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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF LINEAR CONNECTION DETERMINED BY THE FIELD OF A TENSOR OF TYPE (1,1) COVARIANTLY CONSTANT IN THE THREE DIMENSIONAL SPACE

1. Introduction

Let M be a three-dimensional differential manifold and ξ a chart belonging to its atlas atl M. By $\mathcal{F}(M)$, resp. $\mathfrak{X}(M)$, we denote respectively the ring of smooth functions and the module of smooth vector field on M.

Let T be a vector field of type (1,1) and ∇ a linear connection on M. Let V be a vector field belonging to $\mathfrak{X}(M)$. Then $\nabla_V T$ denotes the covariant derivative of the given vector field T along the vector field V.

In the present paper we shall give a necessary and sufficient condition for the existence of a linear connection ∇ determined by the field of a tensor T covariantly constant, i.e. satisfying the condition

$$\nabla_V T = 0 \tag{1.1}$$

for every $V \in \mathfrak{X}(M)$. We call such a tensor field parallel with respect to the linear connection ∇.

Condition (1.1) can be written out with respect to an arbitrary chart $\xi \in$ atl M

$$(1.2)\qquad \nabla_\sigma T^\mu_\lambda := \partial_\sigma T^\mu_\lambda + \Gamma^\mu_{\sigma\alpha} T^\alpha_\lambda - \Gamma^\alpha_{\sigma\lambda} T^\mu_\alpha = 0 \qquad (\sigma, \alpha, \lambda, \mu = 1,2,3),$$

where T^μ_λ are given coordinates of the tensor field T in the chart ξ, $\partial_\sigma T^\mu_\lambda$ are partial derivatives of the field T and $\Gamma^\beta_{\sigma\alpha}$ are unknown coordinates of the linear connection in the given chart. (1.2) can be treated as a system of n^2 equations (for every σ) linear non-homogeneous with respect to $\Gamma^\beta_{\sigma\alpha}$. This problem has been solved for 2-dimensional space in [1] for 3-dimensional space in [2] (not published doctoral thesis of the author). In the present paper we give main stages of this solution.

2. Some scalar concomitants of the tensor field T

Let $\mathcal{T}$ denote the set of all non-zero tensor fields of type (1.1) defined on M.
We shall consider scalar fields which are concomitants of $T \in \mathcal{T}$ [3]

$$(2.1)\quad \begin{cases} \overset{(1)}{S}(T) := T^\alpha_\alpha, \quad \overset{(2)}{S}(T) := T^1_1 T^2_2 - T^2_1 T^1_2 + T^1_1 T^3_3 - T^3_1 T^1_3 + T^2_2 T^3_3 - T^3_2 T^2_3, \\ \overset{(3)}{S}(T) := \det(T^\mu_\lambda) = 3!\, T^1_{[1} T^2_2 T^3_{3]}, \end{cases}$$

$\overset{(1)}{S}(T)$ is called the trace of first order of the matrix (T^μ_λ) of the tensor field T and is denoted by tr T, $\overset{(2)}{S}(T)$ is called the trace of second order of the matrix (T^μ_λ) and it is seen from the definition that $\overset{(3)}{S}(T) = \det(T^\mu_\lambda)$.

The traces $\overset{(i)}{S}(T)$ $(i = 1,2,3)$ are scalar fields $\overset{(i)}{S}(T) \in \mathcal{F}(M)$.
The partial derivatives of these scalar field

$$(2.2)\qquad \overset{(i)}{U}_\sigma(T) := \partial_\sigma \overset{(i)}{S}(T) \qquad (i = 1,2,3;\ \sigma = 1,2,3).$$

are co-vector fields. The vanishing of these fields, i.e. $\overset{(i)}{U}_\sigma(T) = 0$ is an invariant property holding in every chart $\xi \in$ atl M.

