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Anna Szynal, Jan Szynal

# ON SOME PROBLEMS CONCERNING SUBORDINATION AND MAJORIZATION OF FUNCTIONS

## 1. Introduction and notations

In this note we deal with some extensions and generalizations of the well - known problems in the theory of subordination and majorization of functions.

First we shall give some notations.

Let $S$ be the class of all functions of the form

$$F(z) = z + A_2 z^2 + \dots, \tag{1}$$

holomorphic and univalent in the disk $K_1$, where

$$K_r = \{z : |z| < r, \quad 0 < r \leq 1\}.$$

By $S_\alpha^* \subset S$, $0 \leq \alpha < 1$, $(S_0^* = S^*)$ we denote the class of all $\alpha$ -starlike functions, i.e. the functions satisfying the condition

$$\mathrm{Re}\left[\frac{z\,F'(z)}{F(z)}\right] > \alpha, \qquad z \in K_1. \tag{2}$$

$S_\alpha^*(b)$, $0 \leq \alpha < 1$, $0 \leq b \leq 1$, $(S_0^*\{b\} = S\{b\})$, stands for the class of all functions of the form

$$F(z) = z + 2b(1-\alpha)z^2 + \dots, \tag{3}$$

holomorphic and univalent in $K_1$ and satisfying there the condition (2).

By $S^*_{(\beta)}\langle b\rangle$, $0<\beta\le 1$, $0\le b\le 1$, $(S^*_{(1)}\langle b\rangle = S^*\{b\})$ we denote the class of all holomorphic functions in $K_1$ of the form

$$F(z) = z + 2b\beta z^2 + \dots,$$

satisfying the condition

$$\left|\arg \frac{zF'(z)}{F(z)}\right| < \beta\frac{\pi}{2}, \quad z\in K_1.$$

This class has been considered in [3] and [10].

Let us observe that $S^*_\alpha = \bigcup_{-1\le b\le 1} S^*_\alpha\{b\}$ and $S^*_\beta = \bigcup_{-1\le b\le 1} S^*_\beta\langle b\rangle$. We can restrict ourselves no loss generality to the case $0\le b\le 1$, because if $F(z)\in S^*_\alpha$ $(S^*_\beta)$ has the form (1) and $\theta = \arg A_2$, then $e^{i\theta}F(e^{-i\theta}z)\in S^*_\alpha(S^*_\beta)$ has nonnegative second coefficient.

By $\mathfrak{S}^M$, $M\ge 1$ we denote the class so-called quasi-starlike functions, introduced by Dziubiński [4]. Namely, we say that a function $g$ is quasi-starlike in $K_1$, if it satisfies the equation

$$F(g(z)) = \frac{1}{M}F(z), \quad z\in K_1, \tag{4}$$

where $F$ is an arbitrary starlike function in $K_1$ and $M$ is a fixed number such that $M\ge 1$.

Let $\Omega_n$, $n\ge 1$, denotes the class of all holomorphic functions in $K_1$ such that

$$\Phi(z) = \alpha_n z^{n-1} + \alpha_{n+1}z^n + \dots, \quad \alpha_n \ge 0, \qquad |\Phi(z)|<1,$$

and let $\Omega_n\{c\}$ be the subclass of $\Omega_n$ consisting of the functions with fixed coefficient $\alpha_n = c\in[0,1)$.

$H_n$, $n\ge 1$, stands for the class of all holomorphic functions in $K_1$ of the form

$$f(z) = a_n z^n + a_{n+1}z^{n+1} + \dots, \quad a_n\ge 0,$$

and $H_n\{c\}$ the subclass of $H_n$ consisting of the functions with fixed $a_n = c \geqslant 0$.

By $H_n^o$ we denote such subclass of functions of $H_n$ which in addition satisfy the condition $f(z)/z^n \neq 0$ in $K_1$ and $a_n > 0$.

We need also the class $P_n[\alpha, b(1-\alpha)]$, $n \geqslant 1$, $0 \leqslant \alpha < 1$, $0 \leqslant b \leqslant 1$, ($P_n[0,b] = P_n\{b\}$, $P_1\{b\} = P\{b\}$) of all holomorphic functions in $K_1$ of the form

$$p(z) = 1 + 2b(1-\alpha)z^n + p_{n+1}z^{n+1} + \dots ,$$

which satisfy the condition

$$\mathrm{Re}\, p(z) > \alpha, \quad z \in K_1.$$

It can be observed that $\bigcup_{-1 \leqslant b \leqslant 1} P_n[\alpha, b(1-\alpha)] = P_{n,\alpha}$.

