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ON THE EXISTENCE OF SOLUTIONS OF A CERTAIN INTEGRAL-FUNCTIONAL EQUATION

1. Introduction

Let $C(X,Y)$ denote a class of continuous functions defined in X with range in Y, where X,Y are given metric spaces.

Let $I = [0,a]$, where a is a fixed positive real number.

We consider the integral-functional equation

$$(1) \qquad x(t) = F\left(t, \int_0^{\alpha_1(t)} f_1(t,s,x(s))\,ds, \ldots, \int_0^{\alpha_r(t)} f_r(t,s,x(s))\,ds, x(\beta_1(t)), \ldots, x(\beta_m(t))\right), \qquad t \in I,$$

where functions $F \in C(I \times (R^n)^{r+m}, R^n)$, $f_j \in C(I^2 \times R^n, R^n)$, $j = 1,\ldots,r$, α_j, $\beta_i \in C(I,I)$, $j = 1,\ldots,r$, $i = 1,\ldots,m$, are given.

We shall always assume that $0 < \alpha_j(t) \leqslant t$, $j = 1,\ldots,r$, $0 \leqslant \beta_i(t) \leqslant t$, $i = 1,\ldots,m$.

In our paper we give sufficient conditions for the existence at least one solution $\bar{x} \in C(I,R^n)$ of equation (1).

The existence of solutions of equation (1) analogically as for differential-functional equations (see [7], [8], [10]) is proved with using fixed point theorems in the Banach space.

Generally speaking these theorems for a appropriately defined operator require a Lipschitz condition (Banach fixed point theorem) or compactness of the operator (Schauder fixed point theorem).

In the literature we have found some fixed point theorems which combine the results of Schauder and the contraction mapping principle (see [1], [2], [7]).

The existence and uniqueness of solution of equation of type (1) was proved in [6] by the use of the method of successive approximations under rather strong conditions.

The particular case of equation (1) is the equation considered in [5] (the case $r = 1$, $m = 1$).

A differential-functional equation of neutral type can be reduced to the equation of the form (1) (see [3], [7] - - [10]).

It is well known that: if the function $F:(t,u_1,\dots,u_r,v_1,\dots \dots,v_m)\to R^n$ satisfes the Lipschitz condition with respect to the last m variables with constants λ_i, $i = 1,\dots,m$ and $\sum_{i=1}^{m} \lambda_i < 1$, (and other conditions which are not tied) then there exists at least one solution of equation (1).

Results of this type can be found in [2], [7] ($i = 1$), and [8] - [10] (for suitable differential equations).

The result of this paper for $\beta_i(t) \leq \bar{\beta}_i \cdot t$ $i = 1,\dots m$, $\bar{\beta}_i \in [0,1]$ gives the weaker condition $\sum_{i=1}^{m} \lambda_i \bar{\beta}_i^{\varrho_j} < 1$ for some $\varrho_j \geq 0$ $j = 1,\dots,r$.

In the present paper we use Schauder fixed-point theorem in a such way as it was done in the particular case discussed in [5].

It seems that our result cannot be obtained by the use of the notion of measure of noncompactness and k-set contraction theory (see [2] and [8]).

2. Assumptions and Lemmas

In analogy with the paper [6] we define the linear operators $L : C(I,R_+) \to C(I,R_+)$, $K : C(I,R_+) \to C(I,R_+)$,

$$(Lg)(t) = \sum_{i=1}^{m} \lambda_i(t) g(\beta_i(t)),$$

$$(Kg)(t) = \sum_{j=1}^{r} K_j(t) \int_0^{\alpha_j(t)} g(s)ds ,$$

where λ_i, K_j, α_j, β_i are given and $\lambda_i, K_j \in C(I,R_+)$, $\alpha_j, \beta_i \in C(I,I)$, $i = 1,\dots,m$, $j = 1,\dots,r$.

