o if $g(x,\omega)$, $x\in R^n$ is a measurable, L_q-continuous ($q\in\langle 1,\infty\rangle$) and L_q-bounded random function (i.e. $\|g(x,\cdot)\|_q \leqslant$ const., $x\in R^n$), then

(2.4) $$\lim_{t\to\tau}\int_{R^n} Z(x,t,\xi,\tau,\omega)g(\xi,\omega)d\xi = g(x,\omega)\ (L_q)\ ^{3)},$$

where the convergence is uniform with respect to $x\in D$ ($D\subset R^n$ being a bounded domain) and $\tau\in\langle 0,T)$.

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3) I.e. that the limit is taken in the L_q-sense. The integral in (2.4) is an improper integral in the L_q-sense (see sec.4).

The above-mentioned function $Z(x,t,\xi,\tau,\omega)$ is called a fundamental solution of equation (0.1).

P r o o f. Using the same method as in [4] (see also [1]) we prove that

$$(2.5)\quad Z(x,t,\xi,\tau,\omega) = W_{\xi,\tau}(x,t,\xi,\tau,\omega) + \int_\tau^t d\theta \int_{R^n} W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)\phi(\zeta,\theta,\xi,\tau,\omega)d\zeta,$$

where ϕ is a solution of the integral equation

$$(2.6)\quad \phi(x,t,\xi,\tau,\omega) = \phi_1(x,t,\xi,\tau,\omega) + \int_\tau^t d\theta \int_{R^n} \phi_1(x,t,\zeta,\theta,\omega)\phi(\zeta,\theta,\xi,\tau,\omega)d\zeta$$

with

$$\phi_1(x,t,\xi,\tau,\omega) = L_{x,t}\left[W_{\xi,\tau}(x,t,\xi,\tau,\omega)\right].$$

Hence it follows that ϕ is given by formulas

$$(2.7)\quad \phi(x,t,\xi,\tau,\omega) = \sum_{m=1}^{\infty} \phi_m(x,t,\xi,\tau,\omega),$$

$$(2.8)\quad \phi_{m+1}(x,t,\xi,\tau,\omega) = \int_\tau^t \int_{R^n} \phi_1(x,t,\zeta,\theta,\omega)\phi_m(\zeta,\theta,\xi,\tau,\tau,\omega)d\zeta\, d\theta\ ,$$

$$m = 1,2,\ldots$$

Indeed, $b_i(x,t,\omega)$ and $c(x,t,\omega)$ are measurable functions with respect to the variables x,t for each $\omega\in\Omega_1$, $\Omega_1\subset\Omega_0$ being such a set that $P(\Omega_1) = 1$. Consequently the integrals in (2.8) exist (as Lebesgue integrals) in the set

$$(2.9)\quad x,\xi\in R^n,\ \ 0<\tau<t<T,\ \ \omega\in\Omega_1.$$

Since for functions ϕ_m the estimates (4.58) of [4] hold true, therefore the series (2.7) is uniformly convergent for $t-\tau>\delta>0$, $x,\xi\in R^n$, $\omega\in\Omega_1$ and satisfy equation (2.6) in the set (2.9). Moreover, we have

$$(2.10)\quad |\phi(x,t,\xi,\tau,\omega)| < C_2(t-\tau)^{-(n+2-\alpha)/2}\exp\left[-\frac{\mu^*|x-\xi|^2}{t-\tau}\right],\ 0<\mu^*<\mu,$$

Using Lemma 4.1 one can show that

$$(2.11)\quad \phi(x,t,\xi,\tau,\omega) = \varphi(x,t,\xi,\tau,\omega) + $$

$$+ \sum_{i=1}^{n} b_i(x,t,\xi,\tau,\omega)\,\psi_i(x,t,\xi,\tau,\omega) + c(x,t,\omega)\chi(x,t,\xi,\tau,\omega),$$

where φ, ψ_i, χ are random functions with the following properties:

(i) they are defined in the set (2.9) and continuous in x,t,ξ,τ uniformly with respect to $\omega\in\Omega_1$;

(ii) they satisfy inequality (2.10);

(iii) if

$$(2.12)\qquad |x-x'|^2 < a(t-\tau) \quad \text{for some} \quad a>0,$$

then there hold the estimates

$$(2.13)\quad \begin{aligned} &|\varphi(x,t,\xi,\tau,\omega)-\varphi(x',t,\xi,\tau,\omega)|,\ |\psi_i(x,t,\xi,\tau,\omega)-\psi_i(x',t,\xi,\tau,\omega)|,\\ &|\chi(x,t,\xi,\tau,\omega)-\chi(x',t,\xi,\tau,\omega)| \leqslant \\ &\leqslant C_3(t-\tau)^{-(n+2-2\mu_2)/2}|x-x'|^{\mu_3}\exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right], \end{aligned}$$

where $0<\mu_2<\alpha/2$, $0<\mu_3<\alpha-2\mu_2$.

Relations (2.11) and (2.13) immediately imply, under the condition (2.12), the estimate

$$(2.14)\quad \begin{aligned} &\|\phi(x,t,\xi,\tau,\cdot) - \phi(x',t,\xi,\tau,\cdot)\|_p \leqslant \\ &\leqslant C_4(t-\tau)^{-(n+2-2\mu_2)/2}|x-x'|^{\mu_3}\exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]. \end{aligned}$$

Now we shall consider the function

$$(2.15)\quad V(x,t,\xi,\tau,\omega) = \int_\tau^t\int_{R^n} W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)\phi(\zeta,\theta,\xi,\tau,\omega)\,d\zeta\, d\theta.$$

In view of (1.6),(2.11),(i) and (ii) this function possesses, by Lemma 4.1, the property (i) and there holds true the estimate

$$(2.16)\qquad |V(x,t,\xi,\tau,\omega)| < C_5(t-\tau)^{-(n-\alpha)/2}\exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]$$

in the set (2.9). Taking into considerations relations (2.11), (1.7), properties (i),(ii) and Lemma 4.2 we conclude that there exist derivatives V_{x_i} uniform with respect to $\omega\in\Omega_1$, which have the property (i). Moreover, these derivatives are given by formula

$$(2.17)\qquad V_{x_i}(x,t,\xi,\tau,\omega) = \int_\tau^t\int_{R^n}\frac{\partial W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial x_i}\phi(\zeta,\theta,\xi,\tau,\omega)d\zeta d\theta$$

and there holds true the estimate

$$(2.18)\qquad |V_{x_i}(x,t,\xi,\tau,\omega)| \leqslant C_6(t-\tau)^{-(n+1-\alpha)/2}\exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]$$

in the set (2.9).

In order to prove the existence of derivatives $V_{x_ix_j}$ let us introduce the function

$$J(x,t,\theta,\xi,\tau,\omega) = \int_{R^n}W_{\zeta,\theta}(x,t,\xi,\theta,\omega)\phi(\zeta,\theta,\xi,\tau,\omega)d\zeta .$$

Observe that $b_k(\zeta,\theta,\omega)$, $c(\zeta,\theta,\omega)$ are measurable functions with respect to $\zeta\in R^n$ for fixed $\theta\in B\subset(0,T)$, $\omega\in\Omega_\theta\subset\Omega_1$, B being a set with Lebesgue measure $|B| = T$, $P(\Omega_\theta) = 1$. Hence, with the aid of properties (i),(ii) and relations (2.11), (1.6), we obtain, by Lemma 4.3, the following assertions:

(I) the function J is defined in the set

$$x,\xi\in R^n,\ 0<\tau<t<T,\ \theta\in B\cap(\tau,t),\ \omega\in\Omega_\theta ;$$

(II) J is a measurable function;

(iii) $\theta \in B$ being fixed the function J is continuous in x,t,ξ,τ uniformly with respect to $\omega \in \Omega_\theta$;
(IV) J is L_p-continuous in x,t,θ,ξ,τ.

