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ON THE EQUIVALENCE OF CERTAIN RELATIONS OF TANGENCY OF ARCS IN METRIC SPACES

Introduction

In the present paper we consider the connection between relations of tangency of arcs in general metric spaces. W.Waliszewski in [5] gave the definition of tangency of sets in a generalized metric space (E, l). Here E denotes a set and l is a non-negative real function defined on the Cartesian product $E_0 \times E_0$, where E_0 is a family on non-empty subsets of the set E. Under some assumptions, the function l induces a metric $l_0(x,y) = l(\{x\},\{y\})$ on the set E, which allows us to define the notions of a sphere and a ball in the space (E , l). According to this definition a set $A \in E_0$ is (a, b) - tangent of order k to the set $B \in E_0$ at a point p of the space (E, l) if the pair (A, B) is (a, b) - concentrated at the point p in the considered space and we have

$$\frac{1}{r^k}\, l(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0,$$

where k is a positive real number and a and b are non-negative real functions defined in a right-hand neighbourhood of the point 0 such that $a(r) \xrightarrow[r \to 0^+]{} 0$ and $b(r) \xrightarrow[r \to 0^+]{} 0$.

The pair (A, B) is (a, b) - concentrated at the point p of the space (E, l) if 0 is a concentration point of the set of all real numbers $r > 0$ such that the sets

$$A \cap S(p,r)_{a(r)} \quad \text{and} \quad B \cap S(p,r)_{b(r)}$$

are non-empty.
$S(p,r)_{a(r)}$ denotes the $a(r)$ - neighbourhood of the set $S(p,r)$ in the space (E, l) and is equal to $\bigcup_{q \in S(p,r)} K(q,a(r))$. $S(p,r)$ denotes the sphere with the center p and the radius r. $K(q,a(r))$ denotes the open ball with the center q and the radius $a(r)$. Analogously we define $S(p,r)_{b(r)}$.

In the section 4 of the paper [5] W.Waliszewski defined in the space (E,ϱ) the functions

$\varrho_i : E_0 \times E_0 \longrightarrow \langle 0, \infty)$ $(i = 0,1,\dots,6)$ induced by the metric ϱ

$$\varrho_0(A,B) = \sup \{\varrho(x,B) \ ; \ x \in A\},$$

$$\varrho_1(A,B) = \max \{\varrho_0(A,B) \ ; \ \varrho_0(B,A)\},$$

$$\varrho_2(A,B) = \inf \{\operatorname{diam}_\varrho (\{x\} \cup B) \ ; \ x \in A\},$$

$$\varrho_3(A,B) = \max \{\varrho_2(A,B) \ ; \ \varrho_2(B,A)\},$$

$$\varrho_4(A,B) = \min \{\varrho_2(A,B) \ ; \ \varrho_2(B,A)\},$$

$$\varrho_5(A,B) = \operatorname{diam}_\varrho (A \cup B),$$

$$\varrho_6(A,B) = \inf \{\varrho(x,B) \ ; \ x \in A\}$$

for $A,B \in E_0$, where $\varrho(x,B) = \inf \{\varrho(x,y) \ ; \ y \in B\}$ and $\operatorname{diam}_\varrho A$ denotes the diameter of the set A in the metric space.
Similarly, we can consider a function

$$\varrho_7(A,B) = \min \{\varrho_0(A,B) \ ; \ \varrho_0(B,A)\}.$$

The functions ϱ_i $(i = 0,1,\dots,7)$ are special cases of functions l such that $\varrho_i(\{x\},\{y\}) = l_0(x,y) = \varrho(x,y)$. Hence in the space (E,ϱ) we can introduce various relations of tangency of sets and investigate the relation between them.

In this paper we shall give some relations between the functions ϱ_1, as well as we shall prove a theorem on the equivalence of the relations of tangency of arcs with the Archimedean property.

1. Let $p \in E$. We assume (cf. [5])

(1) $T_l(a,b,k,p) \overset{\text{df}}{=} \{(A,B);\ A \cup B \subset E \wedge (A,B)$ is (a,b) - concentrated at the point $p \wedge$

$$\frac{1}{r^k}\, l(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}) \xrightarrow[r \to 0+]{} 0\Big\}.$$

Assume that we are given two non-negative real functions l_1, l_2 defined on the Cartesian product $E_0 \times E_0$.

T h e o r e m 1. If the functions l_1 and l_2 satisfy the conditions

(i) $l_1(\{x\}, \{y\}) = l_2(\{x\}, \{y\}) = l_0(x, y)$,

(ii) $l_1(A,B) \leqslant l_2(A,B)$

for any sets $A,B \in E_0$, then we have

(2) $$(A,B) \in T_{l_2}(a,b,k,p) \Rightarrow (A,B) \in T_{l_1}(a,b,k,p).$$

P r o o f. Suppose that conditions (i) and (ii) hold and $(A,B) \in T_{l_2}(a,b,k,p)$.