Lewandowski [6] and Mac Gregor [8] has considered (among the others) the following problem: let $f \in H_1$, $F \in S^*$, and suppose that $|f(z)| \leqslant |F(z)|$ holds for every $z \in K_1$. Find "the best number" $r \in (0,1)$ such that for every $z \in K_r$ $|f'(z)| \leqslant |F'(z)|$ holds. The term "the best number" means that the last inequality does not take place in any larger disk.

In this note we allow the majorants with fixed second coefficient (the classes $S^*_\alpha\{b\}$ and $S^*_\beta\langle b\rangle$) as well as the class of quasi-starlike functions $\mathfrak{S}^M$.

Moreover, we solve somewhat general problem as in [6] and [8]. Namely, we determine the function

$$(5) \quad \varphi(H_n\{c\}, T_h; r) = \sup\left\{\left|\frac{f'(z)}{F'(z)}\right| : f \in H_n\{c\},\ F \in T_h; \right.$$

$$\left. |f(z)| < |F(z)|,\quad z \in K_1,\quad |z| = r < 1\right\},$$

where $T_h$ denote the class of all holomorphic functions in $K_1$ of the form (1) such that for any $r \in [0,1)$

(6) $$\left|\frac{zF'(z)}{F(z)}\right| \geqslant h(r), \qquad |z| = r<1,$$

holds (exact bound) with $h$ being continous, positive and decreasing function for $r\in[0,1)$, $h(0) = 1$. (Another problem of this kind in the theory of subordination has been considered in [2]).

We also find the largest disk in which the relation $f(z) \prec F(z)$, $f\in H_n^0$, $F\in S_\alpha^*\{b\}$ or $S_\beta^*\langle b\rangle$ taking place in $K_1$ implies the relation $|f(z)|\leqslant |F(z^n)|$ (the sign $\prec$ denotes that $f$ is subordinate to $F$ in $K_1$).

All obtained results are best possible and in special cases reduces to the results of [6] and [7],[8]. Taking into account that in the case $b=0$ the extremal functions are odd, we see that our results solve the problems stated above for subclasses consisting of odd functions.

2. Lemmas

L e m m a 1. The region of variability of $\{p(z)\}$ for fixed $z\in K_1$ and $p$ ranging over $P_n[\alpha, b(1-\alpha)]$ is (if $b\neq 1$) the closed disk

(7) $$|p(z) - w_0|\leqslant R_0,$$

where

(8) $$w_0 = \frac{(1-b^2|z|^2)+(1-2\alpha)|z|^{2n}(|z|^2-b^2)+b(1-|z|^2)\left[(1-2\alpha)z^n-\bar{z}^n\right]}{(1-b^2|z|^2) - |z|^{2n}(|z|^2-b^2)-b(1-|z|^2)(z^n+\bar{z}^n)}$$

$$R_0 = \frac{2(1-\alpha)(1-b^2)|z|^{n+1}}{(1-b^2|z|^2)-|z|^{2n}(|z|^2-b^2)-b(1-|z|^2)(z^n+\bar{z}^n)}.$$

Boundary functions in (7) have the form

(9) $$p_0(z) = \frac{\left[1+(1-2\alpha)bz^n\right]+\varepsilon z\left[(1-2\alpha)z^n+b\right]}{(1-bz^n) - \varepsilon z(z^n-b)}, \qquad |\varepsilon| = 1.$$

If $b=1$, then the domain of variability is the closed disk

$$(7') \qquad \left| p(z) - \frac{1+(1-2\alpha)|z|^{2n}}{1-|z|^{2n}} \right| \leq \frac{2(1-\alpha)|z|^n}{1-|z|^{2n}},$$

and the boundary functions have the form

$$p_0(z) = \frac{1+(1-2\alpha)\varepsilon z^n}{1-\varepsilon z^n}, \quad |\varepsilon| = 1.$$

P r o o f. Suppose that $p(z) \in P_n[\alpha, b(1-\alpha)]$. Then the function

$$(10) \qquad Q(z) = \frac{p(z)-\alpha}{1-\alpha} \in P_n\{b\}$$

and we can write

$$(11) \qquad \frac{Q(z)-1}{Q(z)+1} = bz^n + \ldots = z^n\omega(z),$$

where $\omega(o) = b$ and $|\omega(z)| < 1$ for $z \in K_1$.

Let us observe that the function

$$(12) \qquad G(z) = \frac{1+g(z)}{1-g(z)} \in P_n, \quad (P_n = P_{n,o}),$$

where

$$(13) \qquad g(z) = z^{n-1}w(z) \quad \text{and} \quad w(z) = \frac{\omega(z)-b}{1-b\,\omega(z)}.$$

Substituting (13) into (12) and (10), we get

$$(14) \quad Q(z) = \frac{\left[z^{n-1}(1+bz^n) + (z^n+b)\right]G(z) + \left[z^{n-1}(1+bz^n)-(z^n+b)\right]}{\left[z^{n-1}(1-bz^n) - (z^n-b)\right]G(z) + \left[z^{n-1}(1-bz^n)-(z^n-b)\right]}.$$

It can be observed that the formula (14) give us one-to-one correspondence between the classes $P_n$ and $P_n\{b\}$.

