DEMONSTRATIO MATHEMATICA
Vol. XI No 2 1978

Shyam Kishore Bajpai

INFLUENCE OF ½f'' (0) ON THE α-CONVEXITY OF NORMALISED STARLIKE UNIVALENT ANALYTIC FUNCTIONS OF ORDER β

0. Let α be a real number and $S_\alpha(\varrho,\beta)$ denote the class of normalised α-convex univalent functions f of order β $(0<\beta<1)$ in the open unit disc $D=\{z \mid |z|<1\}$ i.e., f is in $S_\alpha(\varrho,\beta)$ if and only if $f(z)$ is regular in D, $f(0)=f'(0)-1=0$, $\frac{1}{2}f''(0)=\varrho$, $f(z)f'(z)/z \neq 0$ for $z\in D$ and

$$(0.1)\quad \operatorname{Re}\{(1-\alpha)zf'(z)/f(z)+\alpha[1+zf''(z)/f'(z)]\}>\beta\}$$

for $z\in D$ and $0<\beta<1$.
If $\alpha=1$ then f in $S_\alpha(\varrho,\beta)$ is convex of order β in D while if $\alpha=0$ then f in $S_\alpha(\varrho,\beta)$ is starlike of order β in D. Starlike and convex univalent functions of order β in D are introduced by M.S. Robertson [6]. But α-convex functions of order zero are of comparatively recent origin and are introduced by Professor Petru T. Mocanu [3] of Rumania. Mocanu, introduced this class of functions for $-1\le\alpha\le 1$ only and proved each function in this class is univalent and starlike [3]. Thereafter, Miller, Mocanu and Reade [4] extended the definition ano proved that f in $S_\alpha(\varrho,\beta)$ is starlike and univalent for all α, i.e., $-\infty<\alpha<\infty$.

In [2] David E. Tepper obtained the convexity region influenced by ϱ for the functions f in $S_0(\varrho,0)$. In [1] we obtained α-convexity region for starlike functions of order zero and independent of ρ in D. In the present paper we shall obtain what is the α-convexity region for f in $S_0(\rho,\beta)$ influenced by ρ. Because of similarity of the proof we shall also obtain the α-convexity for f in $S_0(\rho,\beta)$ independent of ρ. This generalizes Tepper's result. Our method of proof is similar to that in [1] and has a relevance to the techniques of V. Zmorovič [7]. Note that Tepper's method is not workable for the present generalization. We need the following result in proving of our theorem which I am stating as a lemma in the next section. I shall provide the proof of this, since it is of interest in its ownright.

1. <u>Lemma.</u> Let f be in $S_0(\rho,\beta)$, i.e., f is starlike of order β, $0 < \beta < 1$, in D with $\rho = \frac{1}{2} f''(0) \geq 0$. Further, if we write

$$q(z) = zf'(z)/f(z) \tag{1.1}$$

then we have

$$\left|zq'(z) - \{q(z)-1\}\{q(z)+(1-2\beta)\}/2(1-\beta)\right| < \frac{(A^2-B^2)}{2(1-\beta)}, \tag{1.2}$$

where

$$A = \frac{2(1-\beta)r}{(1-r^2)}, \quad |z| = r, \tag{1.3}$$

$$B = |q(z) - a| < A \tag{1.4}$$

and

$$a = \frac{\{1+(1-2\beta)r^2\}}{(1-r^2)}. \tag{1.5}$$

Also if

$$a_1 = \frac{\{1 + (1-2\beta)c^2r^2\}}{\{1 - c^2r^2\}}, \tag{1.6}$$

$$d_1 = \frac{2(1-\beta)cr}{\{1 - c^2r^2\}} \tag{1.7}$$

and

$$c = \frac{(2r + b)}{(2 + br)}, \quad b = \frac{\rho}{(1-\beta)}, \quad \rho > 0 \tag{1.8}$$

then we have

$$|q(z) - a_1| < d_1. \tag{1.9}$$

Further, if

$$a_2 = \frac{\{1 + 2(1-\beta)t^2r^2 - 2tr - 2t(1-2\beta)r^3 + (1-2\beta)r^4\}}{\{(1-r^2)(1-2tr + r^2)\}}, \tag{1.10}$$

$$d_2 = \frac{\{2(1-\beta)(t-r)(1-tr)r\}}{\{(1-r^2)(1-2tr+r^2)\}}, \tag{1.11}$$

$$d_3 = \frac{\{2(1-\beta)(t+r)(1+tr)r\}}{\{1 + 2tr - 2tr^3 - r^4\}}, \tag{1.12}$$

$$a_3 = \frac{\{1 + 2tr + 2(1-\beta)t^2r^2 + 2(1-2\beta)tr^3 + (1-2\beta)r^4\}}{\{(1-r^2)(1+2tr+r^2)\}} \tag{1.13}$$

and

$$t = \frac{b}{2} = \frac{\rho}{2(1-\beta)}, \quad 0 < \beta < 1, \quad \rho > 0 \tag{1.14}$$

then we have

(1.15) $$|q(z) - a_2| > d_2$$

and

(1.16) $$|q(z) - a_3| < d_3.$$

P r o o f. Since f is in $S_0(\rho,\beta)$, $\mathrm{Re}\{zf'(z)/f(z)\} > \beta$ $0 \leq \beta < 1$ for $z \in D$. There exists a function $w(z)$ with $w(0) = 0$, $|w(z)| \leq |z| < 1$ regular in D such that

(1.17) $$\frac{zf'(z)}{f(z)} = \beta + \frac{\{(1-\beta)(1+w(z))\}}{\{1-w(z)\}}, \; z \in D.$$

With our notations of lemma we have

(1.18) $$q(z) = \frac{\{1+(1-2\beta)w(z)\}}{\{1-w(z)\}} .$$

Now we show that the inequality (1.2) is equivalent to

(1.19) $$|zw'(z) - w(z)| < \frac{(r^2 - |w(z)|^2)}{(1-r^2)}$$

for $z \in D$ and $|z| = r$. Since the assertion (1.19) is well--known, we need only to show the equivalence of (1.2) and (1.19). Differentiating $q(z)$ we get

(1.20) $$q'(z) = \frac{2(1-\beta)w'(z)}{\{1-w(z)\}^2} .$$

We have also

(1.21) $$q(z) + (1 - 2\beta) = 2(1-\beta)/\{1 - w(z)\}$$

and

$$q(z) - 1 = \frac{2(1-\beta)w(z)}{1-w(z)} .$$

Using equations (1.20), (1.21) and (1.22) we get

$$(1.23)\qquad zq'(z) - \left[\{q(z)-1\}\{q(z)+(1-2\beta)\}\right]/2(1-\beta) = \\ = 2(1-\beta)\{zw'(z)-w(z)\}/\{1-\ (z)\}^2.$$

Further we have

$$(1.24)\qquad q(z) - a = -2(1-\beta)\{r^2 - w(z)\}/\{(1-r^2)[1-w(z)]\}$$

$$(1.25)\qquad B^2 = 4(1-\beta)^2\{|r^2-w(z)|\}^2/\{(1-r^2)^2\{|1-w(z)|^2\}\ .$$

By using (1.3), (1.24) and (1.25) we get

$$(1.26)\qquad \frac{(A^2-B^2)(1-r^2)^2}{4(1-\beta)^2} = \frac{(1-r^2)(r^2-|w(z)|^2)}{|1-w(z)|^2}\ .$$

Equations (1.23) and (1.26) then substituted in (1.2) yield (1.19). This completes the proof of (1.2).

