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Włodzimierz Łenski

ON THE NORM-STRONG APPROXIMATION OF 2π-PERIODIC FUNCTIONS

1. Notation

Let L^q $(1<q<\infty)$ be the class of all 2π-periodic, real-valued functions f whose q-th power is Lebesgue-integrable in the interval $<-\pi,\pi>$; the norm of $f\in L^q$ is defined by the formula

$$\|f\|_{L^q} = \|f(x)\|_{L^q} = \left\{\int_{-\pi}^{\pi}|f(x)|^q\,dx\right\}^{\frac{1}{q}}.$$

Consider the Fourier series

$$S[f] = \frac{a_0}{2} + \sum_{\nu=1}^{\infty}(a_\nu\cos\nu x + b_\nu\sin\nu x) \tag{1}$$

of a function $f\in L^q$. Denote by $S_k(x)$ and $\sigma_k^{\delta}(x)$ the partial sums and the Cesàro (C,δ)-means of the series (1), respectively. Let us introduce the integral modulus of continuity of $f\in L^q$:

$$\omega(\varrho)_{L^q} = \omega(\varrho;f)_{L^q} = \sup_{0<h<\varrho}\|f(x+h)-f(x)\|_{L^q}.$$

We shall deal with a regular summability method for real sequence, determined by a triangular matrix $\|\alpha_{nk}/A_n\|$ $(\alpha_{nk}\geq 0$ and $A_n=\sum_{k=0}^{n}\alpha_{nk})$, that is the condition $w_k\to w$ implies

$$\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}w_k \to w \quad \text{as} \quad n\to\infty .$$

The aim of the present paper is to estimate the quantity

$$H_n^p(f)_{L^q} = \left\{\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}\left\|\sigma_k^{\delta-1} - f\right\|_{L^q}^p\right\}^{\frac{1}{p}} \quad \text{for } \delta,\ p>0.$$

For convenience, the suitable positive constants independent of f, n and r will be denoted by C_j ($j = 1,2,3,\dots,20$).

2. Statement of results

In this Section the theorems, relative to the norm-strong approximation, will be presented (cf.[4], p.89).

T h e o r e m 1. Suppose that for a certain $\gamma>1$

$$(2)\qquad \sum_{l=0}^{j}\left\{\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_{nk}^{\gamma}\,\varphi_k^{p\gamma}}{(k+1)^{1-\gamma}}\right\}^{\frac{1}{\gamma}} < C_1 A_n\frac{1}{n+1}\sum_{k=0}^{n}\varphi_k^p \quad (n=0,1,2,\dots),$$

where j satisfies the condition $2^j < n+1 < 2^{j+1}$, $\alpha_l = 2^l-1$, $\beta_l = \min(2^{l+1}-2, n)$, and

$$\varphi_k = \begin{cases} \dfrac{1}{k+1}\displaystyle\sum_{\nu=0}^{k}\omega\left(\frac{1}{\nu+1}\right)_{L^q} & \text{if} \quad q=1,\\[2ex] \omega\left(\dfrac{1}{k+1}\right)_{L^q} & \text{if} \quad q>1,\end{cases}$$

and let $\delta>1/2$ be such that $(1-\delta)p<1-\frac{1}{\gamma}$. Then, for $n = 0,1,2,\dots,$

$$(3)\qquad H_n^p(f)_{L^q} \le \begin{cases} C_2\left(\frac{1}{n+1}\sum_{k=0}^{n}\varphi_k^p\right)^{\frac{1}{p}} & \text{if } p>q \text{ or } p<q<2, \\ C_3\left(\frac{1}{(n+1)^{p/q}}\sum_{k=0}^{n}\omega^p\left(\frac{1}{k+1}\right)_{L^q}\right)^{\frac{1}{p}} & \text{if } q>\max(2,p). \end{cases}$$

Under the assumptions of Theorem 1, the following more precise result holds.

