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Wiesław Grycak

ON TOTALLY UMBILICAL SURFACES IN SOME RIEMANNIAN MANIFOLDS

1. Preliminaries

Let V_m be an m-dimensional Riemannian manifold immersed in an n-dimensional Riemannian manifold V_n, and let $u^\alpha = u^\alpha(w^i)$ be the parametric representation of the submanifold V_m in V_n, where (u^α) are local coordinates in V_n and (w^i) are local coordinates in V_m. Let $B_i^\alpha = \partial_i u^\alpha$, where $\partial_i = \partial/\partial w^i$.

If $G_{\lambda\omega}$ is the fundamental tensor of the manifold V_n, then g_{ij} defined by $g_{ij} = B_i^\lambda B_j^\omega G_{\lambda\omega}$ is the first fundamental tensor of the submanifold V_m. In the sequel Greek indices take values 1,...,n and Latin indices take values 1,...,m $(m<n)$.

Let $N_x{}^\lambda (x,y,z = m+1,\dots,n)$ be pairwise orthogonal unit normals to V_m. Then we have the relations

$$(1)\qquad G_{\alpha\beta} N_x{}^\alpha N_y{}^\beta = \begin{cases} e_x & (x=y) \\ 0 & (x\neq y) \end{cases}, \qquad G_{\alpha\beta} N_x{}^\alpha B_i{}^\beta = 0,$$

$$(2)\qquad g^{ij} B_i{}^\alpha B_j{}^\beta = G^{\alpha\beta} - \sum_x e_x N_x{}^\alpha N_x{}^\beta,$$

where e_x is the indicator of the vector N_x^α.

The Schouten curvature tensor $H_{ij}{}^\alpha$ of the submanifold V_m is defined by

(3) $$H_{ij}^{\alpha} = \nabla_i B_j^{\alpha},$$

where ∇_i denotes covariant differentiation with respect to the fundamental tensor g_{ij} of V_m.

If we put

(4) $$H_{ij}^{\alpha} = \sum_x e_x H_{ijx} N_x^{\alpha},$$

then the second fundamental tensor H_{ijx} for N_x^{α} is given by

(5) $$H_{ijx} = H_{ij}^{\alpha} N_{x\alpha},$$

where $N_{x\alpha} = N_x^{\omega} G_{\omega\alpha}$.

The Gauss and Codazzi equations for V_m can be written in the form

(6) $$R_{lkji} = \bar{R}_{lkji} + \sum_x e_x (H_{lix} H_{kjx} - H_{ljx} H_{kix})$$

and

(7) $$\bar{R}_{lkjx} = \nabla_l H_{kjx} - \nabla_k H_{ljx} + \sum_y e_y (L_{lyx} H_{kjy} - L_{kyx} H_{ljy})$$

respectively ([3], [2]), where L_{ixy} is the third fundamental tensor with respect to the normals N_x^{α}, given by

(8) $$L_{ixy} = (\nabla_i N_x^{\alpha}) N_{y\alpha} \; (= -L_{iyx}),$$

R_{lkji}, $\overset{*}{R}_{\alpha\beta\gamma\delta}$ are curvature tensor for V_m and V_n, respectively, and

(9) $$\bar{R}_{lkij} = \overset{*}{R}_{\alpha\beta\gamma\delta} B_l^{\alpha} B_k^{\beta} B_i^{\gamma} B_j^{\delta}, \quad \bar{R}_{lkjx} = \overset{*}{R}_{\alpha\beta\gamma\delta} B_l^{\alpha} B_k^{\beta} B_j^{\gamma} N_x^{\delta}.$$

We have also the equations of Weingarten

(10) $$\nabla_i N_x^{\alpha} = -H_i{}^r{}_x B_r^{\alpha} + \sum_y e_y L_{ixy} N_y^{\alpha},$$

where $H_i{}^k{}_x = H_{irx} g^{rk}$.

2. <u>A totally umbilical submanifold</u>

If $H_{ij}{}^{\alpha}$ defined by (3) satisfies the relation

(11) $$H_{ij}{}^{\alpha} = g_{ij}H^{\alpha},$$

where the vector H^{α}, called the mean curvature vector, is given by

(12) $$H^{\alpha} = \frac{1}{m} g^{rs} H_{rs}{}^{\alpha},$$

then V_m is called a totally umbilical submanifold.