Put

$$\beta_i^1(t) = \beta_i(t),\ \beta_{n+1}^{i_1,\dots,i_{n+1}}(t) = \beta_n^{i_1,\dots,i_n}\left(\beta_{i_{n+1}}(t)\right) ,$$

$$i_n = 1,\dots,m,\ n = 0,1,\dots,$$

$$\lambda_1^i(t) = \lambda_i(t),\ \lambda_{n+1}^{i_1,\dots,i_{n+1}}(t) = \lambda_{i_{n+1}}(t)\cdot\lambda_n^{i_1,\dots,i_n}(\beta_{i_{n+1}}(t)),$$

$$i,i_n = 1,\dots,m,\ n = 0,1,\dots,\ t\in I,$$

we have for $g \ C(I,R_+)$ and $n = 1,2,\dots,$

$$(L^n g)(t) = \sum_{i_1=1}^{m}\dots\sum_{i_n=1}^{m} \lambda_n^{i_1,\dots,i_n}(t) g\left(\beta_n^{i_1,\dots,i_n}(t)\right) ,$$

where $L^n \overset{df}{=} L \circ L^{n-1}$, $n = 1,2,\dots$, $L^0 = J$ - operator identity.

Put

$$Mg = \sum_{n=0}^{\infty} L^n g$$

with pointwise convergence of the series in I.

In further considerations we need the following lemma.

L e m m a 1. If $h, K_j \in C(I,R_+)$, $\alpha_j, \beta_i \in C(I,I)$ $\alpha_j(t), \beta_i(t) \in [0,t]$, $t \in I$, $j = 1,\dots,r$, $i = 1,\dots,m$, and

$$m = Mh < +\infty ,$$

$$\bar{m} = Mk < +\infty ,$$

where $k(t) = \sum_{j=1}^{r} K_j(t)\alpha_j(t)$, and if $m, \bar{m} \in C(I,R_+)$, $\sup_{t\in I} \bar{m}(t)/t < +\infty$, $t \in I$, then

(a) there exists $g_0 \in C(I,R_+)$ being the unique solution of the equation

$$g = MKg + Mh,$$

in the class of bounded, non-negative and measurable functions defined in I. (This class will be denoted by $M(I,R_+)$),

(b) the function g_0 is the unique solution of the equation

$$g = Kg + Lg + h$$

in the class $M(I,R_+,g_0) \overset{df}{=} \{g : g \in M(I,R_+),\ \|g\|_0 < +\infty\}$, where $\|g\|_0 \overset{df}{=} \inf\{c : 0 < g \leqslant c < g_0,\ c \in R_+\}$,

(c) the function $g = 0$ is the unique solution of the inequality

$$g \leqslant Kg + Lg$$

in the class $M(I,R_+,g_0)$.

The proof of this Lemma appears in the paper [6].

In this paragraph we shall make the following assumption.

A s s u m p t i o n H_1. 1. There exist functions $\lambda_i, \gamma_j, l_j, h_j, H_0 \in C(I,R_+)$, $i = 1,\dots,m$, $j = 1,\dots,r$, such that

(a) $$\|F(t,u_1,\dots,u_r,v_1,\dots,v_m) - F(t,u_1,\dots,u_r,\bar{v}_1,\dots,\bar{v}_m)\| \leqslant \leqslant \sum_{i=1}^{m} \lambda_i(t)\,\|v_i - \bar{v}_i\|,$$

(b) $$\|F(t,u_1,\dots,u_r,v_1,\dots,v_m)\| \leqslant \sum_{j=1}^{r} \gamma_j(t)\,\|u_j\| + + \sum_{i=1}^{m} \lambda_i(t)\,\|v_i\| + H_0(t),$$

(c) $\|f_j(t,s,u)\| < l_j(t)\,\|u\| + h_j(t), \quad j = 1,\dots,r,$

for $t \in I$, $\|u\|, \|v_i\|, \|\bar{v}_i\| < R_0 \overset{df}{=} \max_{t\in I} g_0(t)$, $i = 1,\dots,m$, (where $g_0(t)$ - defined in Lemma 1),

$$\|u_j\| \le R_j \overset{df}{=} a\left[R_0 \max_{t\in I} l_j(t) + \max_{t\in I} h_j(t)\right], \quad j = 1,\dots,r.$$

2. The assumptions of Lemma 1 are fulfilled for $h, K_j \in C(I,R_+)$ defined by $h(t) = H_0(t) + \sum_{j=1}^{r} \gamma_j(t)\alpha_j(t)h_j(t)$ $K_j(t) = \gamma_j(t)l_j(t)$ $j = 1,\dots,r$, $t \in I$.

Let $W \overset{df}{=} [y : y \in C(I,R^n), \|y(t)\| < g_0(t), t \in I]$ with g_0 defined in Lemma 1.