T h e o r e m 2. (i) If $(1-\delta)\max(p,q)<1-\frac{1}{\gamma}$, then

$$(4)\qquad H_n^p(f)_{L^q} < C_4\left(\frac{1}{n+1}\sum_{k=0}^{n}\varphi_k^p\right)^{\frac{1}{p}} \qquad (n=0,1,2,\ldots).$$

(ii) For

$$\psi_k = \begin{cases} (k+1)^{\frac{1}{p}-\frac{1}{q}}\,\omega\left(\frac{1}{k+1}\right)_{L^q} & \text{if } q>\max(2,p), \\ \varphi_k & \text{if } q=1 \text{ or } q>1 \text{ and } (p<q\le 2 \text{ or } p>q) \end{cases}$$

satisfying the condition (2) in place φ_k, we have

$$(5)\qquad H_n^p(f)_{L^q} \le C_5\left(\frac{1}{n+1}\sum_{k=0}^{n}\psi_k^p\right)^{\frac{1}{p}} \qquad (n=0,1,2,\ldots).$$

R e m a r k. It can be easily observed that the condition (2) holds, for $\alpha_{nk} = A_{n-k}^{(\beta-1)}$, whenever $\beta>1-\frac{1}{\gamma}$ (by definition, $A_k^{(\beta)} = \binom{k+\beta}{k}$).

3. Preliminary lemmas

We start with the basic estimates of the Hardy-Littlewood type (cf. [3], p.78 and [6], p.150).

L e m m a 1. Let $u(r,x)$, $v(r,x)$ be the Poisson and the Poisson conjugate integrals of a function $f \in L^q$ having the integral modulus of continuity $\omega(\varrho)_{L^q}$. Write

$$(6)\quad F(z) = u(r,x) + i\, v(r,x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu - i\cdot b_\nu) z^\nu \quad (z=re^{ix},\ 0<r<1),$$

where a_ν, b_ν are the Fourier coefficients of f. Then, for $r \in (0,1)$, we have

$$(7)\qquad \left\| F'(re^{ix}) \right\|_{L^q} \leq \begin{cases} C_6 \dfrac{\omega(1-r)_{L^q}}{1-r} & \text{if} \quad q>1, \\[2ex] C_7 \dfrac{1}{r} \displaystyle\int_{1-r}^{1} y^{-2}\, \omega(y)_{L^q}\, dy & \text{if} \quad q=1. \end{cases}$$

P r o o f. Clearly,

$$\left|F'(z)\right| = \frac{1}{r}\left|\frac{\delta F(z)}{\delta x}\right| \quad \text{and} \quad \frac{\delta F(z)}{\delta x} = \frac{\delta u(r,x)}{\delta x} + i\,\frac{\delta v(r,x)}{\delta x}.$$

Evaluate first the quantity $\left\|\frac{\delta u(r,x)}{\delta x}\right\|_{L^q}$ $(1<q<\infty)$. Since

$$\int_{-\pi}^{\pi} \frac{\delta}{\delta x} P(r,x)dx = 0,$$

where $P(r,x)$ denote the Poisson kernel, we obtain

$$\frac{\delta u(r,x)}{\delta x} = \frac{r(1-r^2)}{\pi} \int_{-\pi}^{\pi} \left\{ f(x) - f(x+y) \right\} \frac{\sin y}{(1-2r\cos y + r^2)^2}\, dy.$$

By the generalized Minkowski inequality ([9], p.38),

$$\left\|\frac{\delta u(r,x)}{\delta x}\right\|_{L^q} \leqslant \frac{r(1-r^2)}{\pi}\left\{\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{|f(x)-f(x+y)||y|}{(1-2r\cos y+r^2)^2}\,dy\right)^q dx\right\}^{\frac{1}{q}} \leqslant$$

$$\leqslant \frac{2r(1-r^2)}{\pi}\int_0^{\pi}\frac{y\;\omega(y)_{L^q}}{((1-r)^2+4r\pi^{-2}y^2)^2}\,dy\,.$$