Now, we verify (1.9). We have

$$(1.27)\qquad q(z) - a_1 = \frac{\{-2(1-\beta)c^2r^2 + 2(1-\beta)w(z)\}}{\{(1-c^2r^2)[1-w(z)]}\ .$$

Using (1.27) in (1.9) we get the inequality

$$(1.27a)\qquad |q(z)-a_1| = \frac{\{2(1-\beta)\,|(c^2r^2 - w(z))|}{|(1-c^2r^2)(1-w(z))|} \leqslant \frac{2(1-\beta)cr}{(1-c^2r^2)}$$

which is equivalent to

$$(1.28)\qquad |c^2r^2 - w(z)| \leqslant cr|1-w(z)|.$$

But inequality (1.28) in turn is equivalent to

$$(1.29)\qquad (1-c^2r^2)(c^2r^2-|w(z)|^2)\geqslant 0.$$

Also, we observe that

$$(1.30) \qquad 1-c^2r^2 > (1-c^2) = \frac{(2-b)(1-r)(1+r)(2+b)}{(2+br)^2} > 0$$

since $0 \leq b = \frac{\varrho}{(1-\beta)} \leq 2$ for starlike functions of order β. Thus (1.28) to hold, we must have

$$|w(z)| \leq cr = \frac{r(b+2r)}{(2+br)}.$$

But the last assertion is well-known for the bounded functions satisfying the conditions of Schwarz's lemma. This completes the proof of (1.9).

Now the proofs of (1.15) and (1.16) are exactly similar. Thus, for completeness we prove (1.16) and leave (1.15) for the reader to verify. By substituting the value of a_3 we have

$$(1.31) \qquad |q(z)-a_3| = \frac{2(1-\beta)\left|r^2(t+r)^2-(1+tr)^2w(z)\right|}{\left|[1-w(z)]\,[1+2tr-2tr^3-r^4]\right|}.$$

Therefore the inequality $|q(z)-a_3| \leq d_3$ is equivalent to

$$(1.32) \qquad \left|\frac{r^2(t+r)^2-(1+tr)^2w(z)}{1-w(z)}\right| \leq (t+r)(1+tr)r.$$

The inequality (1.32) holds if and only if

$$0 \leq \left[(1+tr)^2-r^2(t+r)^2\right]\left[r^2(t+r)^2-(1+tr)^2|w(z)|^2\right].$$

But $(1+tr)^2-r^2(t+r)^2 = (1-r^2)(1+2tr+r^2) \geq 0$ for $r<1$ and $0 \leq t = \frac{\varrho}{2(1-\beta)}$. Therefore (1.32) holds true if and only if $r^2(t+r)^2-(1+tr)^2|w(z)|^2 \geq 0$ or if

$$|w(z)| \leq \frac{r(t+r)}{1+tr} = \frac{[2(1-\beta)r+\varrho]\,r}{2(1-\beta)+\varrho r}.$$

The last assertion is the generalised Schwarz's lemma for bounded functions and is well known (see [5], p.107). Thus the proof is finished.

R e m a r k. The inequality (1.16) gives us

$$(1.33)\qquad a_3 - d_3 = \frac{1 + 2t\beta r - (1-2\beta)r^2}{1 + 2tr + r^2} \leqslant$$

$$\leqslant \operatorname{Re} \frac{zf'(z)}{f(z)} \leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant$$

$$\leqslant a_3 + d_3 = \frac{1+2(1-\beta)tr + (1-2\beta)r^2}{(1-r^2)} .$$

The inequality (1.33) gives us the basic distortion theorem of David E. Tepper [2].

2. T h e o r e m. Having proved our lemma, now, we are set to prove our desired theorem. The theorem which we are going to prove generalizes a result of David E. Tepper for α-starlike functions of order β to the range $0 < \beta < \beta_0(\alpha,\beta) = \beta_0$. The positive number β_0 is the smallest root of the equation

$$(2.1)\qquad \alpha w^2 - 2\beta(2+\alpha-2\beta)w + \alpha\beta^2 = 0 ,$$

where

$$w = \frac{N(\beta)}{D(\beta)}$$

and

$$N(\beta) = 8\beta^2(1-\beta)(1+\alpha-2\beta)l_3 - \alpha^2(1-\beta)^2 l_3 + 4m_3\alpha\beta(1-\beta) -$$

$$- 2\beta l_2^2(2+\alpha-2\beta) - 2l_2 m_2 \alpha ,$$

$$D(\beta) = m_2^2\alpha - \alpha l_2^2\beta^2 + 4\alpha\beta^3(1-\beta)l_3 + \alpha^2(1-\beta)^2 m_3$$

and

$$l_3 = 2l_1^2\beta(2+\alpha-2\beta) + 2l_1 m_1\alpha ,$$

$$m_3 = \alpha m_1^2 - \alpha\beta^2 l_1^2,$$

$$l_2 = l - 32\beta^2(1-\beta)(2+\alpha-2\beta),$$

$$m_2 = m + 16\alpha\beta^3(1-\beta),$$

where

$$l_1 = 2(1+\alpha-\beta)(4t\beta-4\beta) - \alpha(4\beta-2\beta t) - \alpha\beta(-4+2t) + $$
$$+ 2\beta(-4+2t)(2+\alpha-2\beta),$$

$$m_1 = 2(1+\alpha-\beta)(-4+8\beta-4t\beta^2) - \alpha\beta(4\beta-2\beta t) -$$
$$- (-4+2t)\alpha\beta^2,$$

$$l = 2(1-t)(4\beta^2+2\alpha\beta^2-4\beta^3)(2+\alpha-2\beta-\alpha) -$$
$$- 2(1-t)(-\alpha\beta^2+2\alpha\beta-\alpha) + 2\beta(2-2t)(2\beta+2\alpha\beta -$$
$$- 2\beta^2-\alpha\beta)(2+\alpha-2\beta) + \alpha^2\beta^2(2-2t) +$$
$$+ \alpha(4-8\beta+4t\beta^2)(2+2\alpha-2\beta) - \alpha^2\beta^2(-4+4t) -$$
$$- \alpha^2(2-4\beta+4\beta^2+2t-4t\beta),$$