We can now prove the first result.

L e m m a 2. If Assumption H_1 is fulfilled, then for any $y \in W$ there exists the unique $x(\cdot,y) \in W$ being a solution of the equation

$$x(t) = F\left(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, x(\beta_1(t)), \dots, x(\beta_m(t))\right), \quad t \in I. \tag{2}$$

P r o o f. We define the sequences $\{x_n\}$, $\{g_n^*\}$ putting:

$$x_{n+1}(t) = F\left(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, x_n(\beta_1(t)), \dots, x_n(\beta_m(t))\right),$$

$x_0(t) = 0$, $n = 0,1,\dots$, $t \in I$,
and

$$g_{n+1}^*(t) = (Lg_n^*)(t)$$

$$g_0^*(t) = g_0(t), \; n = 0,1,\dots, \; t \in I,$$

with g_0 defined in Lemma 1.

We prove by induction that

(a) $$\|x_n(t)\| \leqslant g_0(t), \quad n = 0,1,\dots, \quad t \in I.$$

We have $\|x_0(t)\| = 0 \leqslant g_0(t)$ and if we suppose that $\|x_n(t)\| \leqslant g_0(t)$ then

$$\|x_{n+1}(t)\| = \left\| F\left(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, x_n(\beta_1(t)), \dots, x_n(\beta_m(t))\right)\right\|$$

$$\leqslant \sum_{j=1}^{r} \gamma_j(t) \left\| \int_0^{\alpha_j(t)} f_j(t,s,y(s))\,ds \right\| + \sum_{i=1}^{m} \lambda_i(t) \left\| x_n(\beta_i(t)) \right\| + H_0(t) \leqslant$$

$$\leqslant \sum_{j=1}^{r} \gamma_j(t) l_j(t) \int_0^{\alpha_j(t)} \|y(s)\|\,ds + \sum_{j=1}^{r} \gamma_j(t)\, h_j(t)\, \alpha_j(t) + \sum_{i=1}^{m} \lambda_i(t) \left\| x_n(\beta_i(t)) \right\| +$$

$$+ H_0(t) \leqslant \sum_{j=1}^{r} K_j(t) \int_0^{\alpha_j(t)} g_0(s)\,ds + \sum_{i=1}^{m} \lambda_i(t)\, g_0(\beta_i(t)) + h(t) = g_0(t), \quad t \in I.$$

From here and by induction we have (a).

From the definition of the sequence $\{g_n^*\}$ we have $g_n^*(t) = (L^n g_0)(t)$.

It follows that $g_n^*(t) \rightrightarrows 0$ as $n \to \infty$, $t \in I$ (see [6]), where $\rightrightarrows$ denotes the uniform convergence in I.

Further by induction we get $\|x_{n+p}(t) - x_n(t)\| \leqslant g_n^*(t)$, $t \in I$, $p,n = 0,1,2,\dots,$.

The proof this fact is very simple because we have $\|x_p(t)\| \leqslant g_0(t) = g_0(t)$, $p = 0,1,2,\dots,$.

Suppose that $\|x_{n+p}(t) - x_n(t)\| \leq g_n^*(t)$, then

$$\|x_{n+p+1}(t) - x_{n+1}(t)\| = \Big\| F\Big(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, x_{n+p}(\beta_1(t)) \dots, x_{n+p}(\beta_m(t))\Big) -$$

$$- F\Big(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, < x_n(\beta_1(t)), \dots, x_n(\beta_m(t))\Big)\Big\| \leq$$

$$\leq \sum_{i=1}^{m} \lambda_i(t) \,\|x_{n+p}(\beta_i(t)) - x_n(\beta_i(t))\| \leq \sum_{i=1}^{m} \lambda_i(t)\, g_n^*(\beta_i(t)) = g_{n+1}^*(t)\ .$$

From this estimation and according to $g_n^*(t) \rightrightarrows 0$, $t \in I$, we have that the sequence $\{x_n\}$ converges in I uniformly to $\bar{x}$ being the continuous solution of equation (2) in the subset W.

