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ON THE PROPERTY W FOR MODULAR GROUPS

In agreement with [1] we assume that

a) $K_w = \{g \in G : o(g) = w,\ w \in \Omega\}$,

b) the group G has property W if for every $w \in \Omega$ we have $K_wK_w \leqslant G$.[1)]

In this note we shall investigate the question whether the property W holds for modular groups.

L e m m a 1. If G_1 is a proper subgroup of G, then the set $(G-G_1)^2$ is a subgroup of G.

P r o o f. If the thesis did not hold, then for some elements $g_1, g_2, g_3, g_4 \in G - G_1$ we would have $g_1g_2g_3 \in G_1$ and $g_1g_2 \in G_1$ or $g_3g_4 \in G_1$. Let $g_1g_2 \in G_1$, then $(g_1g_2)^{-1}g_1g_2g_3 \in G_1$, which contradicts the assumption $g_3 \notin G_1$.

We have to show still that the second condition for subgroup also holds, i.e. if $g_1, g_2 \in G-G_1$ then $g_1g_2 \in (G-G_1)^2 \Rightarrow g_2^{-1}g_1^{-1} \in (G-G_1)^2$. It this were not true, then we would have

$$g_2^{-1}g_1^{-1} \notin (G-G_1)^2 \Rightarrow g_2^{-1} \notin G-G_1$$

or

$$g_1^{-1} \notin G-G_1 \Rightarrow g_2^{-1} \notin G_1 \quad \text{or} \quad g_1^{-1} \notin G_1 .$$

Since G_1 is a group by assumption, we infer that $g_2 \in G_1$ or $g_1 \in G_1$. But this contradicts the assumption that $g_1, g_2 \in G - G_1$.

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1) Here Ω denotes the set of natural numbers, $o(g)$ is the order of g, $A < G$ means that A is a subgroup of G.

D e f i n i t i o n 1. A finite group G is called a P - group if one of the following conditions holds:

1. G is an elementary abelian p - group.

2. G is a group with generators $a_1, a_2, \ldots a_n, b$ and with the defining relations $a_i^p = b^q = 1$, $a_i a_j = a_j a_i$, $b\, a_i b^{-1} = a_i^r$ where $r \neq 1$, $r^q = 1 \pmod{p}$ (see [2] p.28).

D e f i n i t i o n 2. A finite group G is called P_0^* - group if it has a normal divisor N which is an abelian P - group such that the quotient group G/N is a cyclic p - group and interior automorphisms induced by the elements in G have for every $a \in N$ the following form $a \to a^r$, where r does not depend on a and satisfies the conditions $r \neq 1$, $r^q = 1 \pmod{p}$.

L e m m a 2. A P_0^* - group has the property W.

P r o o f. Let G be a P_0^* - group. By definition, we have

$$G/N = \left\{N,\ Nb,\ Nb^2, \ldots, Nb^q, \ldots, Nb^{q^k-1}\right\},$$

where N is an abelian P - group, and the cosets are generated by an element b such that $b^{q^k} = 1$. Hence an arbitrary element $g \in G$ has the form $g = a\, b^i$. Since

$$\forall_{g \in G} \ \forall_{a \in N} \ g\, a\, g^{-1} = a^r,\ r \neq 1,\ r^q = 1 \pmod{p}$$

we have

$$(ab^i)^n = a^{r^{(n-1)i} + r^{(n-2)i} + \ldots + r^i + 1}\, b^{ni} = a^{\frac{r^{ni}-1}{r^i-1}}\, b^{ni}$$

Let us fix the order of the element $ab^i \in G$.

a) if $q \nmid i$, then $o(ab^i) = q^k$, because

$$(ab^i)^{q^k} = a^{\frac{r^{iq^k}-1}{r^i-1}}\, b^{iq^k} = (a^p)^{\frac{s}{r^i-1}} (b^{q^k})^i = 1.$$

b) If $i = mq^{\alpha}$, where $0 < m \leqslant q^{k-\alpha}$, $q \nmid m$, then $o(ab^{i}) = pq^{k-\alpha}$, because

$$\left(a\, b^{mq^{\alpha}}\right)^{q^{k-\alpha}} = a^{\frac{r^{mq^{k}}-1}{r^{mq^{\alpha}}-1}} \cdot 1 = a^{\frac{(sp+1)^{mq^{k-1}}-1}{(sp+1)^{mq^{\alpha-1}}-1}} =$$

$$= a^{\frac{(sp)^{mq^{k-1}-1} + mq^{k-1}(sp)^{mq^{k-1}-2} + \ldots + mq^{k-1}}{(sp)^{mq^{\alpha-1}-1} + \ldots + mq^{\alpha-1}}},$$