$$m = (-\alpha\beta^2+2\alpha\beta-\alpha)(2-2t)(2\beta-2\beta^2+\alpha\beta) -$$
$$- \alpha\beta^2(2-2t)(2\beta+2\alpha\beta-\beta^2-\alpha\beta) +$$
$$+ \alpha(-2\beta+4\beta^2+2t\beta-4t\beta^2)(2+2\alpha-2\beta) -$$
$$- \alpha\beta(2-4\beta+4\beta^2+2t-4t\beta),$$

$$t = \rho/2(1-\beta).$$

For our convenience we also denote $\beta_1 = \beta_1(\alpha,\beta)$ to be the smallest positive root of the equation

$$\alpha V^2 - 2\beta(2+\alpha-2\beta)V + \alpha\beta^2 = 0, \tag{2.2}$$

where

$$V = \frac{N_1(\beta)}{D_1(\beta)}$$

and

$$N_1(\beta) = -4(2+\alpha^2)\beta^3 + 4(5+4\alpha+\alpha^2+\alpha^3)\beta^2 - 8(2+3\alpha+\alpha^2)\beta + 4(1+2\alpha+\alpha^2),$$

$$D_1(\beta) = 8\alpha\beta^3 - 8(2\alpha+\alpha^2)\beta^2 + 8(\alpha+\alpha^2)\beta.$$

It is comparatively easier to find the smallest positive root from the equation (2.2) than from (2.1). But, if $\alpha = 1$ and $0 < \beta < 1$ then it is easy to see that both β_0 and β_1 belong to the interval $\left[\frac{1}{2}, 1\right)$. This therefore implies that Tepper's result is generalised to the case $0 < \beta < \beta_0$ $(= \frac{1}{2}$ say). This, however, will become clear from the following theorem, which I am intending to prove in this paper.

Theorem. Let $f(z)$ be a function in $S_0(\varrho,\beta)$ and $0 < \varrho < 2(1-\beta)$ for $z \in D$. Let $r(\alpha,\beta)$ where $\alpha > 0$ be the radius of the largest disk in which

$$\mathrm{Re}\left[\alpha(1+zf''(z)/f'(z) + (1-\alpha)zf'(z)/f(z)\right] > 0.$$

If α, β, ϱ satisfy the following conditions

$$\alpha a^2 - 2(1+2\alpha-2\beta)a\beta + \alpha\beta^2 = 0. \tag{2.3}$$

$$(1-\beta)^2 - \varrho(1-\beta)(\alpha-\alpha\beta-2\beta)r + (\varrho^2\beta^2 - 2(1-\beta)^2\{(1-2\beta) + 2\alpha(1-\beta)\})r^2 - \varrho(1-\beta)(\alpha+2\beta-4\beta^2-\alpha\beta)r^3 + (1-\beta)^2(1-2\beta)^2r^4 = 0 \tag{2.4}$$

(2.5) $$1 - 2(1+\alpha-2\beta-\alpha\beta)r + (1-2\beta)^2r^2 = 0$$

and

$$a = \frac{1 + (1-2\beta)r^2}{1 - r^2},$$

then $r(\alpha,\beta)$ is the smallest positive root of the equation (2.3) for $\beta_1 < \beta < 1$, the smallest positive root of the equation (2.4) for $0 \leq \beta < \beta_0(\alpha,\beta)$, and the smallest positive root of the equation (2.5) for $0 \leq \beta < \beta_1(\alpha,\beta)$, where $\beta_0(\alpha,\beta)$ and $\beta_1(\alpha,\beta)$ are the smallest positive roots of the equations (2.1) and (2.2) respectively.

Let $\alpha < 0$ and

(2.1)' $$(-1-\alpha) + (\alpha-3+4\beta)r + (\alpha-2\alpha\beta-\beta^2+8\beta-3)r^2 - (1-2\beta)(1+\alpha-2\beta)r^3 = 0.$$

(2.2)' $$(-1-\alpha) - 2\varrho r + \{\alpha+\alpha(1-2\beta) - (2-4\beta) - \varrho^2\}r^2 - 2\varrho(1-2\beta)r^3 + \{-\alpha(1-2\beta) - (1-2\beta)^2\}r^4 = 0.$$

(2.3)' $$1 + 2(1+\alpha-\alpha\beta-2\beta)r + (1-2\beta)^2r^2 = 0$$

(2.4)' $$\alpha(1-\alpha+\alpha^2) + \{2\alpha^2-2\alpha(1-\alpha)^2(1-\beta)\}r^2 + \{\alpha^2 + \alpha(1-\alpha)^2(1-2\beta)\}r^4 = 0$$

(2.5)' $$1 + \varrho(\alpha+\beta)r + \{\varrho^2 + 2\alpha + 2(1+\alpha)(1-2\beta)\}r^2 + \{\alpha\varrho + 2\varrho(1-2\beta)\}r^3 + (1-2\beta)^2r^4 = 0.$$

If y' and h' are the smallest positive roots of the equations (2.1)' and (2.2)' respectively, then $r(\alpha,\beta)$ is the smallest positive root of the equation (2.5)' for

$r(\alpha,\beta) < h'$ and the smallest positive root of the equation (2.3)' for $r(\alpha,\beta) < y'$. Otherwise $r(\alpha,\beta)$ is the smallest positive root of the equation (2.4)'.

P r o o f. Since f is in $S_0(\varrho,\beta), 0 \leq \beta < 1$, we have some $w(z)$ for which

$$\frac{zf'(z)}{f(z)} = \frac{1+(1-2\beta)w(z)}{1-w(z)}, \quad z \in D \tag{2.6}$$

and $w(0) = 0$, $|w(z)| < 1$ in D.

Let us write $q(z)$ for $zf'(z)/f(z)$ and also write

$$J(f) = \mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1+\frac{zf''(z)}{f'(z)}\right]\right\}. \tag{2.7}$$

Then from the inequality (1.2) we get

$$\mathrm{Re}\left[J(f)\right] = J(f) \geq \begin{cases} M(q) & \text{for } \alpha \geq 0 \\ H(q) & \text{for } \alpha \leq 0, \end{cases} \tag{2.8}$$

where

$$M(q) = \mathrm{Re}\left\{q(z) + \frac{\alpha\left[q(z)-1\right]\left[q(z)+1-2\beta\right]}{2(1-\beta)q(z)}\right\} - \frac{\alpha(A^2-B^2)}{2(1-\beta)|q(z)|} \tag{2.9}$$

and

$$H(q) = \mathrm{Re}\left\{q(z) + \frac{\alpha\left[q(z)-1\right]\left[q(z)+1-2\beta\right]}{2(1-\beta)q(z)}\right\} + \frac{\alpha\{A^2-B^2\}}{2(1-\beta)|q(z)|}. \tag{2.10}$$