We must prove that the solution $\bar{x}$ of (2) is unique. Suppose that $\bar{x}, \bar{\bar{x}}$ are solutions of (2) and $\bar{x}, \bar{\bar{x}} \in W$ we have then

$$\|\bar{x}(t) - \bar{\bar{x}}(t)\| = \Big\| F\Big(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, \bar{x}(\beta_1(t)), \dots, \bar{x}(\beta_m(t))\Big) -$$

$$- F\Big(t, \int_0^{\alpha_1(t)} f_1(t,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y(s))\,ds, \bar{\bar{x}}(\beta_1(t)), \dots, \bar{\bar{x}}(\beta_m(t))\Big)\Big\| \leq$$

$$\leq \sum_{i=1}^{m} \lambda_i(t) \|\bar{x}(\beta_i(t)) - \bar{\bar{x}}(\beta_i(t))\|, \quad t \in I.$$

Put $u(t) = \|\bar{x}(t) - \bar{\bar{x}}(t)\|$ we have $u(t) \leq (Lu)(t)$. This implies that $u(t) \leq (L^n u)(t)$. On the other hand since $\bar{x}, \bar{\bar{x}} \in W$, then $u(t) \leq 2\,(L^n g_0)(t) \rightrightarrows 0$ for $n \to \infty$, $t \in I$.

Finally $u(t) \equiv 0$ and we conclude $\|\bar{x}(t) - \bar{\bar{x}}(t)\| \equiv 0$. Thus the proof of Lemma is complete.

Consider the operator $\mathcal{U}: W \to W$

$$\mathcal{U}y \stackrel{df}{=} x(\cdot, y),$$

where $x(\cdot, y)$ is the unique solution of equation (2) for a given $y \in W$.

Under some additional conditions we shall prove that the operator $\mathcal{U}$ has a fixed point in the subset W.

Let $\omega_0, \omega_j, \bar{\omega}_j, \bar{\bar{\omega}}_j, \tilde{\omega}_j \in C(R_+, R_+)$, $j = 1, \ldots, r$ be subadditive and non-decreasing functions and $\omega_0(0) = \omega_j(0) = \bar{\omega}_j(0) = \bar{\bar{\omega}}_j(0) = \tilde{\omega}_j(0) = 0$, $j = 1, \ldots, r$, such that

1. $\|F(t, u_1, \ldots, u_r, v_1, \ldots, v_m) - F(t', u'_1, \ldots, u'_r, v_1, \ldots, v_m)\| \leqslant$

$$\leqslant \omega_0(|t - t'|) + \sum_{j=1}^{r} \omega_j(\|u_j - u'_j\|)$$

for $t, t' \in I$, $\|v\| \leqslant R_0$, $\|u_j\|$, $\|u'_j\| \leqslant R_j$, $j = 1, \ldots, r$,

2. $\|f_j(t, s, v) - f_j(t', s, v')\| \leqslant \bar{\omega}_j(|t - t'|) + \bar{\bar{\omega}}_j(\|v - v'\|)$, $j = 1, \ldots, r$,

for $t, t' \in I$, $0 \leqslant s \leqslant \min(t, t')$, $\|v\|$, $\|v'\| \leqslant R_0$,

3. $|\alpha_j(t) - \alpha_j(t')| \leqslant \tilde{\omega}_j(|t - t'|)$, $j = 1, \ldots, r$, $t, t' \in I$.

Note that such functions always exist, in fact, they are the moduli of continuity of suitable functions.

A s s u m p t i o n H_2. Assume that the following series are convergent

1. $m_1(t, \delta_1, \ldots, \delta_r) \stackrel{df}{=} (Mp)(t, \delta_1, \ldots, \delta_r) < +\infty$,

where $p(t, \delta_1, \ldots, \delta_r) = \sum_{j=1}^{r} \omega_j(\delta_j \alpha_j(t))$, $\delta_j \in R_+$, $j = 1, \ldots, r$,

2. $$m_2(t_1,t_2) \overset{df}{=} \sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m} \lambda_k^{i_1,\dots,i_k}(t_2)\,\omega\left(\left|\beta_k^{i_1,\dots,i_k}(t_1)-\beta_k^{i_1,\dots,i_k}(t_2)\right|\right) < +\infty,$$

where $t_1, t_2 \in I$, $\omega(s) = \omega_0(s) + \sum_{j=1}^{r} \omega_j(s\,\overline{\omega}_j(s) + R_j\,\widetilde{\omega}_j(s))$.

