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Kazimierz Warchulski

ON THE CAUCHY-NEUMANN PROBLEM FOR THE BIPARABOLIC EQUATION

1. In the paper we shall give the solution of the limit problem for the equation

$$(1) \qquad L^2u(X) = F(X)$$

and for the domain

$$Z = \left\{ X:\ x_i > 0\ (i=1,2,\dots,n),\ t > 0 \right\};\ X=(x_1,x_2,\dots,x_n,t),$$

where

$$L = P - bI,\ P = \sum_{i=1}^{n} D^2_{x_i} - D_t = \Delta_x - D_t$$

$b \geqslant 0$ is a positive real number, $F(X)$ is a given function defined for $X \in \bar{Z}$, where

$$\bar{Z} = \left\{ X:\ x_i \geqslant 0\ \ (i = 1,2,\dots,n),\ \ t \geqslant 0 \right\}.$$

2. Let $X_i = (e_{i1}x_1, e_{i2}x_2, \dots, e_{in}x_n, t)$ $(i = 1,2,\dots,2^n)$ where

$$e_{ik} = \begin{cases} 1 \text{ as } i = j2^k - m,\ j = 1,2,\dots,2^{n-k},\ k = 1,2,\dots,n, \\ -1 \text{ as } i \neq j2^k - m,\ m = 2^{k-1}, 2^{k-1}+1,\dots,2^k-1, \end{cases}$$

denote all possible points which are symmetric images of the point X with respect to the coordinate hyperplanes or are iterations of those symmetric images.

In the sequel we shall use the following denotations:

$$Y = (y_1,y_2,\ldots,y_n,s),\ x = (x_1,x_2,\ldots,x_n),\ y = (y_1,y_2,\ldots,y_n),$$

$$X^i = (x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n,t)\ (i = 1,2,\ldots,n),$$

$$Y^i = (y_1,y_2,\ldots,y_{i-1},y_{i+1},\ldots,y_n,s)\ (i = 1,2,\ldots,n),$$

$$X_0^i = (x_1^0,x_2^0,\ldots,x_{i-1}^0,0,x_{i+1}^0,\ldots,x_n^0,t_0)\ (i = 1,2,\ldots,n),$$

$$x_0^i = (x_1^0,x_2^0,\ldots,x_{i-1}^0,x_{i+1}^0,\ldots,x_n^0,t_0)\ (i = 1,2,\ldots,n),$$

$$X_0 = (x_1^0,x_2^0,\ldots,x_n^0,0),\quad x_0 = (x_1^0,x_2^0,\ldots,x_n^0).$$

$$Z^0 = \left\{x : x_i \geqslant 0 \quad (i = 1,2,\ldots,n)\right\},$$

$$Z^k = \left\{x^k : x_i \geqslant 0\ (i = 1,2,\ldots,n;\ i \neq k,\ t \geqslant 0\right\}\ (k=1,2,\ldots,n),$$

$$Q^1 = \left\{X : |x_i| \leqslant c_i\ (i = 1,2,\ldots,n),\quad 0 < T_0 \leqslant t \leqslant T\right\},$$

$$Q^2 = \left\{X : |x_i| \leqslant c_i \quad (i = 1,2,\ldots,n),\quad 0 \leqslant t \leqslant T\right\},$$

$$\Omega^k = \left\{X:\ 0 < c_k \leqslant x_k \leqslant \bar{c}_k,\ |x_i| \leqslant c_i\ (i=1,2,\ldots,n;\ i\neq k),\quad 0 \leqslant t \leqslant T\right\},$$

$(k=1,2,\ldots,n)$.

$$S^0 = \left\{X:\ x_i > 0 \quad (i=1,2,\ldots,n),\ t = 0\right\},$$

$$S^k = \left\{X:\ x_k=0,\ x_i > 0\ (i=1,2,\ldots,n;\ i\neq k),\ t > 0\right\}\ (k=1,2,\ldots,n),$$

where $c_i < \bar{c}_i$ $(i=1,2,\ldots,n)$ and $0 < T_0 < T$ are arbitrary positive numbers.