If $0<r\leqslant 1/2$, then

$$(8)\quad \left\|\frac{\delta u(r,x)}{\delta x}\right\|_{L^q} < \frac{4r}{\pi}\int_0^{\pi} 8y\;\omega(y)_{L^q}dy \leqslant 16r\pi\left(\frac{\pi}{1-r}+1\right)\omega(1-r)_{L^q} <$$

$$\leqslant 32r\pi^2\;\frac{\omega(1-r)_{L^q}}{1-r}\,.$$

In the case $1/2\leqslant r<1$, the substitution $2\sqrt{r}\;y/(\pi(1-r)) = x$ leads to (cf. [2], p.892)

$$(9)\quad \left\|\frac{\delta u(r,x)}{\delta x}\right\|_{L^q} \leqslant \frac{\pi(1+r)}{1-r}\int_0^{2\sqrt{r}/(1-r)}\frac{x\;\omega(\pi x(1-r)/\sqrt{4r})_{L^q}}{(1+x^2)^2}\,dx \leqslant$$

$$\leqslant \frac{\pi(1+r)\;\omega(1-r)_{L^q}}{1-r}\int_0^{2\sqrt{r}/(1-r)}\frac{(\pi x/\sqrt{4r}+1)x}{(1+x^2)^2}\,dx \leqslant C_8 r\;\frac{\omega(1-r)_{L^q}}{1-r}.$$

Estimate now the quantity $\left\|\frac{\delta v(r,x)}{\delta x}\right\|_{L^q}$ for $q>1$. We start with the identity

$$v(r,x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x+y)\cdot P(r,y)\,dy\,,$$

where $\tilde{f}$ is the conjugate function of f; whence

$$\frac{\delta v(r,x)}{\delta x} = \frac{r(1-r^2)}{\pi}\int_{-\pi}^{\pi}\left\{\tilde{f}(x) - \tilde{f}(x+y)\right\} \frac{\sin y}{(1-2r\cos y + r^2)^2}\,dy.$$

Using the generalized Minkowski inequality mentioned above, and the Riesz inequality ([9], p.404), we obtain

$$\left\|\frac{\delta v(r,x)}{\delta x}\right\|_{L^q} < \frac{2r(1-r^2)}{\pi}\int_0^{\pi}\frac{y\,\omega(y;\tilde{f})_{L^q}}{((1-r)^2 + 4r\pi^{-2}y^2)^2}\,dy <$$

$$< C_9\,\frac{2r(1-r^2)}{\pi}\int_0^{\pi}\frac{y\,\omega(y)_{L^q}}{((1-r)^2 + 4r\pi^{-2}y^2)^2}\,dy .$$

Repeating the previous argument, we get

$$\left\|\frac{\delta v(r,x)}{\delta x}\right\|_{L^q} < C_{10}r\,\frac{\omega(1-r)_{L^q}}{1-r} .$$

To estimate the quantity $\left\|\frac{\delta v(r,x)}{\delta x}\right\|_{L^q}$ for $q = 1$ the following generalized form of Theorem 2.30(II) of [9],p.411 will be needed.

Let $U(r,x)$ and $V(r,x)$ be two conjugate harmonic functions in the disc $\{z: |z| < 1\}$ $(z=re^{ix},\ 0<r<1)$, and let

$$\|U(1-\varrho,x)\|_{L^1} \leqslant \chi(\varrho) \quad \text{for} \quad \varrho\in(0,1),$$

where the function $\chi(\varrho)$ is continuous in $(0,1)$. Then

$$\|V(r,x)\|_{L^1} \leqslant 4\int_{1-r}^{1} y^{-1}\chi(y/2)\,dy \qquad \text{if} \qquad 0<r<1.$$