3. The functions m_1, m_2 are continuous.

L e m m a 3. If Assumption H_1 and condition 1 of Assumption H_2 are fulfilled, then the operator U is continuous in W.

P r o o f. Let $y_1, y_2 \in W$ and $x_1 \overset{df}{=} x(\cdot, y_1) = Uy_1$, $x_2 \overset{df}{=} x(\cdot, y_2) = Uy_2$ are solutions of equation (2).

Put $u(t) = \|x_1(t) - x_2(t)\|$, $\delta_j = \overline{\omega}_j(\max_{t\in I} \|y_1(t) - y_2(t)\|)$, $j = 1,\dots,r$, by our assumptions we get

$$u(t) = \|x_1(t) - x_2(t)\| = \Big\| F\Big(t, \int_0^{\alpha_1(t)} f_1(t,s,y_1(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y_1(s))\,ds, x_1(\beta_1(t)), \dots, x_1(\beta_m(t))\Big) -$$

$$- F\Big(t, \int_0^{\alpha_1(t)} f_1(t,s,y_2(s))\,ds, \dots, \int_0^{\alpha_r(t)} f_r(t,s,y_2(s))\,ds, x_2(\beta_1(t)), \dots, x_2(\beta_m(t))\Big)\Big\| \leq$$

$$\leq \sum_{j=1}^{r} \omega_j\left(\Big\|\int_0^{\alpha_j(t)} \left[f_j(t,s,y_1(s)) - f_j(t,s,y_2(s))\right] ds\Big\|\right) + \sum_{i=1}^{m} \lambda_i(t)\,\|x_1(\beta_i(t)) - x_2(\beta_i(t))\| \leq$$

$$\leq \sum_{j=1}^{r} \omega_j(\delta_j\,\alpha_j(t)) + \sum_{i=1}^{m} \lambda_i(t)\,u(\beta_i(t)).$$

Finally, we get

$$u(t) \leqslant p(t,\delta_1,\dots,\delta_r) + (Lu)(t),$$

$$u(t) \leqslant \sum_{k=0}^{n-1} (L^k p)(t,\delta_1,\dots,\delta_r) + (L^n u)(t).$$

But $u(t) \leqslant 2\, g_0(t)$, consequently we have

$$u(t) \leqslant \sum_{k=0}^{n-1} (L^k p)(t,\delta_1,\dots,\delta_r) + 2(L^n\, g_0)(t).$$

Letting $n \to \infty$ and from $(L^n\, g_0)(t) \rightrightarrows 0$ for $n \to \infty$ we obtain $u(t) \leqslant Mp(t,\delta_1,\dots,\delta_r) = m_1(t,\delta_1,\dots,\delta_r)$. By the continuity of the function m_1 we conclude the assertion of Lemma 3.

L e m m a 4. If Assumptions H_1, H_2, are fulfilled and

$$(3)\quad \sum_{i_1=1}^{m} \dots \sum_{i_n=1}^{m} \lambda_n^{i_1,\dots,i_n}(t_2) g_0\left(\beta_n^{i_1,\dots,i_n}(t_1)\right) \rightrightarrows 0 \quad \text{as} \quad n \to \infty,\ t_1, t_2 \in I$$

then the set $U(W)$ is compact.

P r o o f. The set $U(W)$ is a subset of $C(I,R^n)$, hence this Lemma is an easy consequence of the Ascoli-Arzela theorem. Indeed with the following calculation we have:

Let $y \in W$ and $x = Uy$

$$\|x(t_1,y) - x(t_2,y)\| = \Big\| F\Big(t_1, \int_0^{\alpha_1(t_1)} f_1(t_1,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t_1)} f_r(t_1,s,y(s))\,ds, x(\beta_1(t_1)), \dots, x(\beta_m(t_1)\Big) -$$

$$- F\Big(t_2, \int_0^{\alpha_1(t_2)} f_1(t_2,s,y(s))\,ds, \dots, \int_0^{\alpha_r(t_2)} f_r(t_1,s,y(s))\,ds, x(\beta_1(t_2), \dots, x(\beta_m(t_2))\Big)\Big\| \leqslant$$

$$<\omega_0(|t_1-t_2|)+\sum_{j=1}^{r}\omega_j\left(\left\|\int_0^{\alpha_j(t_1)} f_j(t_1,s,y(s))\,ds-\int_0^{\alpha_j(t_2)} f_j(t_2,s,y(s))\,ds\right\|\right)+$$