Using the fact that the boundary functions for the functional $\{G(z)\}$, where $z$ is fixed point from $K_1$ and $G$ ranging over the class $P_n$ are given by

(15) $$G(z) = \frac{1 + \varepsilon z^n}{1 - \varepsilon z^n}, \qquad |\varepsilon| = 1,$$

we obtain (after substituting (15) into (14) and (14) into (10)),that the boundary functions respect to the functional $\{p(z)\}$, $z \in K_1$, and $p$ is ranging over $P_n[\alpha, b(1-\alpha)]$, have the form (9).

The formulas (8) follow from the fact that function (9) maps the disk $|z| \leq r < 1$ onto the disk (7).

Putting in the Lemma 1 $n=1$, and taking into account fact that $F \in S^*_\alpha\{b\}$, if and only if $\frac{zF'(z)}{F(z)} \in P_1[\alpha, b(1-\alpha)]$, we obtain the following

C o r o l l a r y 1. The region of variability of $\left[\frac{z F'(z)}{F(z)}\right]$ for fixed $z \in K_1$ and $F$ ranging over $S^*_\alpha\{b\}$ is the closed disk (7) if $b \neq 1$ and (7') if $b = 1$.

The geometric interpretation of (7) for $n = 1$, after using Corollary 1, implies

L e m m a 2. Let $F \in S^*_\alpha\{b\}$. Then for fixed $z \in K_1$

(16) $$\frac{1-2\alpha b\operatorname{Re}z - (1-2\alpha)|z|^2}{1-2b\operatorname{Re}z + |z|^2} \leq \operatorname{Re}\frac{z F'(z)}{F(z)} \leq$$

$$\leq \frac{(1-|z|^2)\left[1-2\alpha b\operatorname{Re}z-(1-2\alpha)|z|^2\right]+ 4(1-\alpha)(1-b^2)|z|^2}{(1-|z|^2)(1-2b\operatorname{Re}z +|z|^2)}$$

(17) $$\frac{1+2\alpha b|z|-(1-2\alpha)|z|^2}{1+2b|z| + |z|^2} \leq \left|\frac{zF'(z)}{F(z)}\right| \leq \frac{1+2(1-\alpha)b|z|+(1-2\alpha)|z|^2}{1-|z|^2}$$

(18) $$\left|\arg \frac{zF'(z)}{F(z)}\right| \leq \operatorname{arc\,tg} \frac{R_0 \operatorname{Re} w_0 + \operatorname{Im} w_0\sqrt{|w_0|^2 - R_0^2}}{\operatorname{Re} w_0\sqrt{|w_0|^2-R_0^2}-R_0 \operatorname{Im} w_0}$$

hold.

P r o o f. The inequality (7) and Corollary 1 imply:

$$\mathrm{Re}\, w_0 - R_0 \leq \mathrm{Re}\,\frac{zF'(z)}{F(z)} \leq \mathrm{Re}\, w_0 + R_0, \tag{19}$$

$$|w_0| - R_0 \leq \left|\frac{zF'(z)}{F(z)}\right| \leq |w_0| + R_0 \tag{20}$$

$$\left|\arg\frac{zF'(z)}{F(z)}\right| \leq \mathrm{arc\,tg}\,\frac{\mathrm{Im}\, w_0}{\mathrm{Re}\, w_0} + \mathrm{arc\,tg}\,\frac{R_0}{\sqrt{|w_0|^2-R_0^2}}, \tag{21}$$

where $w_0$ and $R_0$ are given by (8) for $n=1$.

The inequality (16) follows from (19) by substituting.

The right hand of the inequality (20) is as follows

$$|w_0| + R_0 = \frac{\left\{\left[A-2\alpha b|z|(1-|z|^2)\cos\varphi\right]^2 + 4b^2(1-\alpha^2)|z|^2(1-|z|^2)^2\sin^2\varphi\right\}^{\frac{1}{2}} + 2(1-\alpha)(1-b^2)|z|^2}{(1-|z|^2)(1+|z|^2-2b|z|\cos\varphi)}, \tag{22}$$

where $z=|z|e^{i\varphi}$ and $A=(1-|z|^2)\left[1-(1-2\alpha)|z|^2\right]+2(1-\alpha)(1-b^2)|z|^2$.