In virtue of Lemma 4.4 there exist derivatives $J_{x_i}, J_{x_i x_j}$ uniform with respect to $\omega \in \Omega_\theta$, which are given by formulas

$$J_{x_i}(x,t,\theta,\xi,\tau,\omega) = \int_{R^n} \frac{\partial W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial x_i} \phi(\zeta,\theta,\xi,\tau,\omega)d\zeta ,$$

(2.19)

$$J_{x_i x_j}(x,t,\theta,\xi,\tau,\omega) = \int_{R^n} \frac{\partial^2 W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial x_i \partial x_j} \phi(\zeta,\theta,\xi,\tau,\omega)d\zeta$$

and possesses properties (I)-(IV).

It results from (1.8) and (2.10) that the estimate

$$\left| J_{x_i x_j}(x,t,\theta,\xi,\tau,\omega) \right| \leq c_7(\theta-\tau)^{-(2-\alpha)/2}(t-\tau)^{-(n+2)/2} \exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]$$

holds in the set

$$x,\xi \in R^n,\ 0<\tau<t<T,\ \theta \in B \cap \left(\tau, \frac{t+\tau}{2}\right),\ \omega \in \Omega_\theta .$$

Observe that relation (4.36) of [4] and the estimates of the moduli $|I_1|, |I_2|, |I_3|, |I_4^{(2)}|$ remain valid in the set

$$G_1 = \left\{x,\xi \in R^n,\ 0<\tau<t<T,\ \theta \in B \cap \left(\frac{t+\tau}{2}, t\right)\right\},\ \omega \in \Omega_\theta .$$

Proceeding as in [4] and using (1.8),(2.14) and generalized Minkowski's inequality ([7], p.21), we obtain the same estimate of the norm $\|I_4^{(1)}\|_p$ (of the integral $I_4^{(1)}$ of [4]) in the set G_1 as that of $|I_4^{(1)}|$ (in [4]). Consequently we have

$$\text{(2.21)}\quad \|J_{x_i x_j}(x,t,\theta,\xi,\tau,\cdot)\|_p < c_8(t-\theta)^{-(2-\mu_3)/2}(t-\tau)^{-(n+2-2\mu_2)/2} \exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]$$

in the set G_1. According to the estimates (2.20),(2.21) and Remark 4.9 the function V possesses L_p-derivatives $V_{x_ix_j}$ in the set

$$(2.22)\qquad x,\ \xi\in R^n,\quad 0<\tau<t<T,\ \omega\in\Omega_{x,t,\xi,\tau}\ \ (P(\Omega_{x,t,\xi,\tau})=1),$$

which are given by formulas

$$(2.23)\qquad V_{x_ix_j}(x,t,\xi,\tau,\omega)=\int_\tau^t J_{x_ix_j}(x,t,\theta,\xi,\tau,\omega)d\theta.$$

Moreover, these derivatives are L_p-continuous in the set

$$(2.24)\qquad x,\ \xi\in R^n,\quad 0<\tau<t<T$$

and we have the estimate

$$(2.25)\quad \|V_{x_ix_j}(x,t,\xi,\tau,\cdot)\|_p\leqslant C_9(t-\tau)^{-(n+2-2\mu_2)/2}\exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]$$

in the set (2.24).

In order to prove the existence of L_p-derivative V_t observe that the function J possesses, by Lemma 4.4, the derivative

$$J_t(x,t,\theta,\xi,\tau,\omega)=\int_{R^n}\frac{\partial W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial t}\phi(\zeta,\theta,\xi,\tau,\omega)d\zeta=$$

$$=\sum_{i,j=1}^n\int_{R^n}\frac{\partial^2 W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial x_i\partial x_j}a_{ij}(\zeta,\theta,\omega)\phi(\zeta,\theta,\xi,\tau,\omega)d\zeta$$

uniform with respect to $\omega\in\Omega_\theta$. The last integral has the form (2.19). Therefore we obtain from the reasoning concerning the integral (2.19) the uniform L_p-convergence of the integral

(2.26) $\int_{\tau}^{t} J_t(x,t,\theta,\xi,\tau,\omega)d\theta$ in $G_2 = \{x,\xi \in R^n, t-\tau \geq \delta > 0\}$.

and the uniform convergence of the integral

(2.27) $$\int_{\tau}^{t} \|J_t(x,t,\theta,\xi,\tau,\cdot)\|_p d\theta \quad \text{in} \quad G_2.$$

Moreover, the function (2.26) is defined in the set (2.22), it is L_p-continuous in (2.24) and there holds true the estimate

(2.28) $$\left\|\int_{\tau}^{t} J_t(x,t,\theta,\xi,\tau,\cdot)d\theta\right\|_p \leq c_{13}(t-\tau)^{-(n+2-2\mu_2)/2} \exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right]$$

in the set (2.24).

Now we prove the existence of L_p-derivative V_t given by formula

(2.29) $$V_t(x,t,\xi,\tau,\omega) = \phi(x,t,\xi,\tau,\omega) + \int_{\tau}^{t} J_t(x,t,\theta,\xi,\tau,\omega)d\theta .$$

Taking $\Delta t > 0$ we have

$$\frac{V(x,t+\Delta t,\xi,\tau,\omega)-V(x,t,\xi,\tau,\omega)}{\Delta t} - \phi(x,t,\xi,\tau,\omega) - \int_{\tau}^{t} J_t(x,t,\theta,\xi,\tau,\omega)d\theta =$$

$$= I + \frac{1}{\Delta t}\int_{t}^{t+\Delta t} J(x,t+\Delta t,\theta,\xi,\tau,\omega)d\theta - \phi(x,t,\xi,\tau,\omega),$$

where

$$I = \int_{\tau}^{t}\left[\frac{J(x,t+\Delta t,\theta,\xi,\tau,\omega)-J(x,t,\theta,\xi,\tau,\omega)}{\Delta t} - J_t(x,t,\theta,\xi,\tau,\omega)\right]d\theta .$$

There is $t' \in (t,t+\Delta t)$ such that

$$I = \int_{\tau}^{t}\left[J_t(x,t',\theta,\xi,\tau,\omega) - J_t(x,t,\theta,\xi,\tau,\omega)\right]d\theta =$$

$$= -\int_{\tau}^{\tau+\eta_1} J_t(x,t,\theta,\xi,\tau,\omega)d\theta - \int_{t-\eta_2}^{t} J_t(x,t,\theta,\xi,\tau,\omega)d\theta +$$

$$+\int_{\tau}^{\tau+\eta_1} J_t(x,t',\theta,\xi,\tau,\omega)d\theta + \int_{t-\eta_2}^{t} J_t(x,t',\theta,\xi,\tau,\omega)d\theta +$$

$$+\int_{t+\eta_1}^{t-\eta_2}\left[J_t(x,t',\theta,\xi,\tau,\omega) - J_t(x,t,\theta,\xi,\tau,\omega)\right]d\theta = -I_1-I_2+I_3+I_4+I_5,$$

$\eta_1,\eta_2>0$ being constants.

The uniform L_p-convergence of the integral (2.26) in G_2 implies that, for any $\varepsilon>0$ and sufficiently small $\eta_1,\eta_2\in(0,\delta/2)$,

$$\|I_k\|_p<\varepsilon, \quad k=1,2,3.$$

Further we have

$$\|I_4\|_p \leqslant \int_{t-\eta_2}^{t}\|J_t(x,t',\theta,\xi,\tau,\cdot)\|_p d\theta \leqslant \int_{t-\eta_2}^{t'}\|J_t(x,t',\theta,\xi,\tau,\cdot)\|_p d\theta .$$

Hence, in view of the uniform convergence of the integral (2.27) in G_2, it follows that $\|I_4\|_p<\varepsilon$ for sufficiently small $\eta_2>0$ and $\Delta t>0$.

