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Julian Musielak

AN APPLICATION OF MODULAR SPACES TO A NON-LINEAR INTEGRAL EQUATION

1. Let $p(t)$ and $r(t)$ be two positive measurable functions defined in the interval $[t_0, \infty)$, where t_0 is an arbitrary real number, and let

$$P = \sup_{t \geq t_0} p(t) < \infty, \qquad Q = \int_{t_0}^{\infty} r(t)dt < \infty .$$

Moreover, let $\varphi(u)$ be an even, nonnegative, convex function on $(-\infty, \infty)$, $\varphi(u) = 0$ iff $u = 0$. We suppose that φ satisfies the condition (Δ_2) for small u, i.e. there exist numbers $\beta, u_0 > 0$ such that $\varphi(2u) < \beta\varphi(u)$ for $0 < u < u_0$. We shall investigate the following integral equation

$$(1) \qquad x(t) = a \int_{t_0}^{t} p(t)r(s)\varphi(x(s))ds + x_0(t),$$

where $x_0(t)$ is a given measurable function. It is easily seen that supposing $p(t)$ to be differentiable a.e. and $y_0(s)/p(s)$ to be locally integrable in $[t_0, \infty)$, the equation (1) is equivalent to the differential equation

$$(2) \quad x'(t) = \frac{p'(t)}{p(t)} x(t) + a\, p(t)r(t)\varphi(x(t)) + y_0(t) \text{ a.e. in } [t_0, \infty),$$

where

$$x_0(t) = p(t) \int_{t_0}^{t} \frac{y_0(s)}{p(s)} ds ,$$

subject to the initial condition $y(t_0) = 0$.

We shall seek bounded solutions of the equation (1) belonging to a space X_{ϱ_0} which will be defined by means of the equation (1) itself, applying the general theory of modular spaces depending on a parameter (see [3]).

2. First, we shall give the necessary notions and results from the theory of modular spaces. Let (Ω, Σ, μ) be a measure space, where Σ is a σ-algebra of subsets of a nonempty set Ω and μ is a finite measure in Σ. Let X be the space of all extended real-valued functions in Ω, Σ-measurable and finite μ-a.e.; equality in X will mean equality μ-a.e. We assume that ϱ is a map of $\Omega \times X$ into $[0, \infty]$ satisfying the following conditions:

1° $\varrho(t,0) = 0$, $\varrho(t,x) = 0$ in Ω implies $x = 0$,
$\varrho(t,-x) = \varrho(t,x)$ for $t \in \Omega$, $x \in X$,
$\varrho(t, \alpha x + \beta y) < \alpha\varrho(t,x) + \beta\varrho(t,y)$ in Ω for $x,y \in X$,
$\alpha, \beta \geq 0$, $\alpha + \beta = 1$,
if $x,y \in X$ and $|x(t)| \leq |y(t)|$ a.e. in Ω, then
$\varrho(t,x) \leq \varrho(t,y) <$ in Ω;

2° $\varrho(t,x)$ is a Σ-measurable function of $t \in \Omega$ for all $x \in X$.

Then ϱ is called a family of convex modulars depending on the parameter t. By means of ϱ, one may define various modulars in X, as e.g.

$$\varrho_s(x) = \int_\Omega \varrho(t,x) d\mu \quad \text{and} \quad \varrho_0(x) = \sup_{t \in \Omega} \varrho(t,x) \text{ (see [3]).}$$

In [1], the modular ϱ_s was applied to solve the general modular equation. Here, we shall apply ϱ_0 which seems to be more suitable to our case. ϱ_0 is a convex modular in X, i.e. $\varrho_0(x)\geqslant 0$, $\varrho_0(x) = 0$ iff $x = 0$, $\varrho_0(\alpha x+\beta y) \leqslant \alpha\varrho_0(x) + \beta\varrho_0(y)$ for $x,y \in X$, $\alpha,\beta \geqslant 0$, $\alpha+\beta = 1$. The linear space

$$X_{\varrho_0} = \{x : \varrho_0(\lambda x) \to 0 \text{ as } \lambda \to 0,\ x \in X\}$$