The function given by (22) (as function of $\varphi$) attains its maximum for $\varphi=0$ and this maximum is equal to the right hand of (17). In the same way we find that minimum of $|w_0|-R_0$ is attained for $\varphi=\pi$ and its value is equal to the left hand of (17).

The inequality (21) is obvious. From Lemma 1 follows that the inequalities (16) - (18) are sharp.

For the class $S^*\{b\}$ the estimate given by (18) takes a simpler form if we observe that in this case the right hand of (18) is increasing function of $\varphi$ and attains its maximum for $\varphi=\frac{\pi}{2}$.

Thus we have, the following corollary.

C o r o l l a r y 2. If $F\in S^*\{b\}$, then for fixed $z\in K_1$ the following sharp estimate

$$(23)\qquad \left|\arg \frac{zF'(z)}{F(z)}\right| \leq \operatorname{arc\,tg} \frac{2b|z|(1-|z|^2)}{(1-|z|^2)^2 + 2(1-b^2)|z|^2} +$$

$$+ \operatorname{arc\,sin} \frac{2(1-b^2)|z|^2}{\left\{\left[(1-|z|^2)^2+2(1-b^2)|z|^2\right]^2+4b^2|z|^2(1-|z|^2)^2\right\}^{\frac{1}{2}}}$$

takes place.

In the case $n=1$, Lemma 1 gives some result from [9] (see also [11]), and the estimates given by (16) - (18) and (23) for $b=0$ are true for odd functions from $S^*$.

Lemma 3. If $\Phi \in \Omega_n\{c\}$, then for every $z \in K_1$

$$(24)\qquad |\Phi(z)| \leq |z|^{n-1} \frac{|z| + c}{1 + c|z|}$$

$$(25)\qquad |\Phi'(z)| \leq \frac{|z|^{2n-2}+(n-1)|z|^{n-2}(1-|z|^2)|\Phi(z)| - |\Phi(z)|^2}{|z|^{n-1}(1-|z|^2)}$$

hold. The signs of the equality in (24) and (25) take place for the function

$$(26)\qquad \Phi_0(z) = z^{n-1} \frac{z + c}{1 + cz}.$$

Proof. The inequality (24) is obvious and can be find in [5]. The inequality (25) can be deduced in the following way. Every function $\Phi \in \Omega_n$ can be written in the form $\Phi(z) = z^{n-1}\hat{\Phi}(z)$, where $\hat{\Phi} \in \Omega_1$. Using the well-known Pick's inequality for $\hat{\Phi} \in \Omega_1$, i.e. $|\hat{\Phi}'(z)| \leq \frac{1-|\hat{\Phi}(z)|^2}{1-|z|^2}$, we get

$$|\Phi'(z)| = \left|\left(z^{n-1}\hat{\Phi}(z)\right)'\right| = \left|(n-1)z^{n-2}\hat{\Phi}(z) + z^{n-1}\hat{\Phi}'(z)\right| \leq$$

$$\leq (n-1)|z|^{n-2}|\hat{\Phi}(z)| + |z|^{n-1}|\hat{\Phi}'(z)| \leq (n-1)\frac{|\Phi(z)|}{|z|} + |z|^{n-1}\frac{1-|\hat{\Phi}(z)|^2}{1-|z|^2} =$$

$$= (n-1)\frac{|\Phi(z)|}{|z|} + \frac{|z|^{2n-2} - |\Phi(z)|^2}{(1-|z|^2)|z|^{n-1}} =$$

$$= \frac{|z|^{2n-2} + (n-1)(1-|z|^2)|z|^{n-2}|\Phi(z)| - |\Phi(z)|^2}{|z|^{n-1}(1-|z|^2)}.$$

Let $v(t)$, $t \in [0,1]$, be real, non-decreasing lower semi-continous and vanishing at $t=0$ function. Let us set

(27) $$r(v) = \sup\left\{x:\ 0 \leq x \leq 1,\ v(x) + 2 \operatorname{arc\,tg} x < \frac{\pi}{2}\right\}.$$

Let $S_v$ be the class of all functions $F$ of the form (1), holomorphic in $K_1$ and such that for any $r \in [0,1)$

(28) $$\left|\arg \frac{zF'(z)}{F(z)}\right| \leq v(r), \quad |z| \leq r < 1.$$

hold.

In [7] it has been proved

L e m m a 4. If $F \in S_v$ and $f \in H_n^0$, then the relation $[f(z) \dashv F(z),\ z \in K_1]$ implies $|f(z)| \leq |F(z^n)|$ in $|z| \leq r(v)$, where $r(v)$ is given by (27).

## 3. The main results

First we are going to consider a particular case of the function $\varphi$.