Note that in integral I_5 we have $t-\theta\geqslant\eta_2$, which implies that $t'-t<t-\theta$ if $\Delta t\in(0,\eta_2)$. Under the condition $t'-t<t-\theta$ there holds true the estimate

$$\left|\frac{\partial W_{\zeta,\theta}(x,t',\zeta,\theta,\omega)}{\partial t} - \frac{\partial W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial t}\right| \leqslant \tag{2.30}$$

$$\leqslant C_{11}(t'-t)(t-\theta)^{-(n+4)/2}\exp\left[-\frac{\mu''|x-\xi|^2}{t-\tau}\right].$$

Constants $\eta_1, \eta_2 > 0$ being fixed, it results from estimates (2.30) and (2.10) that

$$|I_5| \leq \int_{\tau+\eta_1}^{t-\eta_2} d\theta \int_{R^n} \left| \frac{\partial W_{\zeta,\theta}(x,t',\zeta,\theta,\omega)}{\partial t} - \frac{\partial W_{\zeta,\theta}(x,t,\zeta,\theta,\omega)}{\partial t} \right| \cdot \left|\phi(\zeta,\theta,\xi,\tau,\omega)\right| d\zeta < \varepsilon$$

for $\Delta t \in (0,\eta_3)$ provided $\eta_3 > 0$ is sufficiently small. Thus we have proved that $\lim\limits_{\Delta t \to 0+0} \|I\|_p = 0$.

In order to obtain (2.29) it remains to prove the validity of relation

$$\text{(2.31)} \qquad \lim_{\Delta t \to 0+0} \left\| \frac{1}{\Delta t} \int_t^{t+\Delta t} J(x,t+\Delta t,\theta,\xi,\tau,\cdot)d\theta - \phi(x,t,\xi,\tau,\cdot) \right\|_p = 0.$$

At first observe that there is a set $B_0 \subset R^n$, $|R^n \setminus B_0| = 0$ such that for every fixed $\zeta \in B_0$ the functions $b_k(\zeta,\theta,\omega)$, $c(\zeta,\theta,\omega)$ are measurable. This implies the existence of the integral

$$\int_t^{t+\Delta t} \phi(x,\theta,\xi,\tau,\omega)d\theta \quad \text{for} \quad x \in B_0,\ \xi \in R^n,\ 0<\tau<t<T,\ \omega \in \Omega_x, P(\Omega_x) = 1.$$

Therefore, taking advantage of the equality

$$\int_{R^n} W_{x,\theta}(x,t+\Delta t,\zeta,\theta,\omega)d\zeta = 1$$

we can write, for $x \in B_0$, $\xi \in R^n$, $t-\tau \geq \delta > 0$,

$$\frac{1}{\Delta t} \int_t^{t+\Delta t} J(x,t+\Delta t,\theta,\xi,\tau,\omega)d\theta - \phi(x,t,\xi,\tau,\omega) =$$

$$= \frac{1}{\Delta t} \int_t^{t+\Delta t} d\theta \int_{R^n} W_{\zeta,\theta}(x,t+\Delta t,\zeta,\theta,\omega)\, \phi\left[(\zeta,\theta,\xi,\tau,\omega) - \phi(x,\theta,\xi,\tau,\omega)\right] d\zeta +$$

$$+\frac{1}{\Delta t}\int_t^{t+\Delta t}\phi(x,\theta,\xi,\tau,\omega)\left\{\int_{R^n}\left[W_{\zeta,\theta}(x,t+\Delta t,\zeta,\theta,\omega)-W_{x,\theta}(x,t+\Delta t,\zeta,\theta,\omega)\right]d\zeta\right\}d\theta+$$

$$+\frac{1}{\Delta t}\int_t^{t+\Delta t}\left[\phi(x,\theta,\xi,\tau,\omega)-\phi(x,t,\xi,\tau,\omega)\right]d\theta = I_1'+I_2'+I_3'.$$

To evaluate the integral I_1' we break it into two integrals I_{11}' and I_{12}' by formulas

$$I_{11}'=\frac{1}{\Delta t}\int_t^{t+\Delta t}\int_{K_r},\quad I_{12}'=\frac{1}{\Delta t}\int_t^{t+\Delta t}\int_{R^n\setminus K_r},$$

where $K_r=\{\zeta\in R^n:|\zeta-x|\leqslant r\}$, $r\in(0,\sqrt{a\delta})$. Taking arbitrary $\varepsilon>0$ it follows from the estimates (1.6) and (2.14) that $\|I_{11}'\|_p<\varepsilon$ for any $\Delta t>0$ and sufficiently small $r>0$. Now fix r. Then, in view of the estimates (1.6) and (2.10), we conclude that $|I_{12}'|<\varepsilon$ for $\Delta t\in(0,\eta)$ provided $\eta>0$ is sufficiently small. Consequently $\lim_{\Delta t\to 0}\|I_1'\|_p=0$ uniformly with regard to $x_0\in B_0$, $t\in\langle\delta,T'\rangle$ ($\delta>0$, $T'\in(\delta,T)$), $\tau\in\langle 0,t-\delta\rangle$, $\xi\in R^n$. Proceeding in a similar way and using, instead of (2.14), the estimate

$$\left|W_{\zeta,\theta}(x,t+\Delta t,\zeta,\theta,\omega)-W_{x,\theta}(x,t+\Delta t,\zeta,\theta,\omega)\right|\leqslant$$

$$<C_{12}(t+\Delta t-\theta)^{-n/2}|\zeta-x|^{\alpha}\exp\left[-\frac{\mu|\zeta-x|^2}{t+\Delta t-\theta}\right],$$

one can prove for I_2' the same conclusion as for I_1'.

Finally, it follows from the uniform L_p-continuity of $\phi(x,t,\xi,\tau,\omega)$ for $x,\xi\in D$ (D being a bounded domain), $t-\tau\geqslant\delta>0$ that $\|I_3'\|_p<\varepsilon$ for $\Delta t\in(0,\eta)$ provided $\eta>0$ is sufficiently small.

Thus we have proved that (2.31) holds true uniformly with respect to $x\in B_0\cap D$, $\xi\in D$ (D being a bounded domain),

$0<\delta\leq t\leq T'$ $(T'\in(\delta,T))$, $0<\tau\leq t-\delta$. It remains to show the validity of (2.31) for any $x,\xi\in R^n$, $0<\tau<t\leq T$. For this purpose let us write

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} J(x,t+\Delta t,\theta,\xi,\tau,\omega)d\theta - \phi(x,t,\xi,\tau,\omega) = H_1+H_2+H_3,$$

where

$$H_1 = \frac{1}{\Delta t}\int_t^{t+\Delta t}\left[J(x,t+\Delta t,\theta,\xi,\tau,\omega) - J(y,t+\Delta t,\theta,\xi,\tau,\omega)\right]d\theta,$$

$$H_2 = \frac{1}{\Delta t}\int_t^{t+\Delta t} J(y,t+\Delta t,\theta,\xi,\tau,\omega)d\theta - \phi(y,t,\xi,\tau,\omega),$$

$$H_3 = \phi(y,t,\xi,\tau,\omega) - \phi(x,t,\xi,\tau,\omega).$$

Using the estimate

$$|W_{\zeta,\theta}(x,t,\zeta,\theta,\omega) - W_{\zeta,\theta}(y,t,\zeta,\theta,\omega)|\leq$$

$$\leq C_{13}|x-y|\cdot(t-\theta)^{-(n+1)/2}\left[\exp\left(-\frac{\mu|x-\zeta|^2}{t-\theta}\right)+\exp\left(-\frac{\mu|y-\zeta|^2}{t-\theta}\right)\right]$$

and (2.10) we obtain the inequality

$$(2.32)\qquad |H_1|< C_{17}(\Delta t)^{-\frac{1}{2}}|y-x|\cdot(t-\tau)^{-(n+2-\alpha)/2}.$$

Now fix $x,\xi\in R^n$, $0<\tau<t<T$ and let $y\in B_0$. According to the previous considerations for any $\varepsilon>0$, $\eta_0>0$ there is $\eta>0$ such that if $0<\Delta t<\eta$, $|y-x|<\eta_0$, then $\|H_2\|_p<\varepsilon$. Taking $y\in B_0\cap K(x,\Delta t)$ it follows from (2.32) that $|H_1|<\varepsilon$ for $\Delta t\in(0,\eta_1)$ provided $\eta_1\in(0,\eta)$ is sufficiently small. Finally, if $\eta_3\in(0,\eta_0)$ is sufficiently small, then

$$\|H_3\|_p<\varepsilon \quad\text{for}\quad y\in K(x,\eta_3).$$

This completes the proof of the relation (2.31). At the same time we have proved that

$$\lim_{\Delta t\to 0+0}\left\|\frac{V(x,t+\Delta t,\xi,\tau,\cdot)-V(x,t,\xi,\tau,\cdot)}{\Delta t}-\phi(x,t,\xi,\tau,\cdot)-\int_{\tau}^{t}J_t(x,t,\theta,\xi,\tau,\cdot)d\theta\right\|_p=0.$$

The reasoning in the case $\Delta t<0$ is similar to that used for $\Delta t>0$. Thus (2.29) is proved.