Putting $\chi(y) = y^{-1}\,\omega(y)_{L^1}$ and applying the inequalities (8) and (9) for $\left\|\frac{\partial u(r,x)}{\partial x}\right\|_{L^1}$, we obtain

$$\left\|\frac{\partial v(r,x)}{\partial x}\right\|_{L^1} \leqslant C_{11} \int_{1-r}^{1} y^{-2}\,\omega(y)dy.$$

Since $\omega(\varrho)_{L^1}$ is a non-decreasing function of ϱ, we have

$$r\,\frac{\omega(1-r)_{L^1}}{1-r} = \omega(1-r)_{L^1} \int_{1-r}^{1} y^{-2}\,dy \leqslant \int_{1-r}^{1} y^{-2}\,\omega(y)_{L^1}\,dy.$$

Collecting the results we get (7).

Now two further lemmas are going to be proved.

L e m m a 2. Suppose that $\delta > 1/2$ and $(1-\delta)p < 1$. Then

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left\|\sigma_k^{\delta-1} - \sigma_k^{\delta}\right\|_{L^q}^{p}\right\}^{\frac{1}{p}} \leqslant C_{12}\,\psi_n \quad (n=0,1,2,\dots). \tag{10}$$

P r o o f. Consider the power series (6) with $z = \varrho e^{ix}$ $(0<\varrho<1)$. Denote by $\tau_k^{\delta}(x)$ the (C,δ)-means of the series $F(e^{ix})$.

If $p \geqslant q$, then the generalized Minkowski inequality ([12], p.35) gives

$$R_m = \left\{\sum_{k=0}^{m} k^{\delta p}\left\|\tau_k^{\delta-1} - \tau_k^{\delta}\right\|_{L^q}^{p} r^{kp}\right\}^{\frac{1}{p}} \leqslant \left\|\left\{\sum_{k=0}^{m} k^{\delta p}\left|\tau_k^{\delta-1} - \tau_k^{\delta}\right|^{p} r^{kp}\right\}^{\frac{1}{p}}\right\|_{L^q}.$$

As it is known ([4], p.241),

$$\delta\sum_{k=0}^{\infty} A_k^{(\delta)}(\tau_k^{\delta-1}(x) - \tau_k^{\delta}(x))z^k = \frac{ze^{ix}F'(ze^{ix})}{(1-z)^{\delta}},$$

where $A_k^{(\delta)} = \binom{k+\delta}{k}$ and $z = re^{i\varphi}$ $(0 < r < 1)$; whence by the Hausdorff-Young inequality ([10], p.153), for $1 < p' \leqslant 2$, $\delta p' > 1$ and $p = \frac{p'}{p'-1}$ we get

$$R_m \leqslant C_{13} \left\| \left\{ r^{p'} \int_{-\pi}^{\pi} \frac{\left|F'(re^{i(\varphi+x)})\right|^{p'}}{\left|1-re^{i\varphi}\right|^{\delta p'}} d\varphi \right\}^{\frac{1}{p'}} \right\|_{L^q}.$$

Further, applying the first mean value theorem for integrals, or the generalized Minkowski inequality if $q \geqslant p'$, we obtain

$$R_m \leqslant C_{13} r \left\| F'(re^{i(\varphi^0+x)}) \right\|_{L^q} \cdot \left\| (1-re^{ix})^{-\delta} \right\|_{L^{p'}},$$

where $\varphi^0 \in \langle -\pi, \pi \rangle$. From Lemma 1 it follows that

$$R_m \leqslant \begin{cases} C_6 C_{13} r \dfrac{\omega(1-r)_{L^q}}{1-r} \left\| (1-re^{ix})^{-\delta} \right\|_{L^{p'}} & \text{if } q > 1, \\ C_7 C_{13} \displaystyle\int_{1-r}^{1} y^{-2}\, \omega(y)_{L^q}\, dy \cdot \left\| (1-re^{ix})^{-\delta} \right\|_{L^{p'}} & \text{if } q = 1 \end{cases}$$