Let T_Q denote T^μ_λ, where Q is a joint index: $Q = (\mu,\lambda)$, $(Q = 1,2,\dots,9)$. By (2.1) and (2.2) we have

$$(2.3)\qquad \overset{(i)}{U}(T) = \partial_\sigma \overset{(i)}{S}(T) = \sum_{Q=1}^{9} \frac{\partial \overset{(i)}{S}(T)}{\partial T_Q}\, \partial_\sigma T_Q \qquad (i = 1,2,3).$$

Let us consider the matrix

$$(2.4)\qquad A(T) := \left(\frac{\partial \overset{(i)}{S}(T)}{\partial T_Q}\right) \qquad (i = 1,2,3; \quad Q = 1,2,\dots,9),$$

where i denotes the row and Q the column of the matrix $A(T)$. Let

$$(2.5)\qquad r_{A(T)} := \text{the rank of } A(T).$$

It is known that $r_{A(T)}$ is an absolute invariant.

Further, let r_A be the maximal rank of the matrix $A(T)$ where T runs through $\mathcal{T}$, i.e.

$$(2.6)\qquad r_A = \max_T r_{A(T)}.$$

From a theorem in [3] it follows that

T h e o r e m 1. Under the above assumptions

$$r_A = 3.$$

Moreover, let us consider the following scalar fields which are concomitants of given tensor field $T \in \mathcal{T}$ [2]

$$(2.7)\qquad \overset{(1)}{\phi}(T) := T^\alpha_\alpha, \qquad \overset{(2)}{\phi}(T) := T^\beta_\alpha T^\alpha_\beta,$$
$$\overset{(3)}{\phi}(T) := T^\beta_\alpha T^\gamma_\beta T^\alpha_\gamma.$$

Partial derivatives of these scalar fields

(2.8) $$\overset{(i)}{V}_{\sigma}(T) := \partial_{\sigma} \overset{(i)}{\phi}(T) \qquad (i = 1,2,3),$$

are covariant vector fields. The vanishing of these fields, i.e. $\overset{(i)}{V}_{\sigma}(T) = 0$ is an invariant property holding in every chart $\xi \in$ atl M.

From (2.7) and (2.8) we obtain

(2.9) $$\overset{(i)}{V}_{\sigma}(T) = \partial_{\sigma} \overset{(i)}{\phi}(T) = \sum_{Q=1}^{9} \frac{\partial \overset{(i)}{\phi}(T)}{\partial T_Q} \partial_{\sigma} T_Q \qquad (i = 1,2,3),$$

where T_Q denotes the coordinates of the tensor T^{μ}_{λ}.

Now let us form the matrix

(2.10) $$B(T) := \left(\frac{\partial \overset{(i)}{\phi}(T)}{\partial T_Q} \right) \qquad \begin{matrix} (i = 1,2,3; \\ Q = 1,2,\dots,9), \end{matrix}$$

where i denotes the row and Q the column of the matrix B(T).

Let $r_{B(T)}$ denote the rank of this matrix

(2.11) $$r_{B(T)} := \text{rank } B(T) \quad \text{for} \quad T \in \mathcal{T}.$$

The maximal rank of B(T) will be denoted by r_B:

(2.12) $$r_B = \max_{T} r_{B(T)}.$$

In [2] it was proved that the scalar field $\overset{(i)}{\phi}(T)$ defined by (2.7) can be expressed by means of the traces $\overset{(i)}{S}(T)$ defined in (2.1), in the following way:

$$(2.13)\qquad \begin{cases} \overset{(1)}{\phi} = \overset{(1)}{S} \; (= T^{\alpha}_{\alpha}) \\ \overset{(2)}{\phi} = \overset{(1)}{S}{}^2 - 2\overset{(2)}{S} \\ \overset{(3)}{\phi} = -3\overset{(1)}{S}\overset{(2)}{S} + 3\overset{(3)}{S} + \overset{(1)}{S}{}^3. \end{cases}$$