Recalling the relation (2.5) and the results obtained for the function (2.15) we conclude the validity of assertions 1^0–3^0 of Theorem 2.1. The assertion 4^0 easily follows from relations (2.5), (2.15), (2.17), (2.23), (2.29) and (2.6).

To end the proof it remains to show that (2.4) holds true. For this purpose observe that

$$(2.33)\qquad \lim_{t\to\tau}\left\|\int_{R^n}W_{\xi,\tau}(x,t,\xi,\tau,\cdot)g(\xi,\cdot)d\xi-g(x,\cdot)\right\|_q=0$$

uniformly with respect to $x\in D$, $\tau\in\langle 0,T)$, D being a bounded domain.

Indeed, writing

$$\int_{R^n}W_{\xi,\tau}(x,t,\xi,\tau,\omega)g(\xi,\omega)d\xi-g(x,\omega)=$$

$$=\int_{R^n}W_{\xi,\tau}(x,t,\xi,\tau,\omega)\left[g(\xi,\omega)-g(x,\omega)\right]d\xi+$$

$$+\int_{R^n}g(x,\omega)\left[W_{\xi,\tau}(x,t,\xi,\tau,\omega)-W_{x,\tau}(x,t,\xi,\tau,\omega)\right]d\xi=I_1''+I_2''$$

and evaluating integrals I_1'' and I_2'' similarly as I_1' and I_2' in the proof of (2.31) we obtain (2.33). Now the estimate (2.16) yields

$$\left\|\int_{R^n}V(x,t,\xi,\tau,\cdot)g(\xi,\cdot)d\xi\right\|_q\leq$$

$$\leq\int_{R^n}\|V(x,t,\xi,\tau,\cdot)\|_\infty\cdot\|g(\xi,\cdot)\|_q d\xi\leq C_{18}(t-\tau)^{\alpha/2}\xrightarrow[t\to\tau]{}0$$

which together with (2.33) completes the proof of (2.4). At the same time we have proved Theorem 2.1.

3. The Cauchy problem

In this section the fundamental solution $Z(x,t,\xi,\tau,\omega)$ will be used in proving of the existence of a solution of the Cauchy problem (0.2), (0.3). For this purpose we additionally introduce the following assumption.

(H_0) The random functions $f(x,t,\omega)$, $g(x,\omega)$, $x\in R^n$, $t\in\langle 0,T\rangle$ $\omega\in\Omega$ are measurable, L_q-continuous $\left(\frac{1}{p}+\frac{1}{q}\leqslant 1\right)$ and L_q-bounded. Moreover, $f(x,t,\omega)$ satisfies for $x\in R^n$ a local Hölder condition in the sense L_q with exponent α, uniformly with respect to $t\in\langle 0,T\rangle$, i.e. for any bounded domain $D\subset R^n$ there is a constant $M>0$ such that

$$\|f(x,t,\cdot)-f(x',t,\cdot)\|_q\leqslant M|x-x'|^\alpha,\quad x,x'\in D,\quad t\in\langle 0,T\rangle .$$

T h e o r e m 3.1. Let assumptions (H) and (H_0) be satisfied. Then the function

$$(3.1)\quad u(x,t,\omega)=\int_{R^n}Z(x,t,\xi,0,\omega)g(\xi,\omega)d\xi-\int_0^t\int_{R^n}Z(x,t,\xi,\tau,\omega)f(\xi,\tau,\omega)d\xi d\tau$$

has the following properties:

1° it is L_q-continuous in G;

2° there exist L_q-derivatives $u_{x_i}(x,t,\omega)$ which are L_q-continuous in G_0;

3° there exist L_r-derivatives $u_{x_ix_j}$ und u_t which are L_r-continuous in G_0 $\left(\frac{1}{r}=\frac{1}{p}+\frac{1}{q}\right)$;

4° $u(x,t,\omega)$ is a solution of the problem (0.2), (0.3), i.e.

$$(3.2)\quad Lu(x,t,\omega)=f(x,t,\omega),\quad (x,t)\in G_0,\ \omega\in\Omega_{x,t}\ (P(\Omega_{x,t})=1),$$

$$(3.3)\quad u(x,0,\omega)=g(x,\omega),\quad x\in R^n,\omega\in\Omega_x\quad (P(\Omega_x)=1).$$

P r o o f. We use the argumentation of the proof of Theorem 2.1 (see also [1], chapter 1). Let us denote

(3.4) $$v(x,t,\omega) = \int_{R^n} Z(x,t,\xi,0,\omega)g(\xi,\omega)d\xi ,$$

(3.5) $$G(x,t,\xi,\omega) = Z(x,t,\xi,0,\omega)g(\xi,\omega).$$

The L_q-continuity of G in the set

(3.6) $$x,\xi \in R^n, \ t \in (0,T\rangle$$

and the estimate

(3.7) $$\|G(x,t,\xi,\cdot)\|_q \leqslant M_1 t^{-\frac{n}{2}} \exp\left[-\frac{\mu|x-\xi|^2}{t}\right]$$

yield, in view of Lemma 4.6, the uniform L_q-convergence of the integral (3.4) in the set

(3.8) $x \in D$ (D being a bounded domain), $t \in (0,T\rangle$.

Hence, by Lemma 4.7, the function $v(x,t,\omega)$ is L_q-continuous in G_0. With the aid of assertion 4^0 of Theorem 2.1 setting additionally

(3.9) $$v(x,0,\omega) = g(x,\omega), \quad x \in R^n, \ \omega \in \Omega$$

we obtain a random function L_q-continuous in G.

Now consider the integral

(3.10) $$\int_{R^n} Z_{x_i}(x,t,\xi,0,\omega)g(\xi,\omega)d\xi .$$

Since the function (3.5) possesses L_q-derivatives

(3.11) $$G_{x_i}(x,t,\xi,\omega) = Z_{x_i}(x,t,\xi,0,\omega)g(\xi,\omega),$$

which are L_q-continuous in the set (3.6), therefore, by the estimate

$$\left\| G_{x_i}(x,t,\xi,\cdot)\right\|_q \leq M_2 t^{-(n+1)/2} \exp\left[-\frac{\bar{\mu}|x-\xi|^2}{t}\right]$$

and by Lemma 4.6, the integral (3.10) is uniformly L_q-convergent in the set (3.8). In virtue of Lemma 4.8 there exist in G_0 L_q-derivatives

$$(3.12) \qquad v_{x_i}(x,t,\omega) = \int_{R^n} Z_{x_i}(x,t,\xi,0,\omega)g(\xi,\omega)d\xi ,$$

which are L_q-continuous in G_0.

Similarly, observing that function (3.5) possesses L_r-derivatives

$$G_{x_i x_j}(x,t,\xi,\omega) = Z_{x_i x_j}(x,t,\xi,0,\omega)g(\xi,\omega),$$

$$G_t(x,t,\xi,\omega) = Z_t(x,t,\xi,0,\omega)g(\xi,\omega)$$

and using the estimate

$$\left\| G_{x_i x_j}(x,t,\xi,\cdot)\right\|_r, \left\| G_t(x,t,\xi,\cdot)\right\|_r < M_3 t^{-(n+2)/2} \exp\left[-\frac{\bar{\mu}|x-\xi|^2}{t}\right]$$

we conclude the existence in G_0 of L_r-derivatives

$$(3.13) \qquad v_{x_i x_j}(x,t,\omega) = \int_{R^n} Z_{x_i x_j}(x,t,\xi,0,\omega)g(\xi,\omega)d\xi ,$$

$$(3.14) \qquad v_t(x,t,\omega) = \int_{R^n} Z_t(x,t,\xi,0,\omega)g(\xi,\omega)d\xi ,$$

which are L_r-continuous in G_0.