By the assumption (i) we have

$$S(p,r) = S_{l_2}(p,r) = S_{l_1}(p,r),\quad S_{l_2}(p,r)_{a(r)} = S_{l_1}(p,r)_{a(r)} =$$

$$= S(p,r)_{a(r)},\quad S_{l_2}(p,r)_{b(r)} = S_{l_1}(p,r)_{b(r)} = S(p,r)_{b(r)}.$$

Since $(A,B)\in T_{l_2}(a,b,k,p)$, we see that the pair (A,B) is (a,b) - concentrated at the point $p\in E$ and we have

$$(3)\qquad \frac{1}{r^k}\, l_2(A\cap S(p,r)_{a(r)},\ B\cap S(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0.$$

From the assumption (ii) it follows that

$$(4)\qquad 0\leqslant l_1(A\cap S(p,r)_{a(r)},\ B\cap S(p,r)_{b(r)}) \leqslant l_2(A\cap S(p,r)_{a(r)},\ B\cap S(p,r)_{b(r)}).$$

From (3) and (4) we have

$$\frac{1}{r^k}\, l_1(A\cap S(p,r)_{a(r)},\ B\cap S(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0$$

which ends the proof of the theorem.

From the definition of the functions ϱ_i we have

$$\varrho_0(A,B) = \sup_{x\in A}\ (\inf_{y\in B} \varrho(x,y)),$$

$$\varrho_2(A,B) = \inf_{x\in A}\ (\sup_{y\in B} \varrho(x,y)),$$

$$\varrho_5(A,B) = \sup_{x,y\in A\cup B} \varrho(x,y),$$

$$\varrho_6(A,B) = \inf_{x\in A,\, y\in B} \varrho(x,y).$$

Hence taking into account the definitions of inf and sup we obtain

$$(5)\qquad \varrho_6(A,B) \leqslant \varrho_i(A,B) \leqslant \varrho_5(A,B)$$

for $A,B\in E_0$ and $i = 0,1,2,3,4,7$.
From (5) and by Theorem 1 we obtain

$$(6)\qquad (A,B)\in T_{\varrho_5}(a,b,k,p) \Longrightarrow (A,B)\in T_{\varrho_i}(a,b,k,p),$$

(6') $$(A,B) \in T_{\varrho_i}(a,b,k,p) \Rightarrow (A,B) \in T_{\varrho_6}(a,b,k,p)$$

for $i = 0,1,2,3,4,7$.
Hence we get

(7) $$(A,B) \in T_{\varrho_5}(a,b,k,p) \Rightarrow (A,B) \in T_{\varrho_6}(a,b,k,p).$$

2. Let us consider an arbitrary metric space (E,ϱ).
Let $X_r = \{x \in E \;;\; r - a(r) < \varrho(p,x) < r + a(r)\}$.
We shall prove that

(2.1) $$S(p,r)_{a(r)} \subset X_r.$$

Let $x \in S(p,r)_{a(r)}$. Hence $x \in \bigcup_{q \in S(p,r)} K(q,a(r))$.
Therefore there exists $q \in S(p,r)$ such that $x \in K(q,a(r))$.
This implies

(2.1.1) $$\varrho(q,x) < a(r) \quad \text{for} \quad q \in S(p,r).$$

By the triangle inequality we obtain

(2.1.2) $$\varrho(p,q) - \varrho(q,x) \leqslant \varrho(p,x) \leqslant \varrho(p,q) + \varrho(q,x).$$

From (2.1.1) and (2.1.2), taking into account that $\varrho(p,q) = r$, we obtain

$$r - a(r) < \varrho(p,x) < r + a(r),$$

hence $x \in X_r$.
Let A,B be arbitrary arcs in the metric space (E,ϱ) originating at a point $p \in E$, defined respectively by the formulas

$$A = \{\varphi(t) : 0 \le t \le 1\},$$

$$B = \{\psi(t) : 0 \le t \le 1\},$$

where φ and ψ are homeomorphisms and $\varphi(0) = p = \psi(0)$.