where the exponent of the power is not divisible by p even when $p|m$. Hence the group G has the following complexes

$$K_1, K_p = \left\{a \in N,\ a \neq 1\right\},\ K_{q^{k-\alpha}} = \left\{b^{mq^{\alpha}},\ 1 \leqslant m < q^{k-\alpha},\ q \nmid m\right\},$$

$$K_{pq^{k-\alpha}} = \left\{ab^{mq^{\alpha}},\ a \neq 1,\ 1 \leqslant m < q^{k-\alpha},\ q \nmid m,\ 0 < \alpha < k-1\right\},$$

$$K_{q^k} = \left\{ab^{i},\ q \nmid i,\ a \neq 1 \vee a = 1\right\}.$$

It is clear that $K_1K_1 \leqslant G$. We also have $K_pK_p = N \leqslant G$. The complex $K_{q^{k-\alpha}}$ consists of the elements of a cyclic group, hence by [1] we have $K_{q^{k-\alpha}} K_{q^{k-\alpha}} \leqslant G$. Since $G_1 = N \cup K_{q^{k-\alpha}} \cup K_{pq^{k-\alpha}}$ for $0 < \alpha < k-1$ is a subgroup of G, by Lemma 1 we have $K_{q^k} K_{q^k} = (G - G_1)^2 \leqslant G$.

It remains to investigate whether $K_{pq^{k-\alpha}} K_{pq^{k-\alpha}}$ is a subgroup of G. We have

$$a_1\, b^{m_1q^{\alpha}}\, a_2 b^{m_2q^{\alpha}}\, a_3 b^{m_3q^{\alpha}}\, a_4 b^{m_4q^{\alpha}} =$$

$$= a_1 a_2^{m_1q^{\alpha}} a_3^{(m_1+m_2)q^{\alpha}} a_4^{(m_1+m_2m_3)q^{\alpha}} b^{(m_1+m_2+m_3+m_4)q^{\alpha}}$$

Let us denote

$$a_5 = a_2^{m_1q^{\alpha}} a_3^{(m_1+m_2)q^{\alpha}} a_4^{(m_2+m_2+m_3)q^{\alpha}}.$$

Since for i such that $q|i$ we have

$$a^{r^i} = \left(a^{r^q}\right)^s = a^{lp+1} = a, \text{ it follows that}$$

$$(*)=a_1a_5b^{(m_1+m_2+m_3+m_4)q^{\alpha}} = \begin{cases} a_1b^{m_5q^{\alpha}}a_5b^{m_6q^{\alpha}} & \text{when } a_5\neq 1, \\ \bar{a}_1b^{m_5q}\bar{a}_2b^{m_6q} \text{ where } \bar{a}_1\bar{a}_2=a_1, \bar{a}_1\neq 1, \bar{a}_2\neq 1 & \text{when } a_5=1 \end{cases}$$

Hence we see that the lemma holds for all complexes of the group G.

L e m m a 3. Every modular p-group has the property W.

Let $G = G_1$ be a finite modular p-group. Let p^{α_0} denote the highest order of the elements of this group. Then by a corollary to Lemma 1.7 of [2] we have $G_1 - K_{p^{\alpha_0}} = G_2 < G_1$. By Lemma 1 this implies $K_{p^{\alpha_0}} K_{p^{\alpha_0}} < G_1$. Using an analogous argument for modular subgroups of G in the sequence $G_1 > G_2 > \ldots > E$ we obtain that $K_{p^{\alpha_i}} K_{p^{\alpha_i}} < G$ for every $K_{p^{\alpha_i}}$ being the complex of the elements of highest order of the group G_{i+1}.

T h e o r e m 1. Every finite modular group has the property W.

P r o o f. The validity of this theorem follows from [2] th. 13, Lemmas 2 and 3 and from the fact that the direct product of groups having the property W, whose orders are relatively prime, has the property W.

T h e o r e m 2. Every modular group possessing elements of infinite order has the property W.

P r o o f. Let G_1 denote the set of elements of finite order in the group G. By [2] (theorems 1.10 and 1.11), G_1 is abelian. By [1], for all w expressing the orders of abelian elements of G_1, we have $K_w K_w \leqslant G$. Moreover, $G - G_1 = K_\infty$, and from Lemma 1 we have $K_\infty K_\infty \leqslant G$.

T h e o r e m 3. Every locally finite modular group G has the property W.

P r o o f. Any four elements a_i of order w belonging to G generate a finite subgroup $H = (a_1, a_2, a_3, a_4)$. By Theorem 1 we infer that the group H has the property W. Hence $a_1 a_2 a_3 a_4 = a_5 a_6$, where $o(a_5) = o(a_6) = w$.

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