Let K denote the class of all functions $u(X)$, continuous with the derivatives $D_t^v D_x^{|w|} u(X)$, $|w| + 2v \leq 4$ $[w = (w_1, w_2, \dots, w_n)$ being the multiindex for which $w_i \geq 0$ $(i=1,2,\dots,n)$; $|w| = w_1+w_2+\dots+w_n]$ and satisfying the equation (1) in the set Z.

We shall construct the function $u(X)$ belonging to the class K and satisfying the initial conditions

$$(2) \quad \lim D_t^k u(X) = f_k(x_0) \ (k=0,1), \quad \text{as } X \to X_0 \in S^0, \quad X \in Z,$$

and the boundary conditions

$$(3) \quad \lim D_{x_i} P^k u(X) = g_k^i(x_0^i) \ (k=0,1), \text{ as } X \to X_0^i \in S^i, \quad X \in Z, \quad (i=1,2,\dots,n).$$

The problem formulated above we shall call the $[C - N - n]$ problem.

We moreover assume that the functions $f_k(x)$, $g_k^i(x^i)$ $(k = 0,1;\ i = 1,2,\dots,n)$ are given and defined on the sets Z^0 and Z^i $(i = 1,2,\dots,n)$ respectively.

The problem formulated above we shall solve by means of the Green function for the equation

$$(4) \qquad Lu(X) = 0.$$

Consider any point $X \in Z$ and any point $Y \in \bar{Z}$. Let

$$(5) \qquad G(X;Y) = \begin{cases} \sum_{i=1}^{2^n} V(X_i;Y) & \text{for } s < t, \\ 0 & \text{for } s \geq t,\ X \neq Y, \end{cases}$$

where $V(X_i;Y) = \exp[-b(t - s)]\, U(X_i;Y)$,

$$U(X_i;Y) = \begin{cases} (t-s)^{-\frac{n}{2}} \exp\left[-\frac{|X_i;Y|^2}{4(t-s)}\right] & \text{for } s < t, \\ 0 & \text{for } s \geqslant t,\ X \neq Y, \end{cases}$$

and $|X_i;Y|^2 = \sum_{k=1}^{n} (y_k - e_{ik}x_k)^2$.

It is easily to prove the following lemma.

L e m m a 1. The function $G(X;Y)$ given by formula (5) has the following properties:

a) for any point $X \in Z$, $G(X;Y)$ as a function of the point $Y \in \bar{Z}$ is bounded, when $X \neq Y$,

b) for any point $X \in Z$, $G(X;Y)$ as a function of the point $Y \in \bar{Z}$ is of class C^∞ and satisfies the equation

$$L^*_Y G(X;Y) = 0,$$

where $L^*_Y = \sum_{i=1}^{n} D^2_{y_i} + D_s - b$, $X \neq Y$,

c) for any point $Y \in \bar{Z}$, $G(X;Y)$ as a function of the point $X \in Z$ is bounded, when $X \neq Y$,

d) for any point $Y \in \bar{Z}$, $G(X;Y)$ as a function of the point $X \in Z$ is of class C^∞ and satisfies the equation (4), when $X \neq Y$,

e) $D_{y_i} G(X;Y)\big|_{y_i=0} = 0$ $(i = 1,2,\dots,n)$, when $X \neq Y$,

f) $\lim_{s \to t^-} G(X;Y) = 0$ for $X \in Z$, $Y \in \bar{Z}$, $x \neq y$.

In the sequel we shall call the function $G(X;Y)$ Green's function for the problem $[C - N - n]$.