From now on we assume that V_m is a totally umbilical submanifold of V_n.

Putting $H_x = H^{\alpha} N_{x\alpha}$ and using (5) and (11), we obtain

(13) $$H_{ijx} = g_{ij}H_x$$

whence, in view of (4) and (12),

(14) $$H^{\alpha} = \sum_x e_x H_x N_x{}^{\alpha}.$$

Therefore, using (1), we have

(15) $$H_{\alpha}H^{\alpha} = \sum_x e_x (H_x)^2.$$

One can easily show (see [4]) that

(16) $$\nabla_i H^{\alpha} = -(H_{\beta}H^{\beta})B_i^{\alpha} + \sum_x e_x A_{ix} N_x{}^{\alpha},$$

where

(17) $$A_{ix} = \partial_i H_x + \sum_y e_y L_{iyx} H_y.$$

Substituting (13) and (15) into (6) we see that

(18) $$R_{lkji} = \bar{R}_{lkji} + (H_{\alpha}H^{\alpha})(g_{li}g_{kj} - g_{lj}g_{ki}),$$

and substituting (13) and (17) into (7) we get

$$\bar{R}_{kjix} = A_{kx}g_{ji} - A_{jx}g_{ki}. \tag{19}$$

But (19) by making use of (14), (15) and (8) yields

$$\bar{R}_{kji\alpha}H^{\alpha} = \frac{1}{2}\nabla_k(H_\alpha H^\alpha)g_{ji} - \frac{1}{2}\nabla_j(H_\alpha H^\alpha)g_{ki}, \tag{20}$$

where

$$\frac{1}{2}\nabla_l(H_\alpha H^\alpha) = \sum_x e_x H_x A_{lx}. \tag{21}$$

3. Main results

In the sequel we shall consider the condition

$$\nabla_\mu \nabla_\varepsilon \overset{*}{C}_{\alpha\gamma\beta\delta} - \nabla_\varepsilon \nabla_\mu \overset{*}{C}_{\alpha\gamma\beta\delta} = 0, \tag{22}$$

where $\overset{*}{C}_{\alpha\gamma\beta\delta}$ defined by

$$\overset{*}{C}_{\alpha\gamma\beta\delta} = \overset{*}{R}_{\alpha\gamma\beta\delta} - \frac{1}{n-2}(G_{\gamma\beta}\overset{*}{R}_{\alpha\delta} - G_{\gamma\delta}\overset{*}{R}_{\alpha\beta} + G_{\alpha\delta}\overset{*}{R}_{\gamma\beta} - G_{\alpha\beta}\overset{*}{R}_{\gamma\delta}) + \tag{23}$$

$$+ \frac{\overset{*}{R}}{(n-1)(n-2)}(G_{\alpha\delta}G_{\gamma\beta} - G_{\alpha\beta}G_{\gamma\delta})$$

is the Weyl conformal curvature tensor of V_n. ∇_ε denotes here the covariant differentiation with respect to $G_{\alpha\beta}$, and C_{lkji} indicates the Weyl conformal curvature tensor of V_m.

T h e o r e m 1. Let V_m be a totally umbilical surface in a manifold V_n satisfying the condition (22). Then for V_m the following relation holds

$$A_{hx}C_{lijk} + A_{ix}C_{lhkj} + A_{jx}C_{lkhi} + A_{kx}C_{ljih} = 0. \tag{24}$$

P r o o f. Let $\bar{C}_{rijk} = \overset{*}{C}_{\alpha\beta\gamma\delta}B_r^\alpha B_i^\beta B_j^\gamma B_k^\delta$ denote the projection of the conformal curvature tensor onto the submanifold V_m. Using (18) we have

$$(25)\qquad \bar{C}_{rijk} = R_{rijk} - (H_\alpha H^\alpha)(g_{rk}g_{ij} - g_{rj}g_{ik}) +$$

$$- \frac{1}{n-2}(g_{ij}\bar{R}_{rk} - g_{ik}\bar{R}_{rj} + g_{rk}\bar{R}_{ij} - g_{rj}\bar{R}_{ik}) +$$

$$+ \frac{\overset{*}{R}}{(n-1)(n-2)}(g_{rk}g_{ij} - g_{rj}g_{ik}),$$

where $\bar{R}_{rk} = \overset{*}{R}_{\alpha\beta} B_r^\alpha B_k^\beta$.