The mapping $S \longrightarrow \phi$ defined above, where $S = (\overset{(1)}{S}, \overset{(2)}{S}, \overset{(3)}{S})$, $\phi = (\overset{(1)}{\phi}, \overset{(2)}{\phi}, \overset{(3)}{\phi})$ is a bijection. The inverse mapping can be expressed as follows

$$(2.14)\qquad \begin{cases} \overset{(1)}{S} = \overset{(1)}{\phi} \\ \overset{(2)}{S} = \frac{1}{2}\overset{(1)}{\phi}{}^2 - \frac{1}{2}\overset{(2)}{\phi} \\ \overset{(3)}{S} = \frac{1}{6}\overset{(1)}{\phi}{}^3 - \frac{1}{2}\overset{(1)}{\phi}\overset{(2)}{\phi} + \frac{1}{3}\overset{(3)}{\phi}. \end{cases}$$

From Theorem 1 and from the fact that the mapping $S \longrightarrow \phi$ is a bijection it follows that

T h e o r e m 2. Under the assumptions above we have

$$r_B = r_A = 3.$$

3. <u>The matrices of the system of equations (1.2)</u>

In the system of equations (1.2) we take the pairs (μ,λ) in the following order: (11), (12), (13), (21), (22), (23), (31), (32), (33), and for the unknows $\Gamma^{\beta}_{6\alpha}$ the pairs (α,β) in the same order. Let $\mathfrak{M}(T)$ denote the fundamental matrix of the system (1.2) and its elements by $M^{\alpha\mu}_{\lambda\beta}$, where the pair (μ,λ) denotes the row, (α,β) the column of the matrix $\mathfrak{M}$. From (1.2) we obtain

$$(3.1)\qquad M^{\alpha\mu}_{\lambda\beta}(T) := \delta^{\alpha}_{\lambda} T^{\mu}_{\beta} - \delta^{\mu}_{\beta} T^{\alpha}_{\lambda}.$$

From this formula it is seen that the matrix $\mathfrak{M}(T)$ is skew-symmetrix with respect to the pairs (μ,λ) and (α,β), i.e. $M^{\alpha}_{\lambda}{}^{\mu}_{\beta}(T) = -M^{\mu}_{\beta}{}^{\alpha}_{\lambda}(T)$. Writing out the augmented matrix $\mathfrak{M}^*_6(T)$ of the system (1.2) we obtain from (3.1):

$$\text{(3.2)}\quad \mathfrak{M}^*_6(T) := \begin{bmatrix} 0 & T^1_2 & T^1_3 & -T^2_1 & 0 & 0 & -T^3_1 & 0 & 0 & \partial_6 T^1_1 \\ -T^1_2 & 0 & 0 & T^1_1-T^2_2 & T^1_2 & T^1_3 & -T^3_2 & 0 & 0 & \partial_6 T^1_2 \\ -T^1_3 & 0 & 0 & T^2_3 & 0 & 0 & T^1_1-T^3_3 & T^1_2 & T^1_3 & \partial_6 T^1_3 \\ T^2_1 & T^2_2-T^1_1 & T^2_3 & 0 & -T^2_1 & 0 & 0 & -T^3_1 & 0 & \partial_6 T^2_1 \\ 0 & -T^1_2 & 0 & T^2_1 & 0 & T^2_3 & 0 & -T^3_2 & 0 & \partial_6 T^2_2 \\ 0 & -T^1_3 & 0 & 0 & -T^2_3 & 0 & T^2_1 & T^2_2-T^3_3 & T^2_3 & \partial_6 T^2_3 \\ T^3_1 & T^3_2 & T^3_3-T^1_1 & 0 & 0 & -T^2_1 & 0 & 0 & -T^3_1 & \partial_6 T^3_1 \\ 0 & 0 & -T^1_2 & T^3_1 & T^3_2 & T^3_3-T^2_2 & 0 & 0 & -T^3_2 & \partial_6 T^3_2 \\ 0 & 0 & -T^1_3 & 0 & 0 & -T^2_3 & T^3_1 & T^3_2 & 0 & \partial_6 T^3_3 \end{bmatrix}$$

The basic matrix $\mathfrak{M}(T)$ consists of the first nine columns of the matrix $\mathfrak{M}^*_6(T)$. Let us introduce the following notation

(3.3) $r_{\mathfrak{M}(T)} :=$ the rank of $\mathfrak{M}(T)$, $r_{\mathfrak{M}^*_6(T)} :=$ the rank of $\mathfrak{M}^*_6(T)$.