is called a modular space, and

$$\|x\| = \inf\{u>0 : \varrho_0(x/u) \leqslant 1\}$$

is a norm in X_{ϱ_0}. An element $x\in X$ belongs to X_{ϱ_0} iff $\varrho(t,\lambda x)\to 0$ as $\lambda\to 0$ uniformly with respect to $t\in\Omega$. A sequence of $x_n \in X_{\varrho_0}$ tends to zero in X_{ϱ_0} iff for any $\lambda>0$, $\varrho(t, \lambda x_n)\to 0$ as $n\to\infty$ uniformly in Ω; the sequence (x_n) is a Cauchy sequence in X_{ϱ_0} iff for any $\lambda>0$, $\varrho(t,\lambda(x_n - x_m))\to 0$ as $m,n\to\infty$ uniformly in Ω.

Let for any $M>0$, $X^M_{\varrho_0}$ denote the set of functions $x\in X_{\varrho_0}$ such that $|x(t)|\leqslant M$ a.e. in Ω. We consider the operator

$$[T(x)](t) = a\,\varrho(t,x) + x_0(t) \tag{3}$$

for $x\in X^M_{\varrho_0}$, where $x_0 \in X^M_{\varrho_0}$. The following result holds.

T h e o r e m 1. Let us suppose that

$$|a|\,\varrho(t,M) + |x_0(t)| \leqslant M \quad \text{a.e. in } \Omega,$$

and let for every $x\in X^M_{\varrho_0}$ the following condition be satisfied: for every $\lambda_1>0$ there exist positive numbers C and λ_2 such that

(4) $\varrho[t, \lambda_2\varrho(\cdot,x)] \leqslant C \sup_{s\in\Omega} \varrho(s, \lambda_1 x)$ for $t\in\Omega$.

Moreover, let us suppose that there exists a number $\alpha > 0$ such that for any $\eta > 0$ and for all $x,y\in X^M_{\varrho_0}$, there holds the inequality

(5) $\varrho\left[t, \frac{\varrho(\cdot,x) - \varrho(\cdot,y)}{\eta}\right] < \sup_{s\in\Omega} \varrho\left(s, \frac{\alpha}{a\eta}(x - y)\right)$ for $t\in\Omega$.

Then the operator T maps $X^M_{\varrho_0}$ into itself and

$$\|T(x) - T(y)\| < \alpha\|x - y\| \quad \text{for all} \quad x,y\in X^M_{\varrho_0}.$$

This theorem follows from the definition of the norm in X_{ϱ_0}, immediately where it is sufficient to take X_{ϱ_0} instead of X_{ϱ_s} in [1].

3. Now, let ϱ be defined by means of a nonlinear integral operator

(6) $$\varrho(t,x) = \int_\Omega k(t,s,x(s))\, d\mu(s),$$

where $k : \Omega\times\Omega\times(-\infty,\infty)\to[0,\infty)$ is a measurable function, $k(t,s,0) = 0$ and $k(t,s,u) > 0$ for $u > 0$ in $\Omega\times\Omega$, $k(t.s.u)$ is an even, continuous, convex function of u for all $(t,s)\in\Omega\times\Omega$ (see [1], formula (7)). Let

(7)
$$k_1(t,u,v) = \int_\Omega k[t,s,k(s,u,v)]\, d\mu(s),$$
$$\varrho_1(t,x) = \int_\Omega k_1(t,s,x(s))d\mu(s).$$

Then the following result holds.

T h e o r e m 2. Let

$$(8) \qquad |a|\int_\Omega k(t,s,M)d\mu(s) + |x_0(t)| < M \quad \text{a.e. in } \Omega,$$

where $M>0$. Let us suppose that the following assumptions are satisfied:

(a) for every $x \in X^M_{\varrho_0}$ and an arbitrary $\lambda_1 > 0$ there exists a number $C>0$ such that

$$\varrho_1(t,x) < C \sup_{s\in\Omega} \varrho(s,\lambda_1 x) \quad \text{in } \Omega,$$

(b) for every $x,y \in X^M_{\varrho_0}$ and each $\eta > 0$ there holds the inequality

$$\int_\Omega\left\{\frac{1}{\mu(\Omega)}\int_\Omega k\left[t,u,\frac{\mu(\Omega)}{\eta}\Big(k(u,v,x(v)) - k(u,v,y(v))\Big)\right] d\mu(v)\right\}d\mu(u) \leqslant$$
$$\leqslant \sup_{s\in\Omega}\int_\Omega k\left[s,v,\frac{\alpha}{a\eta}(x(v)-y(v))\right]d\mu(v)$$

for all $t \in \Omega$.
Then T defined by (3) maps $X^M_{\varrho_0}$ into itself and
$\|T(x) - T(y)\| \leqslant \alpha\|x-y\|$ for $x,y \in X^M_{\varrho_0}$.