At present we consider the functions

$$(3.15) \qquad w(x,t,\omega) = \int_0^t \int_{R^n} Z(x,t,\xi,\tau,\omega)f(\xi,\tau,\omega)d\xi d\tau ,$$

(3.16) $$F(x,t,\xi,\tau,\omega) = Z(x,t,\xi,\tau,\omega)f(\xi,\tau,\omega).$$

Since the function (3.16) is L_q-continuous in the set

(3.17) $$x,\xi \in R^n, \quad 0 \leqslant \tau < t \leqslant T$$

and there holds the estimate

(3.18) $$\|F(x,t,\xi,\tau,\cdot)\|_q \leqslant M_4(t-\tau)^{-\frac{n}{2}} \exp\left[-\frac{\bar{\mu}|x-\xi|^2}{t-\tau}\right]$$

therefore, by Remark 4.9, the function $w(x,t,\omega)$ is L_q-continuous in G_0. Moreover, (3.18) yields

$$\|w(x,t,\cdot)\|_q \leqslant M_5 t,$$

which implies that

$$\lim_{t\to 0} \|w(x,t,\cdot)\|_q = 0$$

uniformly with respect to $x \in R^n$. Thus, setting additionally

(3.19) $$w(x,0,\omega) = 0, \qquad x \in R^n, \ \omega \in \Omega$$

we find that $w(x,t,\omega)$ is L_q-continuous in G.

Note that function (3.16) possesses in (3.17) L_q-derivatives

$$F_{x_i}(x,t,\xi,\tau,\omega) = Z_{x_i}(x,t,\xi,\tau,\omega)f(\xi,\tau,\omega)$$

which are L_q-continuous in (3.17) and satisfy the inequality

$$\|F_{x_i}(x,t,\xi,\tau,\cdot)\|_q \leqslant M_6(t-\tau)^{-(n+1)/2} \exp\left[-\frac{\bar{\mu}|x-\xi|^2}{t-\tau}\right].$$

Hence, taking advantage of Remark 4.9 we find that there exist in G_0 L_q-derivatives $w_{x_i}(x,t,\omega)$, given by formulas

$$(3.20)\qquad w_{x_i}(x,t,\omega) = \int_0^t \int_{R^n} Z_{x_i}(x,t,\xi,\tau,\omega) f(\xi,\tau,\omega)\, d\xi\, d\tau$$

and L_q-continuous in G_o.

In order to prove the existence of L_r-derivatives $w_{x_i x_j}$ and w_t let us introduce the function

$$(3.21)\qquad J(x,t,\tau,\omega) = \int_{R^n} W_{\xi,\tau}(x,t,\xi,\tau,\omega) f(\xi,\tau,\omega)\, d\xi .$$

Observe that $f(\xi,\tau,\omega)$ is measurable for every fixed $\tau \in B \subset \langle 0,T\rangle$, where $|B| = T$. Hence, in view of Lemma 4.7, the function (3.21) is uniformly L_q-continuous in every set

$$(3.22)\qquad x \in D,\quad t \in \langle \delta_1, T\rangle,\ \tau \in B \cap \langle 0, t-\delta_2\rangle\ (0<\delta_2<\delta_1),$$

D being a bounded domain. Consequently, the function (3.21) is L_q-continuous in the set

$$(3.23)\qquad x \in R^n,\ t \in (0,T\rangle,\ \tau \in B \cap \langle 0,t).$$

This implies, by Lemma 4.8, the existence (in the set (3.23)) of L_q-derivatives

$$(3.24)\qquad \begin{aligned} J_{x_i}(x,t,\tau,\omega) &= \int_{R^n} \frac{\partial W_{\xi,\tau}(x,t,\xi,\tau,\omega)}{\partial x_i} f(\xi,\tau,\omega)\, d\xi , \\ J_{x_i x_j}(x,t,\tau,\omega) &= \int_{R^n} \frac{\partial^2 W_{\xi,\tau}(x,t,\xi,\tau,\omega)}{\partial x_i\, \partial x_j} f(\xi,\tau,\omega)\, d\xi , \end{aligned}$$

which are uniformly L_q-continuous in every set (3.22). The function (3.24) can be treated analogously to (2.19) and (4.36) of [4]. Hence, in the set

$$(3.25)\qquad x \in D\ \ (D \text{ being a bounded domain}),\ t \in (0,T\rangle, \tau \in B \cap \langle 0,t),$$

there holds true the estimate

$$(3.26)\qquad \left\| J_{x_i x_j}(x,t,\tau,\cdot)\right\|_q < M_7 (t-\tau)^{-(2-\alpha)/2},$$

$M_7 > 0$ being a constant depending on D. Using this estimate and Remark 4.9, we see that the integral

$$(3.27)\qquad \int_0^t J_{x_i x_j}(x,t,\tau,\omega)d\tau$$

is uniformly L_q-convergent in every set

$$(3.28)\quad x\in D \quad (D \text{ being a bounded domain}),\ t\in\langle\delta,T\rangle,\ \delta\in(0,T).$$

Thus, in virtue of the above-mentioned remark, the function

$$(3.29)\quad w_1(x,t,\omega) = \int_0^t J(x,t,\tau,\omega)d\tau = \int_0^t \int_{R^n} W_{\xi,\tau}(x,t,\xi,\tau,\omega)f(\xi,\tau,\omega)d\xi\, d\tau$$

possesses in G_0 L_q-derivatives

$$(3.30)\quad \frac{\partial^2 w_1(x,t,\omega)}{\partial x_i\ \partial x_j} = \int_0^t J_{x_i x_j}(x,t,\tau,\omega)d\tau = \int_0^t d\tau \int_{R^n} \frac{\partial^2 W_{\xi,\tau}(x,t,\xi,\tau,\omega)}{\partial x_i\ \partial x_j} f(\xi,\tau,\omega)d\xi,$$

which are L_q-continuous in G_0.

At present, proceeding like in the proof of the formula (2.29), we show that there exists in G_0 L_q-derivative

$$(3.31)\qquad \frac{\partial w_1(x,t,\omega)}{\partial t} = f(x,t,\omega) + \int_0^t J_t(x,t,\tau,\omega)d\tau,$$

which is L_q-continuous in G_0.

At first, taking into considerations the relation

$$J_t(x,t,\tau,\omega) = \sum_{i,j=1}^{n} a_{ij}(x,t,\omega) J_{x_i x_j}(x,t,\tau,\omega)$$

and the estimate (3.26), observe that the integral

(3.32) $$\int_0^t J_t(x,t,\tau,\omega)d\tau$$

is uniformly L_q-convergent in the set (3.28), whereas the integral

(3.33) $$\int_0^t \|J_t(x,t,\tau,\cdot)\|_q d\tau$$

is uniformly convergent in (3.28).