Since $G_{\alpha\beta} B_i^\alpha N_x^\beta = 0$, we obtain, contracting (23) with $N_x^\alpha B_i^\gamma B_j^\beta B_k^\delta$ and using (19),

$$(26)\qquad \bar{C}_{xijk} = A_{kx}g_{ij} - A_{jx}g_{ik} - \frac{1}{n-2}(g_{ij}\bar{R}_{xk} - g_{ik}\bar{R}_{xj}),$$

where $\bar{C}_{xijk} = \overset{*}{C}_{\alpha\beta\gamma\delta} N_x^\alpha B_i^\beta B_j^\gamma B_k^\delta$, $\bar{R}_{xk} = \overset{*}{R}_{\alpha\beta} N_x^\alpha B_k^\beta$.

But the condition (22) can be written in the form

$$\overset{*}{C}_{\mu\beta\gamma\delta}\overset{*}{R}^\mu{}_{\alpha\omega\varrho} + \overset{*}{C}_{\alpha\mu\gamma\delta}\overset{*}{R}^\mu{}_{\beta\omega\varrho} + \overset{*}{C}_{\alpha\beta\mu\delta}\overset{*}{R}^\mu{}_{\gamma\omega\varrho} + \overset{*}{C}_{\alpha\beta\gamma\mu}\overset{*}{R}^\mu{}_{\delta\omega\varrho} = 0,$$

which yields, after transvecting with $B_i^\beta B_j^\gamma B_k^\delta B_h^\alpha B_l^\omega N_x^\varrho$,

$$(27)\qquad \bar{C}_{\mu ijk}\bar{R}^\mu{}_{hlx} + \bar{C}_{h\mu jk}\bar{R}^\mu{}_{ilx} + \bar{C}_{hi\mu k}\bar{R}^\mu{}_{jlx} + \bar{C}_{hij\mu}\bar{R}^\mu{}_{klx} = 0.$$

Making now use of (2), (25), (26) and (19) we obtain

$$(28)\qquad \bar{C}_{\mu ijk}\bar{R}^\mu{}_{hlx} = g_{hl}A_{sx}R^s{}_{ijk} - A_{hx}R_{lijk} +$$

$$+ \frac{A_{hx}}{n-2}(g_{ij}\bar{R}_{lk} - g_{ik}\bar{R}_{lj}) +$$

$$- \frac{1}{n-2} g_{hl}(g_{ij}A^r{}_x\bar{R}_{rk} - g_{ik}A^r{}_x\bar{R}_{rj}) +$$

$$+ g_{ij}\sum_y D_{ky}\bar{R}_{yhlx} - g_{ik}\sum_y D_{jy}\bar{R}_{yhlx} + P_{ijkhlx},$$

where

$$\bar{R}_{yhlx} = \overset{*}{R}_{\alpha\beta\gamma\delta} N_y{}^{\alpha} B_h{}^{\beta} B_l{}^{\gamma} N_x{}^{\delta} ,$$

$$\begin{aligned} P_{ijkhlx} = \frac{A_{hx}}{n-2} (g_{lk}\bar{R}_{ij} - g_{lj}\bar{R}_{ik}) - \frac{1}{n-2} (A_{kx}\bar{R}_{ij} - A_{jx}\bar{R}_{ik}) + \\ + \frac{\overset{*}{R}}{(n-1)(n-2)} g_{hl}(A_{kx}g_{ij} - A_{jx}g_{ik}) + \\ - \frac{\overset{*}{R} A_{hx}}{(n-1)(n-2)} (g_{lk}g_{ij} - g_{lj}g_{ik}) + \\ + (H_{\alpha}H^{\alpha})A_{hx}(g_{lk}g_{ij} - g_{lj}g_{ik}) - \\ - (H_{\alpha}H^{\alpha})(A_{kx}g_{hl}g_{ij} - A_{jx}g_{hl}g_{ik}), \end{aligned}$$