This result is obtained applying Jensen's inequality for convex functions and thus showing that (a) and (b) imply the assumptions (4) and (5) of Theorem 1 to be satisfied.

4. Let us remark that taking $0<\alpha<1$ in Theorem 2, T becomes a contraction operator in $X^M_{\varrho_0}$. Now, since convergence in X_{ϱ_0} implies convergence in measure, so $X^M_{\varrho_0}$ is a closed subset of X_{ϱ_0}. Hence, if we prove X_{ϱ_0} to be complete, we may apply the Banach fix-point principle to the equation (1).

T h e o r e m 3. If ϱ is given by (6), then the space X_{ϱ_0} is complete.

P r o o f. Let (x_n) be a Cauchy sequence in X_{ϱ_0}, i.e. $\varrho(t, \lambda(x_n - x_m)) \to 0$ as $m,n \to \infty$ uniformly in Ω, for any $\lambda > 0$. Let us fix $t \in \Omega$ and let us write $M(s,u) = k(t,s,u)$, then x_n belong to the generalized Orlicz space L^*_M over Ω and (x_n) is a Cauchy sequence in L^*_M. Since L^*_M is complete (see [2], 2.31), so $x_n(\cdot) \to x(t,\cdot)$ as $n \to \infty$ in L^*_M, $x(t,\cdot) \in L^*_M$. Consequently, $x_n(\cdot) \to x(t,\cdot)$ as $n \to \infty$ in μ-measure; but this shows that $x(t,\cdot)$ is independent of t, and we may write $x(t,s) = x(s)$ for $t,s \in \Omega$. Now, we extract a subsequence $x_{n_i}(s) \to x(s)$ μ-a.e. in Ω. Applying Fatou lemma to the sequence of functions $k[t,s, \lambda(x_{n_i}(s) - x_m(s))]$, $m = 1,2,\ldots,$ we obtain

$$\varrho\left[t, \lambda(x_{n_i} - x)\right] \leqslant \varliminf_{m \to \infty} \varrho_0 \left[\lambda(x_{n_i} - x_m)\right].$$

But the right-hand side of the last inequality is so small as we like for sufficiently large i. Hence $x_{n_i} \to x$ in X_{ϱ_0}. Since (x_n) is a Cauchy sequence, we conclude that $x_n \to x$ in X_{ϱ_0}. It is evident that $x \in X_{\varrho_0}$. Thus, X_{ϱ_0} is complete.

Let us remark that in Theorem 3 it is sufficient to assume that $k(t,s,u) \to \infty$ as $u \to \infty$ for $(t,s) \in \Omega \times \Omega$ in place of convexity of $k(t,s,u)$ in the variable u.

From Theorems 2 and 3 and from Banach's fix-point principle it follows immediately that

T h e o r e m 4. If all the assumptions of Theorem 2 are satisfied with $0 < \alpha < 1$ and ϱ is given by (6), then the equation (1) has exactly one solution in $X^M_{\varrho_0}$.

5. We turn now back to the special case considered in § 1. Let us write $r(t) = q(t)\, w(t)$, where $0 < q(t) < Q$ and $\int_{t_0}^{\infty} w(t)dt = 1$, and let Σ be the σ-algebra of Lebesgue

measurable subsets of $\Omega = [t_0, \infty)$. We define $\mu(A) = \int_A w(s)ds$ for any $A \in \Sigma$, then $\mu(\Omega) = 1$. Finally, let us take

$$k(t,s,u) = \begin{cases} p(t)q(s)\varphi(u) & \text{if } t_0 < s < t \\ 0 & \text{if } t_0 \leqslant t < s. \end{cases}$$