Now, let $\Delta t > 0$, $t > \delta > 0$ and $\eta \in (0,\delta)$. Then we can write

$$\frac{w_1(x,t+\Delta t,\omega) - w_1(x,t,\omega)}{\Delta t} - f(x,t,\omega) - \int_0^t J_t(x,t,\tau,\omega)d\tau =$$

$$= I_1 - I_2 + I_3 + I,$$

where

$$I_1 = \int_{t-\eta}^{t} \frac{J(x,t+\Delta t,\tau,\omega) - J(x,t,\tau,\omega)}{\Delta t}\, d\tau ,$$

$$I_2 = \int_{t-\eta}^{t} J_t(x,t,\tau,\omega)d\tau ,$$

$$I_3 = \int_0^{t-\eta} \left[\frac{J(x,t+\Delta t,\tau,\omega) - J(x,t,\tau,\omega)}{\Delta t} - J_t(x,t,\tau,\omega)\right] d\tau ,$$

$$I = \frac{1}{\Delta t} \int_t^{t+\Delta t} J(x,t+\Delta t,\tau,\omega)d\tau - f(x,t,\omega).$$

The uniform L_q-convergence of the integral (3.32) implies that for any $\varepsilon > 0$ and sufficiently small $\eta > 0$ we have $\|I_2\|_q < \varepsilon$. The inequality

$$\left\|\frac{J(x,t+\Delta t,\tau,\cdot) - J(x,t,\tau,\cdot)}{\Delta t}\right\|_q < \left\|J_t(x,t',\tau,\cdot)\right\|_q, \quad t' \in \langle t,t+\Delta t\rangle$$

(following from Lemma 4.5) and the uniform convergence of the integral (3.33) yield

$$\|I_1\|_q \leqslant \int_{t-\eta}^{t} \|J_t(x,t',\tau,\cdot)\|_q d\tau < \varepsilon$$

for sufficiently small $\eta > 0$ and $\Delta t > 0$.

Now fix $\eta > 0$. Then observing that

$$I_3 = \int_0^{t-\eta} d\tau \int_{R^n} \left[\frac{\partial W_{\xi,\tau}(x,t',\xi,\tau,\omega)}{\partial t} - \frac{\partial W_{\xi,\tau}(x,t,\xi,\tau,\omega)}{\partial t}\right] f(\xi,\tau,\omega)d\xi$$

(where $t' \in (t,t+\Delta t)$) and using the estimate (2.30) we have $\|I_3\|_q < \varepsilon$ for sufficiently small $\Delta t > 0$. Consequently

$$\lim_{\Delta t \to 0+0} \|I_1+I_2+I_3\|_q = 0.$$

It remains to show that

$$\lim_{\Delta t \to 0+0} \|I\|_q = 0. \tag{3.34}$$

At first, like in the proof of relation (2.31), we establish (3.34) for $x \in B_0$, where $B_0 \subset R^n$, $|R^n \setminus B_0| = 0$ is such a set that $f(\xi,\tau,\omega)$ is measurable for every fixed $\xi \in B_0$. Indeed, for $x \in B_0$ the expression I can be written as follows

$$I = I_1+I_2+I_3,$$

where

$$I_1 = \frac{1}{\Delta t}\int_t^{t+\Delta t} d\tau \int_{R^n} W_{\xi,\tau}(x,t+\Delta t,\xi,\tau,\omega)\left[f(\xi,\tau,\omega) - f(x,\tau,\omega)\right] d\xi ,$$

$$I_2 = \frac{1}{\Delta t}\int_t^{t+\Delta t} f(x,\tau,\omega)\left\{\int_{R^n}\left[W_{\xi,\tau}(x,t+\Delta t,\xi,\tau,\omega) - W_{x,\tau}(x,t+\Delta t,\xi,\tau,\omega)\right]d\xi\right\} d\tau ,$$

$$I_3 = \frac{1}{\Delta t}\int_t^{t+\Delta t}\left[f(x,\tau,\omega) - f(x,t,\omega)\right]d\tau .$$

The further argumentation is similar to that used for integrals I'_1, I'_2 and I'_3 in the proof of (2.31). Next, like in the above-mentioned proof, one can show the validity of (3.34) for any $x \in R^n$.

Since the case $\Delta t < 0$ can be treated is a similar way as the case $\Delta t > 0$, the formula (3.31) is completely proved.

Now we discuss the existence of L_r-derivatives $\frac{\partial^2 w_2}{\partial x_i \partial x_j}$ and $\frac{\partial w_2}{\partial t}$ of the function

$$w_2(x,t,\omega) = \int_0^t \int_{R^n} V(x,t,\xi,\tau,\omega) f(\xi,\tau,\omega)\, d\xi\, d\tau , \tag{3.35}$$

V being given by formula (2.15). For this purpose note that the function

$$F(x,t,\xi,\tau,\omega) = V(x,t,\xi,\tau,\omega) f(\xi,\tau,\omega)$$

possesses L_r-derivatives

$$F_{x_i x_j}(x,t,\xi,\tau,\omega) = V_{x_i x_j}(x,t,\xi,\tau,\omega) f(\xi,\tau,\omega),$$

which are L_r-continuous in the set (3.17). Thus, applying the estimate (2.25) and Remark 4.9, we conclude that there exist L_r-derivatives

$$\frac{\partial^2 w_2(x,t,\omega)}{\partial x_i \partial x_j} = \int_0^t \int_{R^n} V_{x_i x_j}(x,t,\xi,\tau,\omega) f(\xi,\tau,\omega)\, d\xi\, d\tau , \tag{3.36}$$

which are L_r-continuous in G_0.

Now observe, that substituting the formula (2.15) in (3.35) and using the estimates (1.6) and (2.10) one can change the order of integration in (3.35). So we have

$$w_2(x,t,\omega) = \int_0^t \int_{R^n} W_{\xi,\tau}(x,t,\xi,\tau,\omega)\bar{f}(\xi,\tau,\omega)d\xi\, d\tau ,$$

where

$$\bar{f}(x,t,\omega) = \int_0^t \int_{R^n} \phi(x,t,\xi,\tau,\omega)f(\xi,\tau,\omega)d\xi\, d\tau .$$

With the aid of the estimate (2.10) and by Remark 4.9 the function $\bar{f}(x,t,\omega)$ is L_q-continuous in G. In virtue of the estimate

$$\|\phi(x,t,\xi,\tau,\cdot) - \phi(x',t,\xi,\tau,\cdot)\|_p \leqslant$$

$$\leqslant M_8(t-\tau)^{-(n+2-2\mu_2)/2}|x-x'|^{\mu_3}\left[\exp\left(-\frac{\mu''|x-\xi|^2}{t-\tau}\right)+\exp\left(-\frac{\mu''|x'-\xi|^2}{t-\tau}\right)\right],$$

following from (2.14) and (2.10), we get

$$\|\bar{f}(x,t,\cdot)-\bar{f}(x',t,\cdot)\|_r \leqslant M_9|x-x'|^{\mu_3}, \quad x,x' \in R^n, \quad t \in \langle 0,T\rangle .$$

Moreover, we may assume that $\bar{f}(x,t,\omega)$ is measurable. Therefore, according to the considerations concerning the derivative $\frac{\partial w_1}{\partial t}$, there exists in G_0 L_r-derivative

$$(3.37) \qquad \frac{\partial w_2(x,t,\omega)}{\partial t} = \bar{f}(x,t,\omega) + \int_0^t \bar{J}_t(x,t,\tau,\omega)d\tau ,$$

where

$$(3.38) \qquad \bar{J}_t(x,t,\tau,\omega) = \int_{R^n} \frac{\partial W_{\xi,\tau}(x,t,\xi,\tau,\omega)}{\partial t}\bar{f}(\xi,\tau,\omega)d\xi .$$

Moreover, the derivative $\frac{\partial w_2}{\partial t}$ is L_r-continuous in G_0. This completes the proof of assertions $1^0 - 3^0$ of Theorem 3.1.

Now combining relations (3.1), (3.4), (3.12)-(3.15),(3.20), (3.21),(3.29)-(3.31),(3.35)-(3.38) and (2.5) we find the validity od (3.2). Finally, relation (3.3) immediately follows from (3.1), (3.4), (3.15), (3.9) and (3.19). Thus Theorem 3.1 is proved.

4. Lemmas

In this section we state lemmas which were used in the previous sections. For the sake of simplicity they are often formulated for more particular cases than it follows from their applications. However appropriate generalizations of these lemmas can be easily obtained.