$$+\sum_{i=1}^{m}\lambda_i(t_2)\|x(\beta_i(t_1),y)-x(\beta_i(t_2),y)\|<\omega_0(|t_1-t_2|)+$$

$$+\sum_{j=1}^{r}\omega_j\left[a\cdot\bar{\omega}_j\cdot(|t_1-t_2|)+R_j\bar{\omega}_j(|t_1-t_2|)\right]+\sum_{i=1}^{m}\lambda_i(t_2)\|x(\beta_i(t_1),y)-x(\beta_i(t_2),y)\|\leqslant$$

$$<\omega\,|t_1-t_2|+\sum_{i=1}^{m}\lambda_i(t_2)\,\|x(\beta_i(t_1),\ )-x(\beta_i(t_2),\ )\|.$$

Let $\nu(t_1,t_2)=\|x(t_1,y)-x(t_2,y)\|$, we have

$$\nu(t_1,t_2)\leqslant\sum_{i=1}^{m}\lambda_i(t_2)\,\nu(\beta_i(t_1),\beta_i(t_2))+\omega(|t_1-t_2|)$$

and consequently

$$\nu(t_1,t_2)\leqslant\sum_{k=0}^{n-1}\sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m}\lambda_k^{i_1\ldots i_k}(t_2)\,\omega\left(\left|\beta_k^{i_1,\ldots,i_k}(t_1)-\beta_k^{i_1,\ldots,i_k}(t_2)\right|\right)+$$

$$+\sum_{i_1=1}^{m}\cdots\sum_{i_n=1}^{m}\lambda_n^{i_1,\ldots,i_n}(t_2)\,\nu\left(\beta_n^{i_1,\ldots,i_n}(t_1),\ \beta_n^{i_1,\ldots,i_n}(t_2)\right),$$

for $n=0,1,\ldots,\ t_1,t_2\in I$.

But $\nu(t_1,t_2) \leqslant g_0(t_1) + g_0(t_2)$, thus

$$\sum_{i_1=1}^{m}\cdots\sum_{i_n=1}^{m}\lambda_n^{i_1,\dots,i_n}(t_2)\,\nu\left(\beta_n^{i_1,\dots,i_n}(t_1),\beta_n^{i_1,\dots,i_n}(t_2)\right)\leqslant$$

$$\leqslant\sum_{i_1=1}^{m}\cdots\sum_{i_n=1}^{m}\lambda_n^{i_1,\dots,i_n}(t_2)\,g_0\left(\beta_n^{i_1,\dots,i_n}(t_1)\right)+\sum_{i_1=1}^{m}\cdots\sum_{i_n=1}^{m}\lambda_n^{i_1,\dots,i_n}(t_2)\,g_0\left(\beta_n^{i_1,\dots,i_n}(t_2)\right).$$

By assumptions of the Lemma we get

$$(L^n g_0)(t_2) \rightrightarrows 0 \quad \text{for} \quad n\to\infty .$$

Finally, letting $n\to\infty$ by (3) we obtain the estimation

$$\nu(t_1,t_2)\leqslant\sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m}\lambda_k^{i_1,\dots,i_k}(t_2)\,\omega\left(\left|\beta_k^{i_1,\dots,i_k}(t_1)-\beta_k^{i_1,\dots,i_k}(t_2)\right|\right)=m_2(t_1,t_2).$$

By the continuity of the function m_2 we infer the equicontinuity of all $x(\cdot,y)$ and now by the Ascoli-Arzela theorem we arrive at the assertion of the lemma.

4. Theorem on the existence

T h e o r e m 1. If Assumptions H_1, H_2 are fulfilled and relation (3) holds, then equation (1) has at least one solution $\overline{x}\in W$.

P r o o f. In view of Lemmas 2,3,4 and the Schauder fixed-point theorem the assertion of the theorem is obvious. In fact, we see that the continuous operator U maps the bounded, closed and convex set $W\in C(I,R^n)$ into its compact subset $U(W)$, thus it has at least one fixed point.

This fixed point of the operator U is the solution of the equation (1).

Now we can give some effective conditions under which the assertion of Theorem 1 holds true.