The case $\alpha \geq 0$. Now let $\alpha \geq 0$. Writing $q(z) = a+x+iy$ and $M(x,y)$ for $M(q)$ we have

$$M(x,y) = a+x+\alpha(2-2\beta)^{-1}\left[a+x-2\beta+(2\beta-1)(a+x)R_1^{-2} - (A^2-x^2-y^2)R_1^{-1}\right], \tag{2.11}$$

where

$$R_1 = |q(z)| = \sqrt{(a+x)^2+y^2}.$$

From (2.11) we have

(2.12) $$\frac{\partial M(x,y)}{\partial y} = \frac{-2(2\beta-1)(a+x)\alpha}{2(1-\beta)R_1^3}\frac{\partial R_1}{\partial y} + \frac{2y\alpha}{2(1-\beta)R_1} +$$

$$+ \frac{\alpha(A^2-x^2-y^2)}{2(1-\beta)R_1^2}\frac{\partial R_1}{\partial y} =$$

$$= \frac{\alpha y}{2(1-\beta)R_1^4}\left[2R_1^3 + (A^2-x^2-y^2)R_1 + 2(1-2\beta)(a+x)\right] =$$

$$= \alpha y R_1^{-4}\, W(R_1,x,y)/2(1-\beta),$$

where

(2.13) $$W(R_1,x,y) = 2R_1^3 + (A^2-x^2-y^2)R_1 + 2(1-2\beta)(a+x) > 2R_1^3 + 2(1-2\beta)(a+x) >$$

$$\geqslant 2(a+x)(R_1^2 + 1-2\beta) > 2(a+x)\left\{(A-a)^2 + (1-2\beta)\right\} =$$

$$= 4(1-\beta)(a+x)(1+r)^{-2}\left[1 + (1-2\beta)r^2\right] > 0 .$$

Since $M(x,y)$ is symmetric in y, it follows that

(2.14) $$\min_y M(x,y) = M(x,0) = M(a+x).$$

Now, if we write $a+x = R$, we get

(2.15) $$M(R) = R + \frac{\alpha\left[R-2\beta + (2\beta-1)R^{-1} - (A^2-x^2)R^{-1}\right]}{2(1-\beta)} .$$

But

(2.16) $$A^2 - x^2 = A^2 - a^2 - R^2 + 2aR$$

and

$$a^2 - A^2 = \frac{1-(1-2\beta)^2 r^2}{(1-r^2)}, \tag{2.17}$$

so we have

$$a^2 - A^2 + 2\beta - 1 = \frac{2\beta\left[1+(1-2\beta)r^2\right]}{(1-r^2)} = 2a\beta. \tag{2.18}$$

Therefore, from (2.15) and (2.18) we have

$$M(R) = \left[1+(1-\beta)^{-1}\alpha\right]R + \alpha(1-\beta)^{-1}(a\beta R^{-1}-a-\beta). \tag{2.19}$$

Thus, from (2.19) we find that $M(R)$ attains its absolute minimum for

$$R = R_o = \left(\frac{a\alpha\beta}{1+\alpha-\beta}\right)^{\frac{1}{2}}. \tag{2.20}$$

Since

$$R_o^2 = a\alpha\beta/(1+\alpha-\beta) \leqslant a < a + A < (a+A)^2 \tag{2.21}$$

it follows that $R_o \leqslant (a+A)$. Thus, either R_o lies in the interval $[a-A,\ a+A]$ or not. If R_o lies in the interval $[a-A,\ a+A]$ then $M(R)$ attains its minimum at $R = R_o$ while if R_o lies outside this interval then $M(R)$ attains its minimum at $R = a-A$. Hence we have

$$\min_R M(R) = \begin{cases} M(R_o) & \text{if } R_o \in [a-A,\ a+A] \\ M(a-A) & \text{otherwise.} \end{cases} \tag{2.22}$$

The radius of α-convexity $r(\alpha,\beta)$ is therefore determined either from the equation

$$M(R_o) = 0, \tag{2.23}$$

where R_0 is given by (2.20), or from the equation

$$M(R_2) = 0, \tag{2.24}$$

where

$$R_2 = a - A = \frac{1 - (1-2\beta)r}{1 + r}. \tag{2.25}$$

These two equations coincide for some $\beta_0 = \beta_0(\alpha,\beta)$. Now, we are interested in determining the transition value of β_0 from (2.24) to (2.23). For this aim, we assume that

$$R_0 = R_2 \quad \text{or} \quad a\alpha\beta(1+\alpha-\beta)^{-1} = R_2^2. \tag{2.26}$$

It follows then from (2.24), (2.19) and (2.26) that

$$\left[1+\alpha(1-\beta)^{-1}\right]R_2 = \left[a r\beta-(1+\alpha-\beta)R_2/\alpha\right]\left[\alpha/(1-\beta)\right] \tag{2.27}$$

or

$$2(1+\alpha-\beta)R_2 = \alpha(\alpha+\beta). \tag{2.28}$$

Substituting the value of $R_2 = a-A$ into (2.28), we have

$$2(1+\alpha-\beta)A = (2+\alpha-2\beta)a-\alpha\beta. \tag{2.29}$$

Now squaring (2.29) and substituting the value of A^2 into the resulting equation, we get

$$(2+\alpha-2\beta)^2a^2 - 2\alpha\beta(2+\alpha-2\beta)a + \alpha^2\beta^2 = \tag{2.30}$$

$$= 4(1+\alpha-\beta)^2(a^2-2a\beta-1+2\beta).$$

Equation (2.30) can be written in the form

$$A_1a^2 + B_1a + C_1 = 0 \tag{2.31}$$

where

$$(2.32) \qquad A_1 = \alpha(3\alpha-4\beta+4),$$

$$(2.33) \qquad B_1 = -8\beta(1+\alpha-\beta)^2 + 2\alpha\beta(2+\alpha-2\beta),$$

$$(2.34) \qquad C_1 = -4(1-2\beta)(1+\alpha-\beta)^2 - \alpha^2\beta^2.$$

Also equation (2.23) is equivalent to

$$(2.35) \qquad a^2\alpha^2 - 2\alpha\beta(2+\alpha-2\beta)a + \alpha^2\beta^2 = 0.$$

Solving (2.35) for a we obtain

$$(2.36) \qquad a = \frac{2\alpha\beta(2+\alpha-2\beta) \pm \sqrt{4\alpha^2\beta^2(2+\alpha-2\beta)^2 - 4\alpha^4\beta^2}}{2\alpha^2} =$$

$$= \frac{\beta}{\alpha}\left[(2+\alpha-2\beta) \pm 2\sqrt{(1-\beta)(1+\alpha-\beta)}\right].$$

Now, if a is taken with the negative value, then solving (2.36) we get

$$r(\alpha,\beta) = \left[\frac{\{\beta(2+\alpha-2\beta) - 2\{\beta(1-\beta)(1+\alpha-\beta)\}^{1/2} - \alpha\}}{\{\beta(2+\alpha-2\beta) - 2\{\beta(1-\beta)(1+\alpha-\beta)\}^{1/2} + \alpha(1-2\beta)\}}\right]^{\frac{1}{2}}.$$