3. We shall prove the lemmas on the uniform convergence of the certain integrals. Let

$$U_{p_i q_i}(x_i,t;y_i,s) = \frac{|y_i-x_i|^{p_i}}{(t-s)^{q_i}} \exp\left[-\frac{(y_i-x_i)^2}{4(t-s)}\right] \quad (i=1,2,\dots,n),$$

$$U_{pq}(X;Y) = \prod_{i=1}^{n} U_{p_i q_i}(x_i,t;y_i,s),$$

$$V_{pq}(X;Y) = \exp\left[-b(t-s)\right] U_{pq}(X;Y),$$

where $p_i \geqslant 0$, q_i $(i = 1,2,\dots,n)$ are the real numbers.

L e m m a 2. If the $f(y)$ is bounded and measurable function for the $y \in Z^0$, then the integral

$$I^1_{pq}(X) = \int_{S^0} f(y) V_{pq}(X;Y)\Big|_{s=0} dy,$$

is uniformly convergent in every set Q^1.

P r o o f . Let $M = \{\sup|f(y)| : y \in Z^0\}$. Since $\exp\left[-b(t-s)\right] \leqslant 1$ then

$$\left|I^1_{pq}(X)\right| \leqslant M \prod_{i=1}^{n} \int_0^{\infty} U_{p_i q_i}(x_i,t;y_i,0)dy_i .$$

Let $R > \max\{2c_i \ (i = 1,2,\dots,n)\}$. If $X \in Z^0$, then for $y_i > R$ $(i = 1,2,\dots,n)$ we have the inequality

$$\frac{1}{4} y_i^2 \leqslant (y_i-x_i)^2 \leqslant 4y_i^2 \qquad (i = 1,2,\dots,n). \tag{6}$$

Moreover for $m > 0$, $0 \leqslant k < \infty$ we obtain the inequality

$$k^m e^{-k} \leqslant m^e e^{-m}. \tag{7}$$

According to the formulae (6) and (7) we have the inequality

$$\int_{y_i > R} U_{p_i q_i}(x_i,t;y_i,0)dy_i \leqslant \int_{y_i>R} \frac{(2y_i)^{p_i}}{t^{q_i}} \exp\left[-\frac{(y_i)^2}{16t}\right] dy_i \leqslant$$

$$\leqslant C_1 t^{\frac{1}{2}(p_i-2q_i+2)} \int_{y_i>R} y_i^{-2} dy_i \leqslant C_2 t^{\frac{1}{2}(p_i-2q_i+2)} \qquad (i=1,2,\dots,n),$$

where C_1, C_2 are positive constans.
Besides for $0 < y_i \leqslant R$, $0 < T_0 \leqslant t \leqslant T$ we have the inequality

$$\int_0^R U_{p_i q_i}(x_i,t;y_i,0)dy_i \leqslant 2R(c_i+R)^{p_i} t^{-q_i} \qquad (i=1,2,\dots,n).$$

Then

$$(8) \qquad \int_0^{\infty} U_{p_i q_i}(x_i,t;y_i,0)dy_i \leqslant C_2 t^{\frac{1}{2}(p_i-2q_i+2)} + 2R(c_i+R)^{p_i} t^{-q_i}$$

and

$$(9) \qquad \left| I^1_{pq}(X) \right| \leqslant M \prod_{i=1}^{n} \left\{ C_2 t^{\frac{1}{2}(p_i-2q_i+2)} + 2R(c_i+R)^{p_i} t^{-q_i} \right\}.$$

The inequality (9) implies that the integral $I^1_{pq}(X)$ is uniformly convergent in every set Q^1. Let

$$S^k_t = \left\{ Y:\ y_k = 0,\ y_i > 0\ (i=1,2,\dots,n;\ i\neq k),\ 0 < s < t \right\} \ (k=1,2,\dots,n),$$

$$Z_t = \left\{ Y:\ y_i > 0\ (i=1,2,\dots,n),\ 0 < s < t \right\}.$$

L e m m a 3. If the function $g(y^k)$ is bounded and measurable for $y^k \in Z^k$ $(k = 1,2,\dots,n)$, then the integrals

$$I_{pq}^{2,k}(X) = \int\limits_{S_t^k} g(y^k)V_{pq}(X;Y)\Big|_{y^k=0}\, dy^k$$

are uniformly convergent in every set Ω^k.