T h e o r e m 2. If condition 1 of Assumption H_1 is satisfed and if

$$K_j(t) \leq \bar{K}_j = \text{const}, \quad j = 1,\dots,r, \quad t \in I,$$

$$\alpha_j(t) \leq \bar{\alpha}_j \cdot t, \ \beta_i(t) \leq \bar{\beta}_i \cdot t, \ \bar{\alpha}_j, \bar{\beta}_i \in [0,1], \quad j = 1,\dots,r,$$

$$i = 1,\dots,m$$

$$h(t) \leq H\, t^p \quad \text{for} \quad p \geq 0, \ H \geq 0,$$

$$\lambda_i(t) \leq \bar{\lambda}_i = \text{const}, \quad i = 1,\dots,m$$

$$\omega_0(s) = \Omega_0 \cdot s^{\lambda}, \ \omega_j(s) = \Omega_j \cdot s^{q_j}, \ q_j = \text{const}, \quad j = 1,\dots,r,$$

$$\Omega_0, \ \Omega_j \in R_+, \quad j = 1,\dots,r,$$

$$\bar{\omega}_j(s) = \Omega_j \cdot s^{\nu_j}, \ \nu_j = \text{const}, \quad \tilde{\omega}_j(s) = \Omega_j \cdot s^{\mu_j},$$

$$\mu_j = \text{const}, \quad j = 1,\dots,r, \ \Omega_j, \ \Omega_j \in R_+,$$

$$\varrho_j \stackrel{df}{=} \min\,(p, q_j, \lambda, \mu_j \cdot q_j, \nu_j \cdot q_j), \quad j = 1,\dots,r,$$

and

$$(4) \qquad \sum_{i=1}^{m} \bar{\lambda}_i \bar{\beta}_i^{\varrho_j} < 1, \quad j = 1,\dots,r,$$

then equation (1) has at least one solution $x \in W$.

P r o o f. To prove this theorem it remains to observe that under conditions assumed Assumptions H_2, condition 2 of H_1 and relation (3) are fulfilled.

The results form the following estimations:

$$m_1(t,\delta_1,\dots,\delta_r) = \sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\dots\sum_{i_k=1}^{m}\lambda_k^{i_1,\dots,i_k}(t)\sum_{j=1}^{r}\omega_j\left(\delta_j\alpha_j\left(\beta_k^{i_1,\dots,i_k}(t)\right)\right)\leqslant$$

$$\leqslant\sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\dots\sum_{i_k=1}^{m}\prod_{l=1}^{k}\lambda_{i_l}\sum_{j=1}^{r}\omega_j\left(\delta_j\cdot\bar{\alpha}_j\cdot\prod_{l=1}^{k}\bar{\beta}_{i_l}\cdot t\right) =$$

$$=\sum_{k=0}^{\infty}\sum_{j=1}^{r}\Omega_j(\bar{\alpha}_j\cdot\delta_j\cdot t)^{q_j}\cdot\left(\sum_{i_1=1}^{m}\dots\sum_{i_k=1}^{m}\prod_{l=1}^{k}\bar{\lambda}_{i_l}\cdot\bar{\beta}_{i_l}\right)^{q_j} =$$

$$=\sum_{k=0}^{\infty}\sum_{j=1}^{r}\left(\sum_{i=1}^{m}\bar{\lambda}_i\cdot\bar{\beta}_i^{q_j}\right)^{k}\cdot\left(\bar{\alpha}_j\cdot\delta_j\cdot t\right)^{q_j}.\Omega_j < +\infty$$

and

$$m_2(t_1,t_2) = \sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\dots\sum_{i_k=1}^{m}\lambda_k^{i_1,\dots,i_k}(t_2)\,\omega\left(\left|\beta_k^{i_1,\dots,i_k}(t_1)-{}_k^{i_1,\dots,i_k}(t_2)\right|\right)\leqslant$$

$$<\sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\dots\sum_{i_k=1}^{m}\prod_{l=1}^{k}\bar{\lambda}_{i_l}\left[\omega\left(\prod_{l=1}^{k}\bar{\beta}_{i_l}\cdot t_1\right)+\omega\left(\prod_{l=1}^{k}\bar{\beta}_{i_l}\cdot t_2\right)\right]\leqslant$$

$$\leqslant\sum_{k=0}^{\infty}\sum_{i_1=1}^{m}\dots\sum_{i_k=1}^{m}\prod_{l=1}^{k}\bar{\lambda}_{i_l}\cdot 2\omega\left(\prod_{l=1}^{k}\bar{\beta}_{i_l}\cdot t\right),$$

where $t = \max(t_1,t_2)$.