L e m m a 1. If A is a rectifiable arc satisfying the Archimedean condition at the point $p \in E$, i.e.

$$(8) \qquad \frac{s(p,x)}{\varrho(p,x)} \xrightarrow[A \ni x \to p]{} 1,$$

where $s(p,x)$ is the length of the arc with ends at p and x, and if we have

$$(9) \qquad \frac{a(r)}{r} \xrightarrow[r \to 0+]{} 0$$

then

$$(10) \qquad \frac{1}{r} \operatorname{diam}_\varrho \left(A \cap S(p,r)_{a(r)}\right) \xrightarrow[r \to 0+]{} 0.$$

P r o o f. For arbitrary points $x,y \in A$ such that $\varrho(p,x) < \varrho(p,y)$ the following inequality holds

$$(11) \qquad \varrho(p,y) < \varrho(p,x) + s(x,y) \le s(p,y),$$

where $s(p,y)$ and $s(x,y)$ denote the lengths of the arcs with ends at p,y and x,y, respectively.

The inequality (11) is equivalent to the following inequality

$$(12) \qquad 1 < \frac{\varrho(p,x) + s(x,y)}{\varrho(p,y)} \le \frac{s(p,y)}{\varrho(p,y)}.$$

Let

$$(13) \qquad \begin{cases} \bar{X}_r \ni x_r = \varphi(t'_r) & \text{where } t'_r = \inf\{t; 0 \le t \le 1 \wedge \varphi(t) \in S(p, r-a(r))\}, \\ \bar{X}_r \ni y_r = \varphi(t''_r) & \text{where } t''_r = \sup\{t; 0 \le t \le 1 \ \ \varphi(t) \in S(p, r+a(r))\}, \end{cases}$$

where

$$\bar{X}_r = \left\{x;\ x \in E \wedge r - a(r) < \varrho(p,x) \leqslant r + a(r)\right\}.$$

From (8) we obtain

$$\frac{s(p,y_r)}{\varrho(p,y_r)} \xrightarrow[r \to 0+]{} 1.$$

Hence by (12) we get

$$(14) \qquad \frac{\varrho(p,x_r) + s(x_r,y_r)}{\varrho(p,y_r)} \xrightarrow[r \to 0+]{} 1.$$

From (13) and (14) we obtain

$$\frac{r - a(r) + s(x_r,y_r)}{r + a(r)} \xrightarrow[r \to 0+]{} 1.$$

This implies

$$\frac{1 - \frac{a(r)}{r} + \frac{s(x_r,y_r)}{r}}{1 + \frac{a(r)}{r}} \xrightarrow[r \to 0+]{} 1.$$

Taking into account (9) we have

$$(15) \qquad \frac{s(x_r,y_r)}{r} \xrightarrow[r \to 0+]{} 0.$$

From (13) by the property of Darboux for the function φ in the interval $< t'_r,\ t''_r >$ we obtain

$$(16) \qquad A \cap \bar{X}_r \subset (x_r,y_r),$$

where (x_r,y_r) is an arc with ends at the points x_r,y_r.
From (2.1) we obtain

$$(17) \qquad A \cap S(p,r)_{a(r)} \subset A \cap X_r \subset A \cap \bar{X}_r.$$

From (16) and (17) it follows that

$$\operatorname{diam}_\varrho (A\cap S(p,r)_{a(r)}) \leq \operatorname{diam}_\varrho (A\cap \bar{X}_r) = \sup_{x',x''\in A\cap\bar{X}_r} \varrho(x', x'') \leq$$

$$\leq \sup_{x',x''\in(x_r,y_r)} \varrho(x', x'') \leq s(x_r,y_r).$$

Hence we have

$$0 \leq \frac{1}{r} \operatorname{diam}_\varrho (A\cap S(p,r)_{a(r)}) \leq \frac{1}{r} s(x_r,y_r).$$

Hence by (15) we obtain the lemma.
For arbitrary sets $A,B\in E_0$ we have

$$\operatorname{diam}_\varrho (A\cup B) \leq \operatorname{diam}_\varrho A + \operatorname{diam}_\varrho B + \varrho(A,B),$$

where $\varrho(A,B) = \inf\limits_{x\in A,\, y\in B} \varrho(x,y)$.

From the definition of the functions ϱ_5 and ϱ_6 we obtain

$$\varrho_5(A,B) \leq \operatorname{diam}_\varrho A + \operatorname{diam}_\varrho B + \varrho_6(A,B). \tag{18}$$

The above considerations lead to the following theorem.

T h e o r e m 2. If the functions $a(r)$ and $b(r)$ satisfy the conditions

$$\frac{a(r)}{r} \xrightarrow[r\to 0+]{} 0 \quad \text{and} \quad \frac{b(r)}{r} \xrightarrow[r\to 0+]{} 0 \tag{19}$$

then for arbitrary rectifiable arcs A,B in the metric apace (E,ϱ) with the origin at p having the Archimedean property at p we have

$$(A,B)\in T_{\varrho_i}(a,b,1,p) \Leftrightarrow (A,B)\in T_{\varrho_j}(a,b,1,p) \tag{20}$$

for $i,j = 0,1,\ldots,7$.