Thus $f \in S_0(\rho,\beta)$ is α - convex in $|z| < r(\alpha,\beta)$ if $\beta_0 \leq \beta < 1$, where β_0 is the smallest positive root of the equation obtained by elimination of a from (2.31) and (2.35). Clearly, $r(\alpha,\beta)$ shows that when $0 < \beta < 1$ and $\beta < \alpha$ then is imaginary. Hence in this case the convexity region is determined by $M(R_2) = 0$. Again, considering the equation (2.35) and letting $\alpha > 0$ we have

$$(2.37) \qquad a^2 = \frac{2\beta(2+\alpha-2\beta)a - \alpha\beta^2}{\alpha}.$$

Equations (2.31) and (2.37) gives us

$$(2.38) \qquad a = \frac{A_1 \alpha\beta^2 - C_1\alpha}{2\beta(2+\alpha-2\beta)A_1 + B_1\alpha} = \frac{N_1(\beta)}{D_1(\beta)},$$

where $N_1(\beta)$ and $D_1(\beta)$ are the same expressions as stated in the theorem. Using this value of a in the expression (2.35) we get the desired equation. The smallest positive root of this equation gives the transition value β_0 for the two convexity regions stated in the theorem. In case when β_0 does not exist, $r(\alpha,\beta)$ is determined by the equation (2.24).

In the present paper our main emphasis is on the fact how the second coefficient influences the radius of convexity or α- convexity of $f(z)$.

Therefore we leave to reader all the lengthy details to verify, if the radius of α- convexity is not influenced by the second coefficient of $f(z)$. In the previous analysis we noted that α - convexity depends on the transition value β_0 and on the region of variability of $q(z)$ given by the equation (1.4). Since region of variability of $q(z)$ implies that R_0 may take any value between a-A and a+A, then transition value β_0 depends on $R = |q(z)|$. We also noticed two facts in the above analysis. First, the minimum of $M(x,y)$ occurs at $y=0$ i.e., on the real axis of the region of variability of $q(z)$. Secondly the absolute minimum of $M(R)$ is obtained at R_0 which lies below a+A and may be above or below of a-A. In the circumstances when we start investigating the minimum of $M(x,y)$ and $M(R)$ depending on the second coefficient in the expansion of $f(z)$ we find that the region of variability of $q(z)$ given by the inequality (1.16) plays the role. However in this case we have

$$(2.39) \qquad a-A < a_3-A_3 < |q(z)| < a_3+A_3 < a+A.$$

Hence, as before, we find again that the minimum of $M(x,y)$ occurs on the real axis $y=0$ but R lies between

a_3-A_3 and a_3+A_3. Investigations yield that the absolute minimum of $M(R)$ is for $R = R_0$, where R_0 is again given by the equation (2.20). In this case there are several possibilities for R_0. R_0 may be less than a_3+A_3 or a_3-A_3 or a-A. So in this case we are struck to the situations $R_0 < a_3-A_3$, $R_0 < a_3+A_3$ and $a_3+A_3 < R_0 < a+A$. Now, if $R_0 \leq a_3-A_3$ then minimum of $M(R)$ is obtained when

$$(2.40) \qquad M(R_3) = 0 \quad \text{i.e. for} \quad R_3 = a_3-A_3.$$

Hence the radius of α - convexity is obtained from the equation (2.40). On substituting the value of R_3 into (2.40) we get the equation (2.4).

Now, we determine how $r(\alpha,\beta)$ is obtained from the equation (2.4). For this aim we need to calculate the transition value β_1 which determines which of the equations (2.4) or (2.3) determine the α - convexity for $f(z)$. Unfortunately, the result (2.3) does not depend on the second coefficient of $f(z)$. Because

$$(2.41) \qquad R_3 = a_3-A_3$$

we compute the transitional value β_1 in the following way.

From (2.41), (2.23) and (2.19) we have

$$(2.42) \qquad 2(1+\alpha-\beta)R_3 = \alpha(a+\beta).$$

From equation (2.42) we get

$$(2.43) \qquad 2(1+\alpha-\beta)a_3 - \alpha(a+\beta) = 2(1+\alpha-\beta)d_3.$$

Also, since

$$(2.44) \qquad a - A = \frac{1-(1-2\beta)r}{1+r}$$

we have

$$(2.45) \qquad r = \frac{1+A-a}{a+1-2\beta-A}.$$

Substituting this value into a_3-d_3 we get

$$(2.46)\qquad a_3 - d_3 = \frac{N_2}{D_2},$$

where

$$N_2 = (a-A+1-2\beta)^2 + 2t\beta(1+A-a)(a+1-2\beta-A) - (1-2\beta)(1+A-a)^2,$$

$$D_2 = (a+1-A-2\beta)^2 + 2t(1+A-a)(a+1-2\beta-A) + (1+A-a)^2.$$

Now from (2.44) and (2.46) we get

$$(2.47)\qquad 2(1+\alpha-\beta)\{(a-A+1-2\beta)^2 + 2t\beta(1+A-a)(a+1-2\beta-A) - - (1-2\beta)(1+A-a)^2\} = = \alpha(a+\beta)\left[(a+1-2\beta-A)^2+2t(1+A-a)(a+1-2\beta-A)+(1+A-a)^2\right].$$

Equation (2.47) on simplifying and by means of (2.32) gives us

$$(2.48)\qquad -16\alpha\beta(1-\beta)a^2 + al + m + A\alpha\{(2+2\alpha-2\beta)(-4+8\beta+4at\beta - -4t\beta^2 - 4\alpha\beta) - \alpha(a+\beta)(-4a+4\beta+2at - 2\beta t)\} = 0,$$

where l and m are defined as in the theorem. On further simplifying of the equation (2.48) we have

$$(2.49)\qquad -(al_2 + m_2) = \alpha A(al_1 + m_1).$$

Squaring (2.49) we get

$$(2.50)\qquad \alpha^2A^2\{a[2l_1^2\beta(2+\alpha-2\beta) + 2l_1m_1\alpha] + \alpha m_1^2 - \alpha\beta^2l_1^2\} = = a\{2\beta l_2^2(2+\alpha-2\beta) + 2l_2m_2\alpha\} + m_2^2\alpha - \alpha l_2^2\beta^2.$$

Now substituting the values of the expressions A^2 and a^2 from the equations (2.18) and (2.35), we get

(2.51) $$a = N(\beta)/D(\beta),$$

where $N(\beta)$ and $D(\beta)$ are defined in the statement of the equation (2.1). Now inserting the value of a from equation (2.51) into equation (2.35) or identifying the two values of a given by (2.36) and (2.51), we obtain an equation in β. The smallest positive root (say β_1) of this equation will be the required transition value. As far as dependence of our result on the second coefficient of $f(z)$ is concerned, we observe that our result for the range $0 < \beta \leq \beta_1$ gives the α - convexity influenced by it but not in the range $\beta_1 \leq \beta \leq 1$. Thus, the present method is not sufficiently strong to yield the result in the needed form. Now, we dispose off the other case.