Since the matrix $\mathfrak{M}(T)$ is skew symmetrix, we have

$$r_{\mathfrak{M}(T)} \neq 1,3,5,7,9.$$

In [2] it is proved that for the non-zero tensor field T^{μ}_{λ} the rank $r_{\mathfrak{M}(T)}$ of the matrix $\mathfrak{M}(T)$ can asume the following values

$$\text{(3.4)}\qquad r_{\mathfrak{M}(T)} = 0,4,6,$$

and the following equivalences hold

$$(3.5)\quad \begin{cases} \text{a)}\quad T^{\mu}_{\lambda} = \alpha\,\delta^{\mu}_{\lambda}(\alpha \neq 0) \Leftrightarrow r_B = 1(T^{\mu}_{\lambda} \neq 0) \\ \text{b)}\quad r_B = 1(T^{\mu}_{\lambda} \neq 0) \Leftrightarrow r_{\mathfrak{M}} = 0(T^{\mu}_{\lambda} \neq 0) \\ \text{c)}\quad r_B = 2 \Leftrightarrow r_{\mathfrak{M}} = 4 \\ \text{d)}\quad r_B = 3 \Leftrightarrow r_{\mathfrak{M}} = 6, \end{cases}$$

where δ^{μ}_{λ} is the Kronecker delta.

4. The existence of linear connection

The following theorem holds:

T h e o r e m 4. A necessary and sufficient condition in order that there exist a linear connection ∇, with respect to which the tensor field $T \in \mathcal{T}$ is parallel, is the vanishing of three covector fields $\overset{(i)}{V}_{\sigma}(T)$ defined in (2.8), i.e.

$$\overset{(i)}{V}_{\sigma}(T) = 0 \quad (i = 1,2,3).$$

O u t l i n e o f t h e p r o o f. Necessity follows from the theorem proved by A.Lichnerowicz [7] and A.Zajtz [8]: if a tensor field T is parallel (covariantly constant) with respect to the connection ∇ and the field K is its concomitant, then K is also parallel. Hence in particular for the scalar concomitant K(T) we have

$$\nabla_{\sigma}K(T) = 0 \Leftrightarrow \partial_{\sigma}K(T) = 0.$$

Thus for scalar concomitants (2.7) we have by (2.8):

$$\overset{(i)}{V}_{\sigma}(T) = 0 \text{ for every } i = 1,2,3.$$

Conversely, assume that $\overset{(i)}{V}_{6}(T) = 0$ $(i = 1,2,3)$. We have to show that $r_{\mathfrak{M}(T)} = r_{\mathfrak{M}^*_6(T)}$.
From the assumption by (2.9) is follows that

$$(4.1)\qquad \sum_{Q=1}^{9} \frac{\partial \overset{(i)}{\phi}(T)}{\partial T_Q} \partial_6 T_Q = 0 \qquad (i = 1,2,3).$$

a) In the case $r_{\mathfrak{M}(T)} = 6$ we show that $B(T)\cdot \mathfrak{M}^*_6(T) = [0]$ (see [6],[5]).

In fact, in the last column of the product $B(T)\cdot \mathfrak{M}^*_6(T)$ we obtain

$$\sum_{Q=1}^{9} \frac{\partial \overset{(i)}{\phi}(T)}{\partial T_Q} \partial_6 T_Q .$$

These expressions are equal by means of (4.1). For the remaining columns of this product we also obtain zero (see [6]). This implies that

$$r_{\mathfrak{M}^*_6(T)} = r_{\mathfrak{M}(T)} = 6.$$

b) For the case $r_{\mathfrak{M}} = 4$ the proof is given in [2].