T h e o r e m 1

(29) $$\varphi(H_1, S_\alpha^*\{b\}; r) = \begin{cases} 1 \text{ for } r \in [0, r_0] \\ 1 + \left(\frac{1}{2}\gamma(r) - \gamma^{-1}(r)\right)^2 \text{ for } r \in [r_0, 1), \end{cases}$$

where

$$(30)\qquad \gamma(r) = \left(\frac{(1-r^2)\left[1+2\alpha br - (1-2\alpha)r^2\right]}{r(1+2br+r^2)}\right)^{\frac{1}{2}}, \qquad r = |z| < 1,$$

and $r_0$ is the unique root of the equation

$$(31)\qquad (1-2\alpha)r^4 - 2(1+\alpha b)r^3 - 2\left[(1-\alpha)+2b\right]r^2 - 2(1-\alpha b)r+1 = 0.$$

C o r o l l a r y 3. If: $f \in H_1$, $F \in S^*_\alpha$, and $|f(z)| \leq |F(z)|$ for $z \in K_1$, then $|f'(z)| < |F'(z)|$ for $z \in K_{r_\alpha}$, where

$$(32)\qquad r_\alpha = \left[(2-\alpha) + (\alpha^2-2+3)^{\frac{1}{2}}\right]^{-1}.$$

C o r o l l a r y 4. If: $f \in H_1$, $F \in S^*\{b\}$, and $|f(z)| \leq |F(z)|$ for $z \in K_1$, then $|f'(z)| \leq |F'(z)|$ for $z \in K_{r_b}$, where

$$(33)\qquad r_b = \frac{1}{2}\left[(d+1) - \sqrt{(d-1)(d+3)}\right], \quad d = (4b+3)^{\frac{1}{2}}.$$

C o r o l l a r y 5. If $f \in H_1$ and $F$ is odd starlike function, and if $|f(z)| \leq |F(z)|$ for $z \in K_1$, then $|f'(z)| < |F'(z)|$ for $z \in K_{r_1}$, where

$$(34)\qquad r_1 = 2\left[(1+\sqrt{5}) + \sqrt{2(1+\sqrt{5})}\right]^{-1}.$$

P r o o f. The assumption $|f(z)| < |F(z)|$ for $z \in K_1$ and $f \in H_1$, $F \in S^*_\alpha\{b\}$ is equivalent to the equality

$$(35)\qquad f(z) = \phi(z)F(z),$$

where $\phi \in \Omega_1$.

Differentiating (35) and taking the absolute value, we get

$$(36)\qquad |f'(z)| < |\phi'(z)|\,|F(z)| + |\phi(z)|\,|F'(z)|.$$

Using the inequality (25) for $n=1$ and the left hand of (17), we obtain

$$(37)\qquad |f'(z)|<|F'(z)|\left\{\frac{-r(1+2br+r^2)|\phi(z)|^2+(1-r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]|\phi(z)|+r(1+2br+r^2)}{(1-r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]}\right\}$$

where $r=|z|<1$.

If we set $|\phi(z)| = x \in [0,1]$, and

$$\psi(x) = -r(1+2br+r^2)x^2+(1-r^2)\left[(1+2\alpha br)-(1-2\alpha)r^2\right]x+r(1+2br+r^2)$$

then from (37)

$$(38)\qquad \varphi(H_1,S^*_\alpha\{b\};r) = \left((1-r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]\right)^{-1}\max_{0<x<1}\psi(x).$$

But

$$(39)\qquad \max_{0<x\leq 1}\psi(x) = \begin{cases}\psi(x_0) & \text{if } x_0<1\\ 1 & \text{if } x_0\geq 1,\end{cases}$$

where

$$(40)\qquad x_0 = \frac{(1-r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]}{2r(1+2br+r^2)}.$$

Since $x_0\leq 1$, if and only if $r\in[r_0,1)$, where $r_0$ is the unique root of equation (31), we get

$$(41)\qquad \max_{0<x<1}\psi(x) = \begin{cases}1 & \text{for } r\in[0,r_0)\\ \psi(x_0) & \text{for } r\in[r_0,1).\end{cases}$$

The substituting (39) into (40) gives (29).

Now we shall show that obtained result is sharp.