The case $\alpha < 0$. The situation in this case is comparatively easy but similar as before. From equations (2.4) and (2.5) we have

(2.52) $$\operatorname{Re}\left[J(f)\right] > H(q) = $$
$$= \operatorname{Re}\left[q(z) + \alpha(q(z)-1)(q(z)+1-2\beta)/2(1-\beta)q(z) + \right.$$
$$\left. + \alpha\left\{(A^2 - B^2)/2(1-\beta)\,|q(z)|\right]\right. .$$

Now putting in (2.52)

$$q(z) = R\cos x + i\, R\sin x, \quad 0 < x < 2\pi$$

we get

(2.53) $$H(q) = \left[R + \alpha\frac{R^2 + (2\beta-1)}{2R(1-\beta)}\right]\cos x -$$
$$- \frac{2\alpha\beta}{2(1-\beta)} + \frac{\alpha A^2 + B^2}{2(1-\beta)R}\,.$$

Further, if we denote $H(q)$ by $H(R, x)$ then (2.53) can be written in the form

$$(2.54)\qquad 2(1-\beta)H(R,x) =$$

$$= \left[(2+\alpha-2\beta)R + \frac{\alpha(2\beta-1)}{R} + 2a\alpha\right]\cos x + \alpha\frac{A^2-a^2-R^2}{R} - 2\alpha\beta =$$

$$= C(R)\cos x + D(R),$$

where

$$(2.55)\qquad \begin{cases} C(R) = (2+\alpha-2\beta)R + \alpha\dfrac{(2\beta-1)}{R} + 2a\alpha, \\ D(R) = \alpha\left(\dfrac{A^2-a^2-R^2}{R}\right) - 2\alpha\beta. \end{cases}$$

Of course $C(R) < 0$ if $\frac{1}{2} \leqslant \beta < 1$ and $\alpha \leqslant -2(1-\beta)$ but for $0 \leqslant \beta < \frac{1}{2}$ and $\alpha \leqslant -2(1-\beta)$, we see that

$$(2.56)\qquad C(R) - (2+\alpha-2\beta)R = \alpha\left(\frac{2\beta-1}{R} + 2a\right) \leqslant$$

$$2a\alpha + \frac{\alpha(2\beta-1)}{a-A} = \frac{N_3(r)}{D_3(r)},$$

where

$$(2.57)\qquad \begin{cases} N_3(r) = \alpha\left[(1-2\beta)(1-r)^3 + 4\beta + 4\beta(1-2\beta)r^3\right], \\ D_3(r) = (1-r^2)\left[1-(1-2\beta)r\right]. \end{cases}$$

Therefore from (2.56), we have for $0 < \beta < \frac{1}{2}$ and $\alpha \leqslant 0$ $C(R) - (2+\alpha-2\ )R \leqslant 0$ and so $C(R) \leqslant 0$ for $\alpha \leqslant -2+2\beta$.
Now we have

$$(2.58)\qquad \frac{\partial H(R,x)}{\partial x} = -C(R)\sin x = 0 \quad \text{for} \quad x = 0, \pi$$

and

$$(2.59)\qquad \frac{\partial^2 H(R,x)}{\partial x^2} = -C(R)\cos x \begin{cases} \geq 0 & \text{for } x = 0 \\ < 0 & \text{for } x = \pi. \end{cases}$$

Therefore equations (2.58) and (2.59) give us for $0 \leq x < 2\pi$

$$(2.60)\qquad \min H(R,x) = H(R,0) \equiv H(R) \quad \text{(say)}.$$

From equation (2.60) we have

$$(2.61)\quad \frac{\partial H(R)}{\partial R} = 1 + \alpha\,\frac{(a^2 - A^2 + 1 - 2\beta)}{2(1-\beta)R^2} = \frac{(1-r^2)R^2 + \{1-(1-2\beta)r^2\}\alpha}{(1-r^2)R^2}.$$

From equation (2.61) we have

$$(2.62)\qquad R^2\,\frac{\partial H(R)}{\partial R} = R^2 + \alpha\left\{\frac{1-(1-2\beta)r^2}{1-r^2}\right\} <$$

$$< \frac{\{1+(1-2\beta)r\}^2(1+r) + \alpha\{1-(1-2\beta)r^2\}(1-r)}{(1-r^2)(1-r)} = \frac{N_5(r)}{D_5(r)},$$

where

$$(2.63)\begin{cases} N_5(r) = (1+\alpha) + (3-4\beta-\alpha)r + (3-8\beta+\beta^2-\alpha+2\alpha\beta)r^2 + (1-2\beta)(1+\alpha-2\beta)r^3, \\ D_5(r) = 1 - r - r^2 + r^3; \end{cases}$$

Let y' be the smallest positive root of the equation $(2.1)'$ in the statement of the theorem. Now, if $r < y'$ then from the equations (2.62) and (2.63) we get $\partial H(R)/\partial R < 0$ showing that $H(R)$ is a decreasing function of R. Therefore for $r < y'$ and $R \in [a-A, a+A]$

$$(2.64)\qquad \min H(R) = H(a+A).$$

In case $r < y'$ the absolute minimum of $H(R)$ is obtained at R_0', where

$$R_0' = \left[\frac{-\alpha\{1-(1-2\beta)r^2\}}{1-r^2}\right]^{\frac{1}{2}} . \tag{2.65}$$

Thus, if $\alpha < -2(1-\beta)$ and $r(\alpha,\beta)$ is the smallest positive root of the equation $H(a+d) = 0$, then $f(z)$ is α-convex in $|z| \leqslant r(\alpha,\beta)$ for $r(\alpha,\beta) < y'$.

If $r(\alpha,\beta) < y'$ then α-convexity of $f(z)$ is given by the region $|z| < r'(\alpha,\beta)$, where $r'(\alpha,\beta)$ is the smallest positive root of the equation $H(R_0') = 0$.

Therefore in this case we have as well the transitional value of α which determine which of the equation determine the -convexity region. This transitional value of α can be obtained by eliminating r from the equations $H(a+d) = 0$ and $N_5(r) = 0$. Let α_0 be the desired transitional value, i.e., the largest negative root of the intersection of equations $H(a+d) = 0$ and $N_5(r) = 0$ in α. We emphasise that α_0 is negative. If $\alpha_0 \geqslant \alpha$ then the radius of α-convexity, $r(\alpha,\beta)$ is obtained from the equations $H(a+d)=0$ or from (2.3)'. Otherwise the root is determined from (2.4)'. In the latter case our result is valid for the range $\alpha_0 < \alpha < -2(1-\beta)$. It will not be difficult to see that this result continue to hold for $-2(1-\beta) \leqslant \alpha \leqslant 0$. For this we refer readers to the paper [1]. Now, we return to the case when α-convexity depends on the second coeficient in the expansion of $f(z)$. In this case we find that

$$R \in [a_3-d_3,\ a_3+d_3] \quad \text{and} \quad a-A < a_3-d_3.$$

Using these two facts and starting from the equations (2.52), (2.53) and (2.54) which are also valid in this case, we find for $R \in [a_3-d_3,\ a_3+d_3]$

$$\min H(R,x) = H(R,0) = H(R) \quad \text{(say)}$$

since $C(R) < 0$ for $\alpha \leq -2(1-\beta)$ and $0 < \beta < 1$.