$$D_{ky} = e_y(A_{ky} - \frac{1}{n-2} \bar{R}_{yk}).$$

It is easy to verify that

$$(29) \qquad P_{ijkhlx} + P_{hkjilx} + P_{khijlx} + P_{jihklx} = 0.$$

Using (28) and (29) we may express (22) as

$$\begin{aligned} (30) \quad & g_{hl}A_{sx}R^{s}{}_{ijk} - A_{hx}R_{lijk} + g_{il}A_{sx}R^{s}{}_{hkj} - A_{ix}R_{lhkj} + \\ & + g_{jl}A_{sx}R^{s}{}_{khi} - A_{jx}R_{lkhi} + g_{kl}A_{sx}R^{s}{}_{jih} - A_{kx}R_{ljih} + \\ & + \frac{A_{hx}}{n-2}(g_{ij}\bar{R}_{lk} - g_{ik}\bar{R}_{lj}) - \frac{1}{n-2} g_{hl}(g_{ij}A^{r}{}_{x}\bar{R}_{rk} - g_{ik}A^{r}{}_{x}\bar{R}_{rj}) + \\ & + \frac{A_{ix}}{n-2}(g_{hk}\bar{R}_{lj} - g_{hj}\bar{R}_{lk}) - \frac{1}{n-2} g_{il}(g_{hk}A^{r}{}_{x}\bar{R}_{rj} - g_{hj}A^{r}{}_{x}\bar{R}_{rk}) + \\ & + \frac{A_{jx}}{n-2}(g_{kh}\bar{R}_{li} - g_{ki}\bar{R}_{lh}) - \frac{1}{n-2} g_{jl}(g_{kh}A^{r}{}_{x}\bar{R}_{ri} - g_{ki}A^{r}{}_{x}\bar{R}_{rh}) + \end{aligned}$$

$$+\frac{A_{kx}}{n-2}(g_{ji}\bar{R}_{lh}-g_{jh}\bar{R}_{li})-\frac{1}{n-2}g_{kl}(g_{ji}A^r{}_x\bar{R}_{rh}-g_{jh}A^r{}_x\bar{R}_{ri})+$$

$$+g_{ij}\sum_y D_{ky}\bar{R}_{yhlx}-g_{ik}\sum_y D_{jy}\bar{R}_{yhlx}+g_{hk}\sum_y D_{iy}\bar{R}_{yilx}+$$

$$-g_{hj}\sum_y D_{ky}\bar{R}_{yilx}+g_{ji}\sum_y D_{hy}\bar{R}_{yklx}-g_{ki}\sum_y D_{hy}\bar{R}_{yjlx}+$$

$$+g_{kh}\sum_y D_{iy}\bar{R}_{xjlx}-g_{jh}\sum_y D_{iy}\bar{R}_{yklx}=0,$$

where $A^r{}_x = A_{sx}g^{sr}$.

Contracting (26) with g^{ij} we obtain

$$(31)\quad \frac{1}{n-2}(A_{hx}\bar{R}_{lk}+A_{kx}\bar{R}_{lh}-A^r{}_x R_{rk}g_{lh}-A^r{}_x\bar{R}_{rh}g_{lk})+$$

$$+\sum_y(D_{ky}\bar{R}_{yhlx}+D_{hy}\bar{R}_{yklx})+\frac{2}{m-2}g_{hk}\sum_y D^r{}_y R_{yrlx}=$$

$$=\frac{1}{m-2}(A_{hx}R_{lk}+A_{kx}R_{lh}-A^r{}_x R_{rk}g_{hl}-A^r{}_x R_{rh}g_{kl}),$$

which, after a contraction with g^{hk}, yields

$$(32)\qquad 2(m-1)\sum_y D^r{}_y R_{yrlx}=0,$$

where $D^r{}_y = D_{sh}g^{sr}$.