Then formula (6) takes the form

$$\varrho(t,x) = \int_{t_0}^{t} p(t)r(s)\varphi(x(s))ds. \tag{9}$$

We check that ϱ_1 defined by (7) satisfies then the condition (a) of Theorem 2. Namely, we have

$$\varrho_1(t,x) = \int_{t_0}^{t}\left\{\int_{u}^{t} p(t)q(s)w(s)\varphi[p(s)q(u)\varphi(x(u))]ds\right\}w(u)du$$

and applying the inequalities $p(t) < P$, $q(t) < Q$ twice and then the inequality $\int_u^t w(s)ds < 1$, we obtain

$$\varrho_1(t,x) \leqslant \int_{t_0}^{t} p(t)r(u)\varphi[PQ\ \varphi(x(u))]du.$$

Applying the condition (Δ_2) for small u, we get

$$\varphi(x(u)) < b\ \varphi(\lambda_1 x(u)) \quad \text{for} \quad u \geqslant t_0,$$

where b is a positive constant depending on λ_1 and M. Hence

$$\varphi[PQ\ \varphi(x(u))] \leqslant \frac{\varphi[PQ\ \varphi(M)]}{PQ\ \varphi(M)}\ PQ\ \varphi(x(u)) \leqslant C\ \varphi(\lambda_1 x(u)),$$

where

$$c = \frac{b\varphi[PQ\,\varphi(M)]}{\varphi(M)}.$$

This proves (a).

Now, we are able to prove the following

Theorem 5. Let us suppose that the functions $p(t)$, $r(t)$ and $\varphi(u)$ satisfy the assumptions given in § 1 and that

$$S = \operatorname*{supess}_{t\in\Omega} |x_0(t)| < M. \tag{10}$$

Let a satisfy the inequality

$$|a| < \frac{1}{PQ} \min\left(\frac{M-S}{\varphi(M)}, \frac{1}{K}\right), \tag{11}$$

where $K>0$ is the Lipschitz constant of the function φ in the interval $[0,M]$. Then the integral equation (1) has exactly one solution in $X^M_{\varrho_0}$ and this solution is given by the formula $x = \lim_{n\to\infty} x_n$ in X_{ϱ_0}, where $x_n(t) = a\varrho(t,x_{n-1}) + x_0(t)$ for $n = 1,2,\ldots$

Proof. It is sufficient to applv Theorem 4, i.e. to show that the assumptions of Theorem 2 with $0<\alpha<1$ are satisfied. Applying (10) and (11) we see that the condition (8) is satisfied. Since (a) was proved above, it is sufficient to prove (b). Since φ is convex, it satisfies Lipschitz condition in the interval $[0,M]$ with a constant $K>0$. Hence the left-hand side of the inequality in (b) is

$$\int_{t_0}^{\infty}\left\{\int_{t_0}^{\infty} k\left[t,u,\frac{1}{\eta}\,(k(u,v,x(v)) - k(u,v,y(v)))\right] d\mu(v)\right\} d\mu(u) \leqslant$$

$$\leqslant p(t)\int_{t_0}^{t} q(u)w(u)\left\{\int_{t_0}^{u}\left[\frac{K}{\eta}\,p(u)q(v)(x(v) - y(v))\right] w(v)dv\right\} du \leqslant$$

$$< \frac{p(t)}{P} \int_{t_0}^{t} \frac{q(u)}{Q} w(u) \left\{ \sup_{s \geq t_0} \int_{t_0}^{s} p(s) q(v) \varphi \left[\frac{KPQ}{\eta} (x(v)-y(v)) \right] w(v) dv \right\} du <$$

$$\leq \sup_{s \geq t_0} \int_{t_0}^{\infty} k \left[s, v, \frac{\alpha}{a\eta} (x(v)-y(v)) \right] d\mu(v) ,$$

where $\alpha = |a| KPQ < 1$, because of (11). This proves the theorem.

REFERENCES

[1] T.M. J ę d r y k a, J. M u s i e l a k: On a modular equation (I), Funct. Approximatio Comment. Math. 3 (1976) 101-111.

[2] J. M u s i e l a k, W. O r l i c z: On modular spaces, Studia Math. 18 (1959) 49-65.

[3] J. M u s i e l a k, A. W a s z a k: A contribution to the theory of modular spaces, Proceed. of the Third Prague Topological Symposium 1971, Prague (1972) 315-319.

INSTITUTE OF MATHEMATICS, ADAM MICKIEWICZ UNIVERSITY, POZNAŃ

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