Let $c\in(0,1)$ be so chosen that

$$\frac{r+c}{1+cr} = \frac{(1-r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]}{2r(1+2br+r^2)}. \tag{42}$$

holds, and let us put

$$\phi_0(z) = \frac{z+c}{1+cz}, \quad F_0(z) = z(1+2bz+z^2)^{\alpha-1}. \tag{43}$$

The mentioned choise of $c$ is motivated by the fact that for $\phi_0$ in (25) and $F_0$ in the left of (17) are occured the signs of equality. We have $f_0(z) = \phi_0(z)F_0(z)$, and after simple calculations, we get

$$\left|\frac{f_0'(z)}{F_0'(z)}\right| = \left|\frac{(1-c^2)z(1+2bz+z^2)}{(1+cz)^2\left[1+2\alpha bz-(1-2\alpha)z^2\right]} + \frac{z+c}{1+cz}\right|. \tag{44}$$

Using (42) and putting in (44) $z=r$, we see that

$$\left|\frac{f_0'(r)}{F_0'(r)}\right| = \frac{4r^2(1+2br+r^2)^2+(1-r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]^2}{4r(1-r^2)(1+2br+r^2)\left[1+2\alpha br-(1-2\alpha)r^2\right]},$$

which is equal to $\varphi(H_1, S_\alpha^*\{b\}; r)$, given by (29).

Corollary 3 follows from Theorem 1 by putting $\varphi\equiv1$ and $b=1$. Let us observe that putting $\alpha=0$ in (32), we get the result for starlike functions [6] and putting $\alpha=1/2$, we get the result for 1/2-starlike functions which is also exact for convex functions [8].

Corollary 4 is obvious and the statment in Corollary 5 which is true for $S_0^*\{o\}$ is exact for odd starlike functions as extremal function $F_0$ given by (43) is odd if $b=0$.

Of course all radii (32) - (34) are exact, which is consequence of sharpeness of Theorem 1.

T h e o r e m 2.

$$(45)\quad \varphi(H_n\{c\},T_h;r)=\begin{cases}\dfrac{r^{n-1}\{r(1-c^2)+(r+c)(1+cr)[(n-1)+h(r)]\}}{(1+cr)^2\, h(r)} & \text{for } r\in[0,r_o]\\[2ex] \dfrac{r^{n-2}\{4r^2+(1-r^2)^2[(n-1)+h(r)]^2\}}{4(1-r^2)h(r)} & \text{for } r\in[r_o,1),\end{cases}$$

where $r_o$ is the unique root of the equation

$$(46)\qquad (n-1)+h(r)=\frac{2\,r}{1-r^2}\left(\frac{r+c}{1+cr}\right).$$

R e m a r k. Theorem 2 is more useful and give the exact results for several subclasses of starlike functions defined by subordination

$$\frac{zF'(z)}{F(z)} \prec H(z),$$

where $H$ is fixed holomorphic and univalent functions with positive real part in $K_1$ and such that $H(o)=1$.

P r o o f. Suppose that $|f(z)|<|F(z)|$, $z\in K_1$ where $f\in H_n\{c\}$ and $F\in T_h$. This fact implies that there exists such function $\phi\in \Omega_n\{c\}$ that

$$(47)\qquad f(z)=\phi(z)F(z).$$

holds.

From (47) we get

$$(48)\qquad |f'(z)|<|\phi'(z)|\,|F(z)|+|\phi(z)|\,|F'(z)|.$$

Using the inequality (6) and (25), we get ($x=|\phi(z)|$)

$$(49)\quad |f'(z)|<|F'(z)|\left\{\frac{r^n}{(1-r^2)h(r)}+\frac{[(n-1)+h(r)]}{h(r)}\,x-\frac{1}{r^{n-2}(1-r^2)h(r)}\,x^2\right\}.$$

But, it is obvious that

(50) $\varphi(H_n\{c\}, T_h; r) =$

$$= \max_{0<x<r^{n-1}\frac{r+c}{1+cr}} \left\{ \frac{r^n}{(1-r^2)h(r)} + \frac{[(n-1)+h(r)]}{h(r)} x - \frac{1}{r^{n-2}(1-r^2)h(r)} x^2 \right\},$$

where $r=|z|<1$.

The simple calculations show that maximum of the function given by (50) is determined by (45).

The formula given by (45) is exact for $r \in [0, r_0]$. Namely, let $h_0(r)$ be the function corresponding to the function $F_0$ such that

$$\min_{|z|\leq r<1} \left| \frac{zF'(z)}{F(z)} \right| = \frac{rF'_0(r)}{F_0(r)} = h_0(r),$$

and let $\phi_0$ be given by (26). Further on, let us put $f_0(z) = \phi_0(z)F_0(z)$. Then for $z=r$ we have

$$\left| \frac{f'_0(z)}{F'_0(z)} \right| = \left| \phi'_0(z) \frac{F_0(z)}{F'_0(z)} + \phi_0(z) \right| =$$

$$= \frac{r}{h(r)} \frac{r^{2n-2} + (n-1)(1-r^2)r^{n-2}|\phi_0(z)| - |\phi_0(z)|^2}{r^{n-1}(1-r^2)} + |\phi_0(z)| ,$$

which ends the proof.