which, together with

$$\left\| (1-re^{ix})^{-\delta} \right\|_{L^{p'}} \leqslant C_{14} (1-r)^{\frac{1}{p'}-\delta} \quad \text{for } \delta p' > 1 \quad (\text{cf. [3], p.65}),$$

shows that

$$R_m \leqslant \begin{cases} r C_6 C_{13} C_{14}\, \omega(1-r)_{L^q} (1-r)^{-\delta-\frac{1}{p}} & \text{if } q > 1, \\ C_7 C_{13} C_{14} (1-r) \displaystyle\int_{1-r}^{1} y^{-2}\, \omega(y)_{L^q}\, dy \cdot (1-r)^{-\delta-\frac{1}{p}} & \text{if } q = 1. \end{cases}$$

Setting $1-r = 1/(m+1)$ $(m \geqslant 1)$ and using the inequality

$$\left(1-\frac{1}{m+1}\right)^k > \left(1-\frac{1}{m+1}\right)^{m+1} > e^{-1} \quad \text{for} \quad k < m+1,$$

we obtain

$$\left\{\sum_{k=0}^{m} k^{\delta p}\left\|\tau_k^{\delta-1} - \tau_k^{\delta}\right\|_{L^q}^p\right\}^{\frac{1}{p}} \leqslant R_m < C_{15}\varphi_m\cdot(m+1)^{\delta+\frac{1}{p}} \quad (\text{see } [5], \text{p.}239).$$

Finally,

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left\|\sigma_k^{\delta-1} - \sigma_k^{\delta}\right\|_{L^q}^p\right\}^{\frac{1}{p}} \leqslant \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left\|\tau_k^{\delta-1} - \tau_k^{\delta}\right\|_{L^q}^p\right\}^{\frac{1}{p}} \leqslant$$

$$\leqslant \left\{\frac{1}{(n+1)^{1+\delta p}}\sum_{k=n}^{2n} k^{\delta p}\left\|\tau_k^{\delta-1} - \tau_k^{\delta}\right\|_{L^q}^p\right\}^{\frac{1}{p}} \leqslant (n+1)^{-\delta-\frac{1}{p}} R_{2n} \leqslant$$

$$< 2^{\delta+\frac{1}{p}} C_{15}\,\varphi_{2n} \quad \text{if} \quad p \geqslant 2, \quad (1-\delta)p < 1,$$

and so, because

$$\varphi_k < \varphi_{k'} \qquad \text{for} \quad k \geqslant k',$$

we have (10) under the restrictions above.

Since the left-hand side of (10) is a non-decreasing function of p, the estimation (10) is also true for any $p \geqslant q$ such that $(1-\delta)p < 1$.

In the case $q \geqslant p$, $1 \leqslant q \leqslant 2$, taking p_1 such that $q \leqslant p_1 \leqslant 2$ $((1-\delta)p_1 < 1)$ and reasoning as before we get (10) for $p = p_1$. Now, by the previous observation, the inequality (10) remains valid for $p \leqslant p_1$, as required.

We have still to examine the case $q > \max(2,p)$. Here applying the following inequality of Tôyama-type ([7], p.285, 3.6.47)

$$\left\{\sum_{\mu=0}^{m}\left(\int_{-\pi}^{\pi}|g_\mu(x)|^t dx\right)^{\frac{s}{t}}\right\}^{\frac{1}{s}} \leqslant (m+1)^{\frac{1}{s}-\frac{1}{t}}\left\{\int_{-\pi}^{\pi}\left(\sum_{\mu=0}^{m}|g_\mu(x)|^s\right)^{\frac{t}{s}}dx\right\}^{\frac{1}{t}}, \quad 0<s\leqslant t<\infty,$$

for $g_\mu \in L^t$ $(\mu=0,1,2,\dots,m)$ $m=0,1,2,\dots,$ we obtain

$$\left\{\sum_{k=0}^{m} k^{\delta p}\left\|\tau_k^{\delta-1}-\tau_k^{\delta}\right\|_{L^q}^{p} r^{kp}\right\}^{\frac{1}{p}} \leqslant (m+1)^{\frac{1}{p}-\frac{1}{q}}\left\|\left\{\sum_{k=0}^{m} k^{\delta p}\left|\tau_k^{\delta-1}-\tau_k^{\delta}\right|^p r^{kp}\right\}^{\frac{1}{p}}\right\|_{L^q}.$$