Lemma 4.1. Suppose $g(x,t,\omega)$, $x \in R^n$, $t \in \langle 0,T \rangle$, $\omega \in \Omega$ is a bounded measurable random function, whereas $f(x,t,\theta,\tau,\zeta,\omega)$ is a random function defined in the set

$$(4.1)\quad G_3 = \{x \in D \quad (D \subset R^k \text{ being a bounded closed domain}),$$

$$\zeta \in R^n,\ 0 < \tau \leq T-\delta (\delta > 0),\ \tau+\delta < t \leq T,\ \tau < \theta < t\},\ \omega \in \Omega$$

and continuous in x,t,θ,τ,ζ uniformly with respect to $\omega \in \Omega_1$, where $P(\Omega_1) = 1$. Moreover we assume that for any $\beta \in (0, \delta/2)$, $\gamma > 0$ the function f is bounded in the set

$$(4.2)\quad G_4 = \{x \in D, |\zeta| < \gamma, 0 < \tau < T-\delta, \tau+\delta \leq t < T,\ \tau+\beta < \theta \leq t-\beta\},\ \omega \in \Omega_1$$

and that the improper Lebesgue integral

$$(4.3)\qquad h(x,t,\tau,\omega) = \int_\tau^t \int_{R^n} f(x,t,\theta,\tau,\zeta,\omega) g(\zeta,\theta,\omega)\, d\zeta\, d\theta$$

is uniformly convergent in the set

(4.4) $G_5 = \{x \in D,\ 0 < \tau < T-\delta,\ \ \tau+\delta \leq t < T\}\ ,\ \ \omega \in \Omega_1$ [4].

Under these assumptions the function $h(x,t,\tau,\omega)$ is continuous in the set G_5 uniformly with respect to $\omega \in \Omega_1$.

L e m m a 4.2. Let all the assumptions of Lemma 4.1 be satisfied. Suppose there exist in the set $\{G_3;\ \omega \in \Omega_1\}$ derivatives f_{x_i} uniform with respect to $\omega \in \Omega_1$, which are continuous in G_3 uniformly with respect to $\omega \in \Omega_1$. Moreover we assume that for any $\beta \in (0, \delta/2)$, $\gamma > 0$ the functions f_{x_i} are bounded in the set (4.2) and that the integrals

$$(4.5)\qquad \int_{\tau}^{t} \int_{R^n} f_{x_i}(x,t,\theta,\tau,\zeta,\omega) g(\zeta,\theta,\omega)\,d\zeta\,d\theta$$

are uniformly convergent in the set (4.2). Under these assumptions there exist in the set (4.4) derivatives h_{x_i} uniform with respect to $\omega \in \Omega_1$, continuous in G_5 uniformly with respect to $\omega \in \Omega_1$ and they are equal to the integrals (4.5).

L e m m a 4.3. We assume that $f(x,y,z,\omega)$, $x \in \langle a_1,b_1\rangle$, $y \in \langle a_2,b_2\rangle$, $z \in (a_3,b_3)$, $\omega \in \Omega_1$ $(P(\Omega_1) = 1)$ is a random function continuous in x,y,z uniformly with respect to $\omega \in \Omega_1$ and bounded in every set

$$(4.6)\qquad x \in \langle a_1,b_1\rangle\ ,\ y \in \langle a_2,b_2\rangle\ ,\ z \in \langle \alpha\ ,\ \beta\rangle \subset (a_3,b_3),\ \omega \in \Omega_1.$$

Let $h(y,z,\omega)$, $y \in \langle a_2,b_2\rangle$, $z \in (a_3,b_3)$, $\omega \in \Omega_1$ be a bounded measurable random function, L_p-continuous in y,z. Suppose that the improper Lebesgue integral

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[4] I.e. for any $\varepsilon > 0$ there are $\eta_1, \eta_2 > 0$ such that for any γ_1, γ_2 $(0, \eta_1)$ and any bounded domain $\Delta \supset \{\zeta \in R^n : |\zeta| < \eta_2\}$ there holds in the set (4.4) the inequality

$$\left| h(x,t,\tau,\omega) - \int_{\tau+\gamma_1}^{t-\gamma_2} \int_{\Delta} f(x,t,\theta,\tau,\zeta,\omega) g(\zeta,\theta,\omega)\,d\zeta\,d\theta \right| < \varepsilon.$$

$$(4.7) \qquad g(x,y,\omega) = \int_{a_3}^{b_3} f(x,y,z,\omega)h(y,z,\omega)dz$$

is uniformly convergent in the set

$$(4.8) \qquad x \in \langle a_1,b_1\rangle \ , \ (y,\omega) \in A_0,$$

$A_0 \subset \langle a_2,b_2\rangle \times \Omega_1$ being such a set that for every fixed $(y,\omega) \in A_0$ the function $h(y,z,\omega)$ is measurable with respect to $z \in (a_3,b_3)$ [5]. Then the function $g(x,y,\omega)$ possesses the following properties:

1^0 it is defined for $x \in \langle a_1,b_1\rangle$, $(y,\omega) \in A_0$;

2^0 measurability in the set $\langle a_1,b_1\rangle \times \langle a_2,b_2\rangle \times \Omega_1$;

3^0 uniform L_p-continuity for $x \in \langle a_1,b_1\rangle$, $y \in B_0 \subset \langle a_2,b_2\rangle$, where B_0 is some set with Lebesgue measure $|B_0| = b_2-a_2$;

4^0 continuity in $x \in \langle a_1 b_1\rangle$ uniform with respect to $(y,\omega) \in A_0$.

L e m m a 4.4. Let assumptions of Lemma 4.3 be fulfilled. Suppose that

(i) there exists a derivative $f_x(x,y,z,\omega)$ uniform with respect to $\omega \in \Omega_1$,

(ii) the derivative f_x is continuous in x,y,z uniformly with respect to $\omega \in \Omega_1$ and bounded in every set (4.6),

(iii) the integral

$$(4.9) \qquad \int_{a_3}^{b_3} f_x(x,y,z,\omega)h(y,z,\omega)dz$$

is uniformly convergent in the set (4.8).

Under these assumptions the function (4.7) possesses a derivative $g_x(x,y,\omega)$ uniform with respect to $\omega \in \Omega_y \subset \Omega_1$ ($y \in B_0$, $P(\Omega_y)=1$) which is equal to the integral (4.9). Moreover, for this derivative properties 1^0–4^0 of Lemma 4.3 hold true.

[5] Obviously $|A_0| = (l \times P)(A_0) = b_2-a_2$, where l is the Lebesgue measure in R.

Proofs of Lemmas 4.1 - 4.4 are similar to those of appropriate theorems for nonrandom functions.

L e m m a 4.5. If a random function $f(x,\omega)$, $x\in\langle a,b\rangle$, $\omega\in\Omega$ has L_p-derivative $f'(x,\omega)$ L_p-bounded in $\langle a,b\rangle$, then

$$(4.10) \qquad \|f(x,\cdot)-f(y,\cdot)\|_p \leqslant A|x-y|, \quad x,y\in\langle a,b\rangle,$$

where $A=\sup_{x\in\langle a,b\rangle}\|f'(x,\cdot)\|_p$.

P r o o f. Take an arbitrary $\varepsilon>0$. Since for any $x\in\langle a,b\rangle$ we have

$$A \geqslant \|f'(x,\cdot)\|_p = \lim_{y\to x}\frac{\|f(y,\cdot)-f(x,\cdot)\|_p}{|y-x|},$$

therefore exists $\delta=\delta(x,\varepsilon)>0$ such that

$$(4.11)\ \|f(y,\cdot)-f(x,\cdot)\|_p \leqslant (A+\varepsilon)|y-x| \quad \text{if} \quad y\in(x-\delta,x+\delta)\cap\langle a,b\rangle \equiv K(x,\delta).$$

For any $x,y\in\langle a,b\rangle$ $(x<y)$ there exist intervals $K_i = K(x_i,\delta_i)$, $i=1,\dots,n$ $(x=x_1<\dots<x_n=y)$ such that $\langle x,y\rangle\subset\bigcup_{i=1}^{n}K_i$. Choosing arbitrary $y_i\in(x_i,x_{i+1})\cap K_i\cap K_{i+1}$, $i=1,\dots,n-1$ it follows from (4.11) that

$$\|f(x,\cdot)-f(y,\cdot)\|_p \leqslant \|f(x_1,\cdot)-f(y_1,\cdot)\|_p + \|f(y_1,\cdot)-f(x_2,\cdot)\|_p +$$

$$+\|f(x_2,\cdot)-f(y_2,\cdot)\|_p+\dots+\|f(y_{n-1},\cdot)-f(x_n,\cdot)\|_p \leqslant$$

$$\leqslant (A+\varepsilon)\left[|x_1-y_1|+|y_1-x_2|+\dots+|y_{n-1}-x_n|\right] = (A+\varepsilon)|x-y|.$$

This yields (4.10).