One can observe that the inequality

$$\frac{r^{n-1}\{r(1-c^2) + (r+c)(1+cr)[(n-1)+h(r)]\}}{(1+cr)^2 h(r)} \leq 1$$

is equivalent to

(51) $$h(r) \geq \frac{r^{n-1}[r(1-c^2)+(n-1)(r+c)(1+cr)]}{(1+cr)[(1+cr)-r^{n-1}(r+c)]} ,$$

and the right hand of (51) is increasing function of $r$. This yields

Corollary 6. If $f \in H_n\{c\}$ and $F \in T_h$ and $|f(z)| \le |F(z)|$ for $z \in K_1$, then $|f'(z)| \le |F'(z)|$ for $z \in K_{r_c}$, where $r_c$ is the unique root of the equation

$$(52) \qquad h(r) = \frac{r^{n-1}\left[r(1-c^2) + (n-1)(r+c)(1+cr)\right]}{(1+cr)\left[(1+cr)-r^{n-1}(r+c)\right]}$$

and this result is best possible.

Corollary 7. If $f \in H_n\{c\}$ and $T_h = S$ or $S^*\left(h(r) = \frac{1-r}{1+r}\right)$, then the relation $|f(z)| < |F(z)|$, $z \in K_1$ implies the relation $|f'(z)| \le |F'(z)|$ for $z \in K_{r(n,c)}$, where $r(n,c)$ is the unique root of the equation

$$(53) \qquad (n-2)cr^{n+2} + \left[(n-3)c^2 + nc + (n-1)\right]r^{n+1} + \left[(n-1)c^2 + (n-2)c + (n+1)\right]r^n + cnr^{n-1} + c^2r^3 - c(c-2)r^2 + (1-2c)r-1 = 0,$$

and this result is exact.

Putting $n=1$ in (53), we obtain a result from [7].

As an application of Theorem 2, we have the following

Theorem 3.

$$(54) \quad \varphi(H_1\{c\}, S^*_\beta\langle b\rangle; r) = \begin{cases} \dfrac{r(1-c^2)(1+2br+r^2)^\beta + (r+c)(1+cr)(1-r^2)^\beta}{(1+cr)^2(1+2br+r^2)^\beta} & \text{for } r \in [0, r_{(\beta)}] \\[2ex] 1 + \left(\varkappa(r) - \frac{1}{2}\varkappa^{-1}(r)\right)^2 & \text{for } r \in [r_{(\beta)}, 1), \end{cases}$$

where

$$\varkappa(r) = \left[\frac{r(1+2br+r^2)}{(1-r^2)^{\beta+1}}\right]^{\frac{1}{2}}$$

and $r_{(\beta)}$ is the unique root of the equation

$$(1+cr)(1-r^2)^{\beta+1} = 2r(r+c)(1+2br+r^2)^{\beta}.$$

P r o o f. If $T_h = S^*_\beta\langle b\rangle$, then from the relation that $F\in S^*_{(\beta)}\langle b\rangle$, if and only if $\frac{zF'(z)}{F(z)} = p^\beta(z)$, where $p(z)\in P\{b\}$ and from (17) we find that

$$h(r) = \left(\frac{1-r^2}{1+2br+r^2}\right)^{\beta} \tag{55}$$

Theorem 2 implies that

$$\varphi(H_1\{c\},\ T_h;r) = \tag{56}$$

$$= \begin{cases} \dfrac{r(1-c^2)+(r+c)(1+cr)h(r)}{(1+cr)^2\ h(r)} & \text{for } r\in[0,r_0] \\[2ex] 1+\left(\sqrt{\dfrac{r}{(1-r^2)h(r)}} - \dfrac{1}{2}\sqrt{\dfrac{(1-r^2)h(r)}{r}}\right) & \text{for } r\in[r_0,1), \end{cases}$$

where $r_0$ is the unique root of the equation

$$h(r) = \frac{2r}{1-r^2}\left(\frac{r+c}{1+cr}\right).$$

The formulas (55) and (56) imply (54).

T h e o r e m 4. If $f\in H_1$ and $G\in \hat{\mathfrak{S}}^M$ and $|f(z)|\leqslant|G(z)|$ for $z\in K_1$, then $|f'(z)|\leqslant|G'(z)|$ for $z\in K_{r_M}$, where $r_M$ is the unique root of the equation

$$2r\left\{(1+2r) - M(1+r)\left[(1+r) - \sqrt{(1+r)^2-\tfrac{4r}{M}}\right]\right\} = \tag{57}$$

$$= (1-r)^2\left\{(1-2r) + M(1+r)\left[(1+r) - \sqrt{(1+r)^2-\tfrac{4r}{M}}\right]\right\},$$

and $\hat{\mathfrak{G}}^M$ denotes the normalized family of quasi-starlike functions in $K_1$ i.e. such family of functions $G$, that $G(z) = Mg(z)$, and $g(z)$ is defined by (4).