Consequently, arguing similarly as in the first case, we have the last part of (10).

L e m m a 3. Suppose that $\delta>1/2$ and $(1-\delta)\max(p,q)<1$. Then

$$(11)\qquad \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left\|\sigma_k^{\delta-1}-\sigma_k^{\delta}\right\|_{L^q}^{p}\right\}^{\frac{1}{p}} \leqslant C_{16}\varphi_n \qquad (n=0,1,2,\dots).$$

P r o o f. Confine our attention to the case $q>\max(2,p)$, because for $p\geqslant q$ and $q>p, 1<q\leqslant 2$ the inequality (11) is a consequence of (10). In view of Hölder's inequality, we have

$$(12)\qquad \left\{\frac{1}{m+1}\sum_{k=0}^{m} k^{\delta p}\left\|\tau_k^{\delta-1}-\tau_k^{\delta}\right\|_{L^q}^{p} r^{kp}\right\}^{\frac{1}{p}} \leqslant$$

$$\leqslant \left\{\frac{1}{m+1}\sum_{k=0}^{m} k^{\delta q}\left\|\tau_k^{\delta-1}-\tau_k^{\delta}\right\|_{L^q}^{q} r^{kq}\right\}^{\frac{1}{q}} =$$

$$= \left\|\left\{\frac{1}{m+1}\sum_{k=0}^{m} k^{\delta q}\left|\tau_k^{\delta-1}-\tau_k^{\delta}\right|^q r^{kq}\right\}^{\frac{1}{q}}\right\|_{L^q} = Q_m(m+1)^{-\frac{1}{q}}.$$

To the last expression we apply the Hausdorff-Young inequality, the generalized Minkowski inequality and Lemma 1, successively. Then for $q'\in(1,2\rangle$ such that $\delta q'>1$ and $q=\frac{q'}{q'-1}$ $(q/q'\geqslant 1)$, the argument similar to that of Lemma 2 gives

$$Q_m < rC_{17}\left\|\left\{\int_{-\pi}^{\pi}\frac{|F'(re^{i(\varphi+x)})|^{q'}}{|1-re^{i\varphi}|^{\delta q'}}\,d\varphi\right\}^{\frac{1}{q'}}\right\|_{L^q} \leqslant$$

$$\leqslant rC_{17}\left\{\int_{-\pi}^{\pi}\frac{\left\|F'(re^{i(\varphi+x)})\right\|_{L^q}^{q'}}{|1-re^{i\varphi}|^{\delta q'}}\,d\varphi\right\}^{\frac{1}{q'}} \leqslant C_{18}\,\omega(1-r)_{L^q}\cdot(1-r)^{-\delta-\frac{1}{q}}.$$

Further, for $1-r=(m+1)^{-1}$ $(m\geqslant 1)$, we obtain

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left\|\sigma_k^{\delta-1}-\sigma_k^{\delta}\right\|_{L^q}^{q}\right\}^{\frac{1}{q}} \leqslant \left\{\frac{1}{(n+1)^{1+\delta q}}\sum_{k=o}^{2n}k^{\delta q}\left\|\tau_k^{\delta-1}-\tau_k^{\delta}\right\|_{L^q}^{q}\right\}^{\frac{1}{q}} \leqslant$$

$$\leqslant \frac{2^{\delta+1/q}}{(2n+1)^{\delta+1/q}}\,Q_{2n} \leqslant C_{18}\,2^{\delta+1/q}\,\omega\left(\frac{1}{2n+1}\right)_{L^q} \leqslant C_{18}\,2^{\delta+1/q}\,\omega\left(\frac{1}{n+1}\right)_{L^q},$$

whenever $(1-\delta)q<1$. Finally, by (12) the estimate (11) follows.