In view of (32) the equation (31) takes the form

$$(33)\quad \frac{1}{n-2}(A_{hx}\bar{R}_{lk}+A_{kx}\bar{R}_{lh}-A^r{}_x\bar{R}_{rk}g_{lh}-A^r{}_x\bar{R}_{rh}g_{lk})+$$

$$+\sum_y(D_{ky}\bar{R}_{yhlx}+D_{hy}\bar{R}_{yklx})=$$

$$=\frac{1}{m-2}(A_{hx}R_{lk}+A_{kx}R_{lh}-A_{rx}R^r{}_k g_{hl}-A_{rx}R^r{}_h g_{kl}).$$

Hence, in view of (33), the relation (30) can be written as

$$(34)\quad g_{hl}A_{rx}R^{r}{}_{ijk} + g_{il}A_{rx}R^{r}{}_{hkj} + g_{jl}A_{rx}R^{r}{}_{khi} + g_{kl}A_{rx}R^{r}{}_{jih} +$$
$$- A_{hx}R_{lijk} - A_{ix}R_{lhkj} - A_{jx}R_{lkhi} - A_{kx}R_{ljih} +$$
$$+ \frac{1}{m-2} g_{ij}(A_{hx}R_{lk} + A_{kx}R_{lh} - A_{rx}R^{r}{}_{k}g_{hl} - A_{rx}R^{r}{}_{h}g_{kl}) +$$
$$- \frac{1}{m-2} g_{ik}(A_{hx}R_{lj} + A_{jx}R_{lh} - A_{rx}R^{r}{}_{j}g_{lh} - A_{rx}R^{r}{}_{h}g_{lj}) +$$
$$+ \frac{1}{m-2} g_{hk}(A_{ix}R_{lj} + A_{jx}R_{li} - A_{rx}R^{r}{}_{j}g_{li} - A_{rx}R^{r}{}_{i}g_{lj}) +$$
$$- \frac{1}{m-2} g_{hj}(A_{ix}R_{lk} + A_{kx}R_{li} - A_{rx}R^{r}{}_{k}g_{li} - A_{rx}R^{r}{}_{i}g_{lk}) = 0.$$

Contracting now this with g^{hl} we get

$$(35)\quad A_{rx}R^{r}{}_{ijk} - \frac{1}{m-2}(g_{ij}A_{rx}R^{r}{}_{k} - g_{ik}A_{rx}R^{r}{}_{j}) =$$
$$= \frac{1}{m-2}(A_{kx}R_{ij} - A_{jx}R_{ik}) - \frac{R}{(m-1)(m-2)}(A_{kx}g_{ij} - A_{jx}g_{ik}).$$

Applying the last relation to (34) we conclude our assertion.

In an analogous manner as above we may prove

T h e o r e m 2. If V_n satisfies the condition obtained from (22) by replacing $\overset{*}{C}_{\alpha\beta\gamma\delta}$ by $\overset{*}{R}_{\alpha\beta\gamma\delta}$, and V_m is totally umbilical, then (24) holds.

C o r o l l a r y 1. If both V_n and V_m are analytic and V_m is connected, then either

(i) $C_{hijk} = 0$ or

(ii) $A_{ix} = 0$, and the vectors $\nabla_i H^{\alpha}$ $(i=1,\dots,m)$ are tangent to V_m.

The proof follows immediately from Lemma 4 of ([5]) and (16).

R e m a r k 1. Adati and Miyazawa introduced [1] the concept of a conformally recurrent space. It is defined as an n-dimensional $(n>3)$ Riemannian manifold whose Weyl's conformal curvature tensor C_{hijk} satisfies the conditions

(iii) $$\nabla_l C_{hijk} = \phi_l C_{hijk}$$

for some vector ϕ_j. Differentiating (iii) covariantly and using (iii) again, we obtain

$$C_{hijk,lm} - C_{hijk,ml} = (\phi_{l,m} - \phi_{m,l})C_{hijk},$$

which, in view of Theorem 1, leads immediately to

C o r o l l a r y 2. Let V_m be a totally umbilical surface in a conformally recurrent space V_n. If the recurrence vector ϕ_j of V_n is locally a gradient, then for V_m the condition (24) is satisfied.

R e m a r k 2. Tanno proved [6] the following remarkable theorem: Let M be a Riemannian manifold with positive definite matric. If for some integer $p \geq 1$ the equation

(iv) $$\nabla_{k_1} \cdots \nabla_{k_p} C_{hijk} = 0$$

holds, then

$$\nabla_k C_{hijk} = 0.$$

Combining Tanno's theorem with Theorem 1, we have

C o r o l l a r y 3. Let V_m be a totally umbilical surface in a Riemannian manifold V_n. If the matric of V_n is positive definite and the Weyl's conformal curvature tensor satisfies (iv), then (24) holds.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, WROCŁAW

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