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A NOTE ON STRICT CONVEXITY AND STRAIGHT LINES IN NORMED SPACES

In his paper "Straight Lines in Metric Spaces" ([1]), C. Reda presented a theory of straight lines based on degeneracy in Hero's formula for the area of a triangle. One of the main results of that work demonstrates that in a Hilbert space, the concept of metric straight line is equivalent to the usual algebraic description of a line. In this note, we show that this is also true in a strictly convex normed space. Further, strictly convex normed spaces are characterized by this equivalence.

1. We begin with some pertinent definitions from [1]. For a real normed space $(N, \|\cdot\|)$, define

$$(1)\qquad d(x,y) = \|x-y\|,$$

$$(2)\qquad p(x,y,z) = \frac{1}{2}\left[d(x,y) + d(y,z) + d(z,x)\right],$$

and

$$(3)\qquad S(x,y,z) = \Big[p(x,y,z)(p(x,y,z)-d(x,y))(p(x,y,z) - d(y,z))(p(x,y,z) - d(z,x))\Big]^{\frac{1}{2}}.$$

A subset L of N is said to be linear if L contains more than one point and if $X(x,y,z) = 0$ for all $x,y,z \in L$.

If $\mathcal{L}$ is the set of all linear subsets of N, define the relation $\leq$ on $\mathcal{L}$ by $L_1 \leq L_2$ if $L_1 \subseteq L_2$. By Zorn's Lemma, for each $L \in \mathcal{L}$ there is a maximal set $L_0 \in \mathcal{L}$ such that $L \subseteq L_0$. The maximal sets of $\mathcal{L}$ are called metric straight lines.

If $x,e \in N$ and $e \neq 0$, the set $L(x,e)=\{x+te: t \text{ is real}\}$ is called an algebraic straight line. It is easily shown that if $x,y \in L(z,e)$ and $x \neq y$, then $L(z,e) = L(x,x-y)$. Theorem 4.2 of [1] shows that every algebraic straight line is also a metric straight line. On the other hand, Example 4 of [1] demonstrates that the converse is false in a general normed space.

Finally, $(N,\|\cdot\|)$ is strictly convex if the conditions $\|x+y\| = \|x\| + \|y\|$ and $y \neq 0$ imply that $x = \alpha y$ for some $\alpha \geqslant 0$.

T h e o r e m . *A normed space $(N, \|\cdot\|)$ is strictly convex if and only if the concepts of algebraic and metric straight lines are equivalent.*

P r o o f . 1. If $(V, \|\cdot\|)$ is strictly convex, then the equivalence of algebraic and metric straight lines follows from Theorem 4.2 and from a proof identical to that of Theorem 4.4 of [1].

2. To prove the converse, assume that algebraic and metric straight lines are equivalent in $(N, \|\cdot\|)$ and let $x,y \in N$ satisfy the conditions $\|x+y\| = \|x\| + \|y\|$ and $y \neq 0$. Then, since $d(x,0) + d(0,-y) = d(x,-y)$, $S(x,-y,0) = 0$ and hence, $\{x,-y,0\}$ is a linear set. By our remarks above, $\{x,-y,0\}$ is contained in a metric straight line L. Since L is also an algebraic straight line and $-y,0 \in L$ with $-y \neq 0$, we may write $L = L(0,y)$. Finally, the conditions $x \in L$ and $\|x+y\| = \|x\| + \|y\|$ imply that $x = \alpha y$ for some $\alpha \geq 0$. Therefore, $(N, \|\cdot\|)$ is strictly convex and the proof is complete.

BIBLIOGRAPHY

[1] C. R e d a : Straight lines in metric spaces, Demonstratio Math. 6(1973) 809-819.

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