4. P r o o f of Theorem 1. We begin with the obvious inequality

$$\left\{H_n^p(f)_{L^q}\right\}^p \leqslant 2^p\left(\frac{1}{A_n}\sum_{k=o}^{n}\alpha_{nk}\left\|\sigma_k^{\delta-1}-\sigma_k^{\delta}\right\|_{L^q}^{p}+\right.$$

$$\left.+\frac{1}{A_n}\sum_{k=o}^{n}\alpha_{nk}\left\|\sigma_k^{\delta}-f\right\|_{L^q}^{p}\right) = 2^p(A+B).$$

It can be easily observed that

$$\left\|\sigma_k^{\delta}-f\right\|_{L^q} \leqslant C_{19}\varphi_k \quad \text{for} \quad \delta>0.$$

(see [1], p.546 if $0<\delta<1, q>1$; [8], p.424 if $\delta\geqslant 1, q>1$ and [11], p.53 if $\delta>0, q=1$). Hence and from the condition (2) it follows that

$$B \leq C_{20}\,\frac{1}{A_n}\sum_{k=o}^{n}\alpha_{nk}\varphi_k^p \leq C_{20}\,\frac{1}{A_n}\sum_{l=o}^{j} 2^l\left\{\frac{1}{2^l}\sum_{k=\alpha_l}^{\beta_l}\alpha_{nk}\,\varphi_k^p\right\} <$$

$$< C_{20}\,\frac{1}{A_n}\sum_{l=o}^{j}\left\{2^{l(\gamma-1)}\sum_{k=\alpha_l}^{\beta_l}\alpha_{nk}^{\gamma}\,\varphi_k^{\gamma p}\right\}^{\frac{1}{\gamma}} \leq$$

$$< C_{20}\,\frac{1}{A_n}\sum_{l=o}^{j}\left\{\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_{nk}^{\gamma}\,\varphi_k^{\gamma p}}{(k+1)^{1-\gamma}}\right\}^{\frac{1}{\gamma}} \leq C_1 C_{20}\,\frac{1}{n+1}\sum_{k=o}^{n}\varphi_k^p .$$

Further if $1/\gamma + 1/\gamma' = 1;\ \gamma, \gamma' > 1$, Hölder's inequality gives

$$A \leq \frac{1}{A_n}\sum_{l=o}^{j}\left\{\sum_{k=\alpha_l}^{\beta_l}\alpha_{nk}\left\|\sigma_k^{\delta-1} - \sigma_k^{\delta}\right\|_{L^q}^{p}\right\} \leq$$

$$< \frac{1}{A_n}\sum_{l=o}^{j}\left\{\left(\sum_{k=\alpha_l}^{\beta_l}\alpha_{nk}^{\gamma}\cdot(k+1)^{\frac{\gamma}{\gamma'}}\right)^{\frac{1}{\gamma}}\left(\sum_{k=\alpha_l}^{\beta_l}(k+1)^{-1}\left\|\sigma_k^{\delta-1}-\sigma_k^{\delta}\right\|_{L^q}^{p\gamma'}\right)^{\frac{1}{\gamma'}}\right\} <$$

$$< \frac{1}{A_n}\sum_{l=o}^{j}\left\{\left(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_{nk}^{\gamma}}{(k+1)^{1-\gamma}}\right)^{\frac{1}{\gamma}}\left(\frac{1}{2^l}\sum_{k=\alpha_l}^{\beta_l}\left\|\sigma_k^{\delta-1} - \sigma_k^{\delta}\right\|_{L^q}^{p\gamma'}\right)^{\frac{1}{\gamma'}}\right\}.$$