P r o o f. Let us observe that from (56) follows that

$$\varphi(H_1\{c\},\ T_h;r) \leqslant 1,$$

if and only if

$$h(r) \geqslant \frac{r(1+c)}{(1-r)(1+cr)}. \tag{58}$$

The right hand of (58) is an increasing function of $c \in [0,1)$, which implies that the root of the equation

$$h(r) = \frac{r(1+c)}{(1-r)(1+cr)}$$

is decreasing function of $c \in [0,1)$.

This fact in turn gives that the relation $(c \to 1)$ $\left[|f(z)| < |F(z)|,\ z \in K_1,\ f \in H_1,\ F \in T_h\right]$ implies $\left[|f'(z)| < |F'(z)|,\ z \in K_{\hat{r}}\right]$, where $\hat{r}$ is the unique root of the equation

$$h(r) = \frac{2r}{1-r} \tag{59}$$

For the class $\hat{\mathfrak{G}}^M$, we have [1]

$$h(r) = \frac{1-r}{1+r}\,\frac{(1-2r)+M(1+r)\left[(1+r)-\sqrt{(1+r)^2-\frac{4r}{M}}\right]}{(1+2r)-M(1+r)\left[(1+r)-\sqrt{(1+r)^2-\frac{4r}{M}}\right]} \tag{60}$$

which implies (57).

Finally we can observe that from Theorem 3, by putting $\beta = 1$ and $b = 1$, follows Corollary 7, and from Theorem 4 in the limit case $M \to \infty$, we get once again the result of starlike majorants.

Now, we use Lemma 4 to obtain a theorem concerning majorization-subordination theory. Namely, using the relation between classes $S^*_\beta\langle b\rangle$ and $P\{b\}$ as well as (23), we get that for every $F\in S^*_\beta\langle b\rangle$ the exact estimate

$$(61)\quad \left|\arg\frac{zF'(z)}{F(z)}\right|<\beta\left(\operatorname{arc\,tg}\frac{2br(1-r^2)}{(1-r^2)^2+2(1-b^2)\cdot r^2}\right)+$$

$$+\operatorname{arc\,sin}\frac{2(1-b^2)r^2}{\left\{\left[(1-r^2)^2+2(1-b^2)r^2\right]^2+4b^2r^2(1-r^2)^2\right\}^{\frac{1}{2}}},$$

where $r=|z|<1$, takes place.

Lemma 4 and (61) imply the following result

T h e o r e m 5. Let $f\in H^o_n$ and $F\in S^*_\beta\langle b\rangle$, then the relation $[f(z)\dashv F(z)]$, $z\in K_1$ implies $|f(z)|\leqslant|F(z)|$ for $z\in K_r$, where $r$ is the unique root of the equation

$$(62)\quad \beta\left(\operatorname{arc\,tg}\frac{2br(1-r^2)}{(1-r^2)^2+2(1-b^2)r^2}\right)+$$

$$+\operatorname{arc\,sin}\frac{2(1-b^2)r}{\left\{\left[(1-r^2)^2+2(1-b^2)r^2\right]^2+4b^2r^2(1-r^2)^2\right\}^{\frac{1}{2}}}+$$

$$+2\operatorname{arc\,tg} r=\frac{\pi}{2}$$

and this result is exact.

Putting $n=1$, $\beta=1$ and $b=1$, we obtain the well-known result [7].

C o r o l l a r y 8. If $f(z)=a_1z+\ldots$, $a_1>0$ is holomorphic in $K_1$ and $f(z)\neq 0$ for $z\in K_1\setminus\{0\}$ and $F\in S^*$, then $f(z)\dashv F(z)$, $z\in K_1$, implies $|f(z)|\leqslant|F(z)|$ for $|z|<\sqrt{2}-1$.

Putting $n=1$, $\beta=1$ and $b=0$ we get

C o r o l l a r y 9. If $f(z)=a_1\cdot z+\ldots$, $a_1>0$ is holomorphic in $K_1$ and $f(z)\neq 0$ for $z\in K_1\setminus\{0\}$ and $F$ is arbitrary odd starlike function, then $f(z) \prec F(z)$, $z \in K_1$ implies $|f(z)| \leq |F(z)|$ for $|z| < \hat{r}$, where $\hat{r}$ is the unique root of the equation

$$r^3 + r^2 + r - 1 = 0. \tag{63}$$

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TECHNICAL UNIVERSITY, LUBLIN;

INSTITUTE OF MATHEMATICS, MARIA CURIE-SKŁODOWSKA UNIVERSITY, LUBLIN

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