c) If $r_{\mathfrak{M}(T)} = 0$, from (3.1) and (3.5) it follows that for the non-zero field T^{μ}_{λ} the following equivalences hold: $(r_{\mathfrak{M}(T)} = 0) \Longleftrightarrow (r_{B(T)} = 1) \Longleftrightarrow T^{\mu}_{\lambda} = \alpha\delta^{\mu}_{\lambda}$, where $\alpha = \alpha(\xi^1,\xi^2,\xi^3)$ and δ^{μ}_{λ} is Kronecker's symbol. Then $\overset{(i)}{V}_{6(T)} = 0$ (for $i = 1,2,3$), i.e. $\overset{(i)}{V}_{6}(\alpha\delta^{\mu}_{\lambda}) = 0$ implies $\alpha = \text{const}$, and consequently $r_{\mathfrak{M}^*_6(T)} = r_{\mathfrak{M}(T)} = 0$ for $\alpha = \text{const}$.

From (3.4) we obtain the following corollary.

C o r o l l a r y 1. If $r_{\mathfrak{M}(T)} = m$ (where $m = 0,4,6$) then for any non-zero field T with respect to $\Gamma^{\beta}_{6\alpha}$ the solutions of the system (1.2) depend upon $3\cdot(9-m)$ arbitrary functions.

Making use of Theorems 1 and 2 we can express Theorem 4 in the following equivalent form.

T h e o r e m 5. A sufficient and necessary condition in order that there exist a linear connection ∇ with respect to which a non-zero tensor field $T \in \mathcal{T}$ is parallel is the vanishing of the covector fields $\overset{(i)}{U}_{\sigma}(T)$ defined in (2.2), i.e.

$$\overset{(i)}{U}_{\sigma}(T) = 0 \qquad (i = 1,2,3).$$

5. The relation to the results of Y.C. Wong

Let M_3 be a three-dimensional connected differential manifold of class C^{∞} with a non-zero smooth tensor field $T^{\mu}_{\lambda}(p)$, $p \in M_3$. Clearly, any system of 3rd linearly independent vectors lying in the space M_p tangent to the manifold M_3 at the point p is a basis of M_p.

The following theorem is a special case of the theorem proved by Y.C. Wong ([10], p.72): A necessary and sufficient condition in order that these exist a linear connection ∇ with respect to which the tensor field T^{μ}_{λ} is parallel (i.e. satisfying the system of equation (1.2)) is that at every point $p \in M_3$ there exist a basis (in general non-holonomic)

$$\underset{\varrho}{b}^{\sigma}(p) \qquad (\varrho, \sigma = 1,2,3) \tag{5.1}$$

in which the coordinates of the tensor $T^{\mu}_{\lambda}(p)$ are constants C^{μ}_{λ} which are not simultaneously zero and such that the following identities hold:

$$T^{\beta}_{\alpha}(p)\, \underset{\beta}{b}^{\mu}(p) \overset{\alpha\ -1}{b_{\lambda}}(p) = C^{\mu}_{\lambda}, \tag{5.2}$$

where $\overset{\alpha\ -1}{b_{\lambda}}(p)$ is the system inverse to $\underset{\beta}{b}^{\mu}(p)$, i.e.

$$\underset{\varrho}{b}^{\mu}(p)\, \overset{\varrho\ -1}{b_{\lambda}}(p) = \delta^{\mu}_{\lambda}. \tag{5.3}$$

From the above-mentioned theorem and from Theorem 4 we obtain the following corollary.

C o r o l l a r y. A necessary and sufficient condition in order that in a three-dimensional space for the tensor field $T^{\mu}_{\lambda}(p)$ at every point p there exist a basis (5.1) in which the coordinates of the tensor $T^{\mu}_{\lambda}(p)$ are constants C^{μ}_{λ} which are simultaneously non-zero and satisfy (5.2) and (5.3) is the vanishing of the covector fields $\overset{(i)}{V}_{\sigma}(T)$, i.e. the relations $\overset{(i)}{V}_{\sigma}(T) = 0$ $(i = 1,2,3)$ (or in the equivalent form $\overset{(i)}{U}_{\sigma}(T) = 0$, where $\overset{(i)}{U}_{\sigma}(T)$ is defined in (2.2)).

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