Now we need to determine the minimum of $H(R)$ given by (2.60) when $a_3-d_3 < R < a_3+d_3$. As before, we have

$$(2.66) \qquad R^2 \frac{\partial H(R)}{\partial R} = R^2 + \alpha \frac{[1-(1-2\beta)r^2]}{1-r^2} <$$

$$< \left[\frac{1+2(1-\beta)tr+(1-2\beta)r^2}{1-r^2}\right]^2 + \alpha\frac{[1-(1-2\beta)r^2]}{1-r^2} = \frac{N_6(r)}{D_6(r)},$$

where

$$\begin{cases} N_6(r) = (1+\alpha)+4(1-\beta)tr + \{4(1-\beta)^2t^2 + 2(1-2\beta) - \\ \qquad -\alpha-\alpha(1-2\beta)r^2\} + 4(1-\beta)(1-2\beta)tr^3 + \\ \qquad + \{(1-2\beta)^2 + \alpha(1-2\beta)\}r^4, \\ D_6(r) = 1-r^2. \end{cases}$$

Now, let h' he the smallest positive root of the equation (2.2)' stated in the theorem. If $r < h'$ we have $\partial H/\partial R \leq 0$ from (2.66). Thus, it follows that $H(R)$ is a decreasing function of R in $[a_3-d_3,\ a_3+d_3]$, so in this region

$$(2.67) \qquad \min H(R) = H(a_3+d_3).$$

If r is not always less than h' then $H(R)$ gets its absolute minimum at $R = R_0'$ given by the equation (2.65). Now, by eliminating r from the equations

$$H(a_3+d_3) = 0 \quad \text{and} \quad N_6(r) = 0$$

we obtain an equation in α. If α_0' denotes the largest negative root of the resulting equation, then for $\alpha < \alpha_0'$ $f(z)$

is α-convex in $|z| < r(\alpha,\beta)$ where $r(\alpha,\beta)$ is the smallest positive root of the equation $H(a_3+d_3) = 0$ and the α-convexity is influenced by ϱ. If $\alpha'_0 < \alpha \leq 0$ then α-convexity region $|z| < r(\alpha,\beta)$ is obtained by determining the smallest positive root $r(\alpha,\beta)$ for the equation $(2.4)'$ and is the same as before. This result is independent of ϱ. Determination of ϱ-influenced α-convexity region in this case remained unsolved.

R e m a r k s. One may notice from the proof analysis that our result is just an exploitation of the region of variability and Schwarz's inequality for bounded functions. In fact, the inequality (1.2) is the sole inequality yielding the result of the present paper, after performing the elementary operations. Thus, if one may succeed in replacing (1.2) by some other inequality which depends from the very beginning on the second coefficient of $f(z)$ then most likely the above analysis will give the result too.

3. S h a r p n e s s o f t h e t h e o r e m

The case $\alpha > 0$. We first establish the sharpness for $r(\alpha,\beta)$ when it is obtained from the equation (2.4). Let us consider the equation (2.19) and write $-z$ for z in it. Now, if there exists some function $f(z)$ for which the equality (2.4) holds, then we must have

$$(3.1)\qquad \alpha\left[1+zf''(z)/f'(z)\right] + (1-\alpha)zf'(z)/f(z) =$$
$$= \left[1+\alpha(1-\beta)^{-1}\right]R^* + \left[(1-\beta)^{-1}(a^*\beta R^{*-1} - a^* - \beta)\right],$$

where

$$(3.2)\qquad R^* = \frac{(1-\beta) - \varrho\beta z - (1-\beta)(1-2\beta)z^2}{(1-\beta) - \varrho z + (1-\beta)z^2}$$

and

$$(3.3)\qquad a^* = \frac{1+(1-2\beta)z^2}{1-z^2}.$$

Now dividing by z and simplifying the right side of (3.1) we get

$$(3.4)\quad (1-\beta)\text{R.H.S.}\left\{\frac{1}{z}\,(3.1)\right\} = (1+\alpha-\beta)I_1 + \alpha\beta I_3 - I_2 - \alpha\beta\,,$$

where

$$I_1 = \frac{(1-\beta)-\alpha\beta z-(1-\beta)(1-2\beta)z^2}{z\left[(1-\beta)-\varrho z+(1-\beta)z^2\right]} =$$

$$= \frac{1}{z} - \frac{(1-\beta)\left[2(1-\beta)z-\varrho\right]}{(1-\beta)-\varrho z+(1-\beta)z^2}\,,$$

$$I_2 = \frac{1+(1-2\beta)z^2}{z(1-z^2)} = \frac{1}{z} + \frac{(1-\beta)}{(1-z)} - \frac{(1-\beta)}{1+z}\,,$$

$$I_3 = \frac{\left[1+(1-2\beta)z^2\right]\left[(1-\beta)-\ z+(1-\beta)z^2\right]}{z(1-z)(1+z)\left[(1-\beta)-\varrho z-(1-\beta)(1-2\beta)z^2\right]} =$$

$$= \frac{1}{z} + \frac{(1-\beta)}{\beta(1-z)} - \frac{(1-\beta)}{\beta(1+z)} + \frac{-2(1-\beta)^2(1-2\beta)z-\varrho\beta(1-\beta)}{\beta\left[(1-\beta)-\varrho\beta z-(1-\beta)(1-2\beta)z^2\right]}\,.$$

From (3.1) and (3.4) we have

$$(3.5)\qquad (1-\alpha)\log\frac{f(z)}{z} + \alpha\log f'(z) =$$

$$= -(1+\alpha-\beta)\log\left[(1-\beta)-\varrho z+(1-\beta)z^2\right] +$$

$$+\alpha\log\left[(1-\beta)-\varrho\beta z-(1-\beta)(1-2\beta)z^2\right]+(1-\beta)\log(1-\beta).$$

The solution of the above differential equation will be the required extremal function if that function belongs to the class $S_0(\varrho,\beta)$. We claim that such a function is

$$(3.6)\qquad F(z) = z\left[\frac{(1-\beta)}{(1-\beta)-\varrho z+(1-\beta)z^2}\right]^{(1-\beta)}.$$