Finally, we get

$$m_2(t_1,t_2) \leq 2 \sum_{k=0}^{\infty} \sum_{j=1}^{r} \left(\sum_{i=1}^{m} \bar{\lambda}_i \, \bar{\beta}_i^{q_j} \right)^k \times$$

$$\times \left[t^{\lambda} \Omega_o + \Omega_j \cdot a^{\nu_j \cdot q_j} \cdot t^{\nu_j \cdot q_j} \tilde{\Omega}_j + (R_j \tilde{\Omega}_j \, t)^{\mu_j \cdot q_j} \right] < +\infty .$$

It is clear that m_1 and m_2 are continuous. Relation (3) of Lemma 4 is implied by

$$\sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \lambda_n^{i_1,\dots,i_n}(t_2) \, g_o\left(\beta_n^{i_1,\dots,i_n}(t_1)\right) \leq c \cdot H \cdot t_1^{\,p} \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \left(\prod_{l=1}^{n} \bar{\lambda}_{i_l} \cdot \bar{\beta}_{i_l}^{p} \right) =$$

$$= c \cdot H \cdot t_1^{p} \left(\sum_{i=1}^{m} \bar{\lambda}_i \, \bar{\beta}_i^{p} \right)^n \rightrightarrows 0 \quad \text{as} \quad n \to \infty .$$

Now we find that Assumptions H_1, H_2 and relation (3) are fulfilled if (4) holds.

5. <u>Some remarks</u>

The following initial problem for functional differential equation of neutral type of the form

$$x^{(n)}(t) = f\left(t, x(\alpha_1(t)), x'(\alpha_2(t)), \dots, x^{(n-1)}(\alpha_n(t)), x^{(n)}(\beta_1(t)) \dots, x^{(n)}(\beta_m(t))\right), \tag{5}$$

for $t \in I$ and

$$x(0) = x'(0) = \dots = x^{(n-1)}(0) = 0, \tag{6}$$

(the general initial condition $x(t_o) = c_o$, $x'(t_o) = c_1, \dots, x^{(n-1)}(t_o) = c_{n-1}$ by the substitution $y(t - t_o) =$

$= x(t_0) - \sum_{j=0}^{n-1} \frac{c_j}{j!}(t - t_0)^j$ reduces to the condition considered here), can be reduced to the integral-functional equation of the form (1).

If we want to investigate whether there exists $x \in C^n([0,a])$ ($C^n([0,a])$ consider the Banach space of n times continuously differentiable functions x from $[0,a]$ to R) which satisfies equation (5) with conditions (6). We can reduce by the substitution $x^{(n)}(t) = y(t), \ldots, x(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds$ the initial problem (5) - (6) to the equivalent equation

$$(7) \qquad y(t) = F\left(t, \int_0^{\alpha_1(t)} \frac{(\alpha_1(t)-s)^{n-1}}{(n-1)!} y(s)ds, \ldots, \int_0^{\alpha_n(t)} y(s)ds, y(\beta_1(t)), \ldots, y(\beta_n(t))\right).$$

Now it easy to formulate the sufficient conditions for the existence of solution of the problem (5) - (6).

In this way we can obtain the result more general than that one stated by Nussbaum [8].

This fact can be motivated by the simple functional-differential equation

$$(8) \qquad x'(t) = k\cdot x'(m\cdot t) + g(t),\ 0 \leqslant t < H,\ x(0) = 0.$$

In this equation k and m are constant, $0 < m < 1$ and g is a C^{n-1} function.

The result of Nussbaum can be applied to this equation, it yields the answer: If $|k\, m^{n-1}| < 1$, $n > 2$ and $k\, m^j \neq 1$ for $0 \leqslant j \leqslant n-2$, there exists a unique solution of (8) in $C^n([0,H])$.

The results of this paper used to this equation give the answer: If $|k\, m^n| < 1$, $n \geqslant 2$ and $k\, m^j \neq 1$ for $0 \leqslant j \leqslant n-2$, there exists a unique solution of (8) in $C^n([0,H])$.

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