Now we state definitions and lemmas concerning integrals of random functions in the L_p-sense. These L_p-integrals involve as a particular case the Lebesgue integrals of random functions occurring in Lemmas 4.1-4.4.

Let $f(x,\omega)$, $x \in X \subset \langle a,b\rangle$ ($|X| = b-a$), $\omega \in \Omega$ be a random function uniformly L_p-continuous in X. Then the integral

$$\int_a^b f(x,\omega)dx = \int_X f(x,\omega)dx \tag{4.12}$$

is taken in the L_p-sense, i.e. as a strong Riemann integral (see [5], p.192 and [6] p.17, cf. [3], pp.267-269).

Now let $f(x,\omega)$, $x \in X \subset \langle a,b)$ ($b \in (a,\infty)$ or $b = \infty$, $|\langle a,b)\setminus X| = 0$), $\omega \in \Omega$ be a measurable random function uniformly L_p-continuous in every set $X \cap \langle a,\beta\rangle$, $\beta \in (a,b)$. In this case we understand the integral (4.12) as an improper strong Riemann integral, i.e.

$$\int_a^b f(x,\omega)dx = \lim_{\beta\to b} \int_a^\beta f(x,\omega)dx \ (L_p).$$

If this L_p-limit exists, then we say that the integral (4.12) is L_p-convergent. It is easy to see that the convergence of the integral $\int_a^b \|f(x,\cdot)\|_p dx$ is a sufficient condition for the L_p-convergence of the integral (4.12).

We introduce the following assumption.

(H_1) A random function $f(x,y,z,\omega)$, $x \in \langle a_1,b_1\rangle$, $y \in Y \subset \langle a_2,b_2\rangle$ ($|Y| = b_2-a_2$), $z \in Z \subset \langle a_3,b_3)$ ($|\langle a_3,b_3)\setminus Z| = 0$), $\omega \in \Omega$ is measurable for every fixed x,y and uniformly L_p-continuous in every set

$$x \in \langle a_1,b_1\rangle,\ y \in Y,\quad z \in Z \cap \langle a_3,\beta\rangle,\ \beta \in (a_3,b_3). \tag{4.13}$$

D e f i n i t i o n. Let assumption (H_1) be satisfied and let the integral

$$g(x,y,\omega) = \int_{a_3}^{b_3} f(x,y,z,\omega)dz \tag{4.14}$$

be L_p-convergent for every fixed $(x,y) \in \langle a_1,b_1\rangle \times Y$. We say that the integral (4.14) is uniformly L_p-convergent in $\langle a_1,b_1\rangle \times Y$ if for any $\varepsilon > 0$ there is $\eta \in (a_3,b_3)$ such that the inequality

$$\left\| g(x,y,\cdot) - \int_{a_3}^{\beta} f(x,y,z,\cdot)dz \right\|_p = \left\| \int_{\beta}^{b_3} f(x,y,z,\cdot)dz \right\|_p < \varepsilon$$

is fulfilled for all $\beta \in (\eta,b_3)$, $(x,y) \in \langle a_1,b_1\rangle \times Y$.

One can easily obtain the following lemma.

L e m m a 4.6. If assumption (H_1) is satisfied and the integral

$$\int_{a_3}^{b_3} \|f(x,y,z,\cdot)\|_p dz$$

is uniformly convergent in $\langle a_1,b_1\rangle \times Y$, then the integral (4.14) is uniformly L_p-convergent in $\langle a_1,b_1\rangle \times Y$.

L e m m a 4.7. If assumption (H_1) is fulfilled and the integral (4.14) is uniformly L_p-convergent in $\langle a_1,b_1\rangle \times Y$, then the function (4.14) is uniformly L_p-continuous in $\langle a_1,b_1\rangle \times Y$.

L e m m a 4.8. Let the assumption of Lemma 4.7 be satisfied. Suppose that there exists L_p-derivative $f_x(x,y,z,\omega)$, uniformly L_p-continuous in every set (4.13) and that the integral

$$\int_{a_3}^{b_3} f_x(x,y,z,\omega)dz \tag{4.15}$$

is uniformly L_p-convergent in $\langle a_1,b_1\rangle \times Y$. Then the function (4.14) possesses L_p-derivative $g_x(x,y,\omega)$, uniformly L_p-continuous in $\langle a_1,b_1\rangle \times Y$ and equal to the integral (4.15).

The above two lemmas can be proved in the standard manner like the appropriate theorems for nonrandom functions. Namely, at first we prove that they hold true for proper integrals in the L_p-sense. Hence, by Theorem on uniform L_p-continuity and

L_p-differentiability of L_p-limit of random functional sequence, Lemmas 4.7 and 4.8 follow.

R e m a r k 4.9. Let us introduce the following definitions of uniform L_p-convergence for integrals

$$(4.16)\qquad g(x,y,\omega) = \int_{a_2}^{y} f(x,y,z,\omega)dz,$$

$$(4.17)\qquad g(x,y,\omega) = \int_{a_2}^{y}\int_{a_3}^{b_3} f(x,y,z,s,\omega)dsdz.$$

D e f i n i t i o n. Assume that $f(x,y,z,\omega)$, $x\in\langle a_1,b_1\rangle$, $y\in(a_2,b_2\rangle$ $(-\infty<a_2<b_2)$, $z\in Z_y=\langle a_2,y)\cap Z$ $(Z\subset\langle a_2,b_2\rangle$, $|Z|=b_2-a_2)$, $\omega\in\Omega$ is a random function measurable for every fixed (x,y) and uniformly L_p-continuous in every set

$$x\in\langle a_1,b_1\rangle,\ y\in\langle a_2',b_2\rangle,\ z\in Z_{y-\beta}(a_2<a_2'<b_2,\quad 0<\beta<a_2'-a_2).$$

We say that the integral (4.16) is uniformly L_p-convergent in the set

$$E=\langle a_1,b_1\rangle\times\langle a_2',b_2\rangle\quad(a_2'\in(a_2,b_2))$$

if for any $\varepsilon>0$ there is $\delta>0$ such that

$$\left\|g(x,y,\cdot)-\int_{a_2}^{y-\eta} f(x,y,z,\cdot)dz\right\|_p<\varepsilon,\ \eta\in(0,\delta),(x,y)\in E.$$

D e f i n i t i o n. Let a random function $f(x,y,z,s,\omega)$, $x\in\langle a_1,b_1\rangle$, $\in(a_2,b_2\rangle$, $z\in Z_y$, $s\in S\subset\langle a_3,b_3)$ $(|\langle a_3,b_3)\setminus S|=0)$, $\omega\in\Omega$ be measurable for every fixed (x,y) and uniformly L_p-continuous in every set

$$x\in\langle a_1,b_1\rangle,\ y\in\langle a_2',b_2\rangle,\ z\in Z_{y-\beta},\quad s\in S\cap\langle a_3,\gamma\rangle,$$

where $a_2' \in (a_2,b_2)$, $\beta \in (0,a_2'-a_2)$, $\gamma \in (a_3,b_3)$. The integral (4.17) is called uniformly L_p-convergent in the set E if for any $\varepsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 \in (a_3,b_3)$ such that

$$\left\| g(x,y,\cdot) - \int_{a_2}^{y-\eta_1} \int_{a_3}^{\eta_2} f(x,y,z,s,\cdot)\,ds\,dz \right\|_p < \varepsilon$$

for all $\eta_1 \in (0,\delta_1)$, $\eta_2 \in (\delta_2,b_3)$, $(x,y) \in E$.

Using these definitions one can find that Lemmas 4.6–4.8 hold true (with obvious modifications) also for integrals (4.16) and (4.17).

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Received March 12, 1977.