Since actual computations for this function give the following

$$(3.7)\quad \log\left\{\frac{F(z)}{z}\right\} = -(1-\beta)\log\left[(1-\beta)-\varrho z+(1-\beta)z^2\right]+(1-\beta)\log(1-\beta)$$

$$(3.8)\qquad \frac{zF'(z)}{F(z)} = \frac{(1-\beta)-\varrho\beta z-(1-\beta)(1-2\beta)z^2}{(1-\beta)-\varrho z+(1-\beta)z^2}$$

we obtain

$$(3.9)\quad \log F'(z) - \log\left\{\frac{F(z)}{z}\right\} = \log\left[(1-\beta)-\varrho\beta z-(1-\beta)(1-2\beta)z^2\right] - \log\left[(1-\beta)-\varrho z+(1-\beta)z^2\right].$$

From (3.7) and (3.9) we have

$$\alpha\log F'(z) + (1-\alpha)\log\left\{\frac{F(z)}{z}\right\} = \text{R.H.S. of } (3.5).$$

This shows clearly that $F(z)$ is the solution of the differential equation (3.5). Further we find that

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right]-\beta = \operatorname{Re}\frac{(1-\beta)^2-(1-\beta)^2z^2}{(1-\beta)-\varrho z+(1-\beta)z^2} = \frac{N_4(z)}{D_4(z)},$$

where

$$\begin{cases} N_4(z) = (1-\beta)^2(1-|z|^2)\left[(1-\beta)(1+|z|^2)-\varrho\operatorname{Re}(z)\right], \\ D_4(z) = \left|(1-\beta)-\varrho z+(1-\beta)z^2\right|^2. \end{cases}$$

But

$$N_4(z) > (1-\beta)^3(1-|z|^2)(1-|z|)^2 > 0,$$

therefore $F(z)$ belongs to the class $S_0(\varphi,\beta)$. This completes the sharpness of $r(\alpha,\beta)$ which is obtained from the equation (2.4). Similarly, we can show that

$$F(z) = \frac{z}{(1-z)^{2(1-\beta)}}, \quad 0 \leq \beta \leq \beta_1$$

is an extremal function for $r(\alpha,\beta)$ obtained from the equation (2.5). Lastly, following the analysis of Zmorovic [7] we can show that the function

$$F(z) = z\left[\frac{1}{1 - 2z\cos\theta + z}\right]^{1-\beta},$$

where θ is obtained from the equation

$$\sqrt{\frac{a\alpha\beta}{1+\alpha-\beta}} = \frac{1-2r\cos\theta-(1-2\beta)r^2}{1-2r\cos\theta+r^2}$$

and $r(\alpha,\beta)$ is the root of (2.3), is an extremal function for the α-convexity result obtained from (2.3).

The case $\alpha < 0$. If the result (2.5)' is sharp then we must have the equality in the equation (2.57) after R is replaced by a_3+d_3 and r by z. This is so if we have for some $f \in S_0(\varphi,\beta)$

$$2(1-\beta)H(R,0) \equiv 2(1-\beta)H(R) =$$

$$= 2(1-\beta)R + \frac{\alpha}{R}\left\{2\beta-1+A^2-a^2\right\} + 2\alpha(a-\beta)|,$$

where

$$a = \frac{1+(1-2\beta)z^2}{1-z^2},$$

$$A = \frac{2(1-\beta)z}{1-z^2},$$

$$R = \frac{1 + (1-2\beta)z^2 + \rho z}{1 - z^2},$$

and

$$H(R) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right].$$

After substituting these values of a, A and R, the above equation takes the form

$$(3.10) \qquad H(R) = \frac{1+\rho z + (1-2\beta)z^2}{1 - z^2} + \alpha\frac{1 + z^2}{1 - z^2} - -\alpha\frac{1 - (1-2\beta)z^2}{1 + \rho z + (1-2\beta)z^2}.$$

$$(3.11) \quad \frac{zy'(z)}{y(z)} = \frac{1+\rho z + (1-2\beta)z^2}{1 - z^2}, \quad 0<\rho<2-2\beta, \quad 0<\beta<1.$$

Now, we look for $f(z)$ which satisfy the equation (3.10). Let $f(z)$ be a solution of the differential equation (3.11).

Our claim is to prove that this $f(z)$ is the required function. Since $f(z)$ satisfies (3.11), we have

$$(3.12) \quad 1 + \frac{zf''(z)}{f(z)} = \frac{1 + z^2}{1 - z^2} + \frac{1 + \rho^2 + (1-2\beta)z^2}{1 - z^2} - -\frac{1 - (1-2\beta)z^2}{1 + \rho z + (1-2\beta)z^2}.$$

Use of (3.11) and (3.12) for $y = f$ yields (3.10). Therefore $f(z)$ given by (3.11) is an extremal function. Now, we determine the solution of (3.11). We have

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{(1-\beta+\frac{1}{2}\varrho)}{1-z} - \frac{(1-\beta-\frac{1}{2}\varrho)}{1+z}$$

and

$$f(z) = z\left[\frac{1+z}{1-z}\right]^{\frac{\varrho}{2}}\left[\frac{1}{1-z^2}\right]^{1-\beta}. \tag{3.13}$$

It is easy to see that $f \in S_0(\varrho,\beta)$. This finishes the verification in this case. Similarly we find that

$$f(z) = \frac{z}{(1+z)^{2-2\beta}} \tag{3.14}$$

is an extremal function for the result given by the equation (2.3)'.

Acknowledgements. I, the author of the present paper, greatly appreciate and thank the referee for his critical comments and for pointing out various typing mistakes in the original manuscript which finally led the paper in the present form.

REFERENCES

[1] S.K. B a j p a i: On the regions of α-convexity for starlike functions; Proc.Amer.Math.Soc. 44 (1974) 365-368.

[2] D.E. T e p p e r: On the radius of convexity and boundary distortion of Schlicht functions; Trans.Amer. Math.Soc., 150 (1970) 519-528.

[3] P.T. M o c a̧ n u: Une proprieté de convexité generaliseé dans la représentation conforme. Mathematica (Cluj) 11 (1969) 127-133.

[4] S.S. M i l l e r, P.T. M o c a n u, M.O. R e a d e: All α-convex functions are starlike. Rev.Roum.Math. Pures. Appl. 17 (1972), 1305-1307.

[5] Z. N e h a r i: Conformal Mapping, McGraw-Hill, New York. 1952.

[6] M.S. R o b e r t s o n: On the theory of univalent functions. Ann. of Math. (2) 37 (1936), 374–408.

[7] V. Z m o r o v i č: On bounds of convexity for starlike functions of order α in the circle $|z| < 1$ and the circular region $0 < |z| < 1$. Mat. Sb. 68 (1965), 518-526. English translation. Amer.Math.Soc. Translation (2) 80 (1969), 103-213.

DEPARTAMENTO DE MATEMÁTICA, C.C.E.N.
CIDADE UNIVERSITÁRIA 58.000, JOÃO PESSOA, PARAIBA, BRAZIL

Received June 15, 1976.