P r o o f. We shall prove that

$$(21) \qquad (A,B) \in T_{\varrho_6}(a,b,1,p) \Rightarrow (A,B) \in T_{\varrho_5}(a,b,1,p).$$

Let us suppose that $(A,B) \in T_{\varrho_6}(a,b,1,p)$.
Then we have

$$\frac{1}{r}\,\varrho_6(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}) \xrightarrow[r \to 0+]{} 0.$$

From (18) we obtain

$$0 \leqslant \frac{1}{r}\,\mathrm{diam}_{\varrho}((A \cap S(p,r)_{a(r)}) \cup (B \cap S(p,r)_{b(r)})) \leqslant$$

$$\leqslant \frac{1}{r}\,\mathrm{diam}_{\varrho}(A \cap S(p,r)_{a(r)}) + \frac{1}{r}\,\mathrm{diam}_{\varrho}(B \cap S(p,r)_{b(r)}) +$$

$$+ \frac{1}{r}\,\varrho_6(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}).$$

Taking into account (19) and Lemma 1 we infer that

$$\frac{1}{r}\,\mathrm{diam}_{\varrho}\ ((A \cap S(p,r)_{a(r)}) \cup (B \cap S(p,r)_{b(r)})) \xrightarrow[r \to 0+]{} 0.$$

Hence $(A,B) \in T_{\varrho_5}(a,b,1,p)$. From (6), (6'), (7) and (21) we obtain the thesis of the theorem.

R e m a r k. If in Theorem 2 the condition (19) is formulated in the form of an alternative, then the theorem will be false.

E x a m p l e. In the Cartesian space R^2 let us consider the arcs

$$A = \left\{(t,\ t^2);\ \ 0 \leqslant t \leqslant 1\right\},$$

$$B = \left\{(t,\ 0)\ ;\ \ 0 \leqslant t \leqslant 1\right\}.$$

Let $a(r) = 0$ and $b(r) = r$, then we have

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \qquad \text{and} \qquad \frac{b(r)}{r} \xrightarrow[r \to 0+]{} 1.$$

We also have

$$A \cap S(p,r)_{a(r)} = \left\{\left(\sqrt{\frac{1}{2}\sqrt{1+4\ r^2} - \frac{1}{2}}, \quad \frac{1}{2}\sqrt{1+4\ r^2} - \frac{1}{2}\right)\right\},$$

$$B \cap S(p,r)_{b(r)} = \left\{(t,0);\quad 0 \leqslant t < 2r\right\}.$$

It is easy to show

$$\varrho_6(A \cap S(p,r)_{a(r)},\quad B \cap S(p,r)_{b(r)}) = \frac{1}{2}\sqrt{1 + 4r^2} - \frac{1}{2},$$

$$\varrho_5(A \cap S(p,r)_{a(r)},\quad B \cap S(p,r)_{b(r)}) = \sqrt{5r^2 - 4r\sqrt{\frac{1}{2}\sqrt{1+4r^2} - \frac{1}{2}}}.$$

Hence we have

$$\frac{1}{r}\ \varrho_6(A \cap S(p,r)_{a(r)},\quad B \cap S(p,r)_{b(r)}) \xrightarrow[r \to 0+]{} 0,$$

$$\frac{1}{r}\ \varrho_5(A \cap S(p,r)_{a(r)},\quad B \cap S(p,r)_{b(r)}) \xrightarrow[r \to 0+]{} 1.$$

This implies

$$(A,B) \in T_{\varrho_6}(a,b,1,p),$$

$$(A,B) \notin T_{\varrho_5}(a,b,1,p).$$

BIBLIOGRAPHY

[1] S. G o ł ą b, Z. M o s z n e r: Sur le contact des courbes dans les espaces metriques generaux, Colloq. Math. 10 (1963) 305-311.

[2] S. M i d u r a: O porównaniu definicji styczności łuków prostych w ogólnych przestrzeniach metrycznych, Rocznik Nauk.Dydakt. WSP w Krakowie, 25 (1966) 91-122.

[3] W. W a l i s z e w s k i: On the tangency of sets in a metric space, Colloq.Math. 15 (1966) 127-131.

[4] W. W a l i s z e w s k i: O symetrii relacji styczności zbiorów w przestrzeni metrycznej, Zeszyty Nauk.Uniw.Łódz. Nauki Mat.-Przyr. seria II (1966) 185-190.

[5] W. W a l i s z e w s k i: On the tangency of sets in generalized metric spaces, Ann.Polon.Math. 28 (1973) 275-284.

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