Since $(1-\delta)\,\gamma' p < 1$, Lemma 2 leads to

$$A < C_{12}^p\,\frac{1}{A_n}\sum_{l=o}^{j}\left\{\left(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_{nk}^{\gamma}}{(k+1)^{1-\gamma}}\right)^{\frac{1}{\gamma}}\cdot\psi_{\alpha_l}^p\right\}.$$

It is clear that

$$\psi_{2^l-1} \leq 2\,\psi_k \quad \text{for} \quad 2^l-1 \leq k \leq 2^{l+1}-2.$$

This and the above inequality for A imply

$$A < (2C_{12})^p \frac{1}{A_n} \sum_{l=0}^{j} \left\{ \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_{nk}^{\gamma} \varphi_k^{\gamma p}}{(k+1)^{1-\gamma}} \right\}^{\frac{1}{\gamma}}.$$

Now, by the condition (2), we obtain

$$A < C_1 (2C_{12})^p \frac{1}{n+1} \sum_{k=0}^{n} \varphi_k^p \quad \text{if} \quad p > q \quad \text{or} \quad p < q < 2,$$

and

$$A < (2C_{12})^p (n+1)^{1-\frac{p}{q}} \frac{1}{A_n} \sum_{l=0}^{n} \left\{ \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_{nk}^{\gamma} \cdot \omega^{\gamma p} (\frac{1}{k+1})_{L^q}}{(k+1)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} \leq$$

$$< C_1 (2C_{12})^p \frac{1}{(n+1)^{p/q}} \sum_{k=0}^{n} \omega^p (\tfrac{1}{k+1})_{L^q} \quad \text{if} \quad q > p \quad \text{and} \quad q > 2.$$

Collecting the results, we get the desired assertion.

5. <u>P r o o f of Theorem 2.</u> The argument runs along the lines of the proof of Theorem 1, namely

$$\left\{ H_n^p(f)_{L^q} \right\}^p < 2^p (A + B),$$

where A, B are as before.

To prove case (i) we apply the same calculation as in the previous proof and thus

$$B \leq C_1 C_{20} \frac{1}{n+1} \sum_{k=0}^{n} \varphi_k^p .$$

Consequently, by Lemma 3 and condition (2) we obtain

$$A < (2C_{16})^p \frac{1}{A_n} \sum_{l=0}^{j} \left\{ \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_{nk}^{\gamma} \varphi_k^{\gamma p}}{(k+1)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} \leq C_1 (2C_{16})^p \frac{1}{n+1} \sum_{k=0}^{n} \varphi_k^p ,$$

where γ and γ' are as before. These estimates imply (4), and thus Theorem 2 (i) is proved.

In the case (ii), Lemma 2 and condition (2) with ψ_k instead of φ_k leads to

$$A \leqslant (2C_{12})^p \frac{1}{A_n} \sum_{l=0}^{j} \left\{ \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_{nk}^{\gamma} \psi_k^{\gamma p}}{(k+1)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} \leqslant C_1 (2C_{12})^p \frac{1}{n+1} \sum_{k=0}^{n} \psi_k^p ,$$

and

$$B \leqslant C_{20} \frac{1}{A_n} \sum_{l=0}^{j} \left\{ \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_{nk}^{\gamma} \varphi_k^{\gamma p}}{(k+1)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} \leqslant$$

$$\leqslant C_{20} \frac{1}{A_n} \sum_{l=0}^{j} \left\{ \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_{nk}^{\gamma} \psi_k^{\gamma p}}{(k+1)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} \leqslant C_1 C_{20} \frac{1}{n+1} \sum_{k=0}^{n} \psi_k^p ,$$

which gives the estimate (5).

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INSTITUTE OF MATHEMATICS, ADAM MICKIEWICZ UNIVERSITY, POZNAŃ

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