P r o o f . Let $k = 1$. For $k = 2,3,\dots,n$ the proof is similar. Since $\exp\left[-b(t-s)\right] \leqslant 1$, then

$$\left|I_{pq}^{2,1}(X)\right| \leqslant M\int\limits_0^t \prod_{i=1}^{n}\int\limits_0^\infty U_{p_iq_i}(x_i,t;y_i,s)dy_i U_{p_1q_1}(x_1,t;0,s)ds,$$

where $M = \left\{\sup\left|g(y^1)\right| : y^1 \in Z^1\right\}$.

Introducing the new variables of integration

$$z_i = (y_i-x_i)\left[2(t-s)^{\frac{1}{2}}\right]^{-1} \quad (i = 1,2,\dots,n) \tag{10}$$

we obtain

$$\int\limits_0^\infty U_{p_iq_i}(x_i,t;y_i,s)dy_i \leqslant 2^{p_i+1}(t-s)^{\frac{1}{2}(p_i-2q_i+1)}\int\limits_{-\infty}^{\infty}\left|z_i\right|^{p_i} e^{-z_i^2}dz \leqslant \tag{11}$$

$$\leqslant C_3(t-s)^{\frac{1}{2}(p_i-2q_i+1)} \quad (i=1,2,\dots,n)$$

and

$$\left|I_{pq}^{2,1}(X)\right| \leqslant MC_3\int\limits_0^t (t-s)^{\frac{1}{2}\left[\sum\limits_{i=2}^{n}(p_i-2q_i)+n-1\right]} U_{p_1q_1}(x_1,t;0,s)ds,$$

where C_3 is a positive constant.

From inequality (7) we obtain that there exist numbers $\mu_1 > 1$ and $\eta_1 > -1$ such that

$$(12)\quad \left|I^{2,1}_{pq}(X)\right| \leq MC_3 c_1^{-\mu_1} \int_0^t (t-s)^{\eta_1}\, ds \leq C_4 t^{\eta_1+1} \leq C_4 T^{\eta_1+1},$$

where C_4 is a positive constant.

Then the integral $I^{2,1}_{pq}(X)$ is uniformly convergent in every set Ω^1.

L e m m a 4. If a) the function $F(Y)$ is bounded and measurable for $Y \in \bar{Z}$, b) $p_i \geqslant 0$ and q_i $(i = 1,2,\dots,n)$ are real numbers satisfying the inequality $\sum_{i=1}^{n}(p_i-2q_i)+n+2>0$, then the integral

$$I^3_{pq}(X) = \int_{Z_t} F(Y)V_{pq}(X;Y)dY$$

is uniformly convergent in every set Q^2.

P r o o f . We have

$$\left|I^3_{pq}(X)\right| \leq M\int_0^t \prod_{i=1}^{n} \int_0 U_{p_i q_i}(x_i,t;y_i,s)dy_i\, ds,$$

where $M = \left\{\sup \left|F(Y)\right| : Y \in \bar{Z}\right\}$.

Introducing the variable (10) and using the inequality (11) we obtain

$$(12a)\quad \left|I^3_{pq}(X)\right| < MC_5\int_0^t (t-s)^{\frac{1}{2}\left[\sum_{i=1}^{n}(p_i-2q_i)+n\right]} ds \leqslant$$
$$\leqslant MC_5 t^{\frac{1}{2}\left[\sum_{i=1}^{n}(p_i-2q_i)+n+2\right]} \leqslant C_6 T^{\frac{1}{2}\left[\sum_{i=1}^{n}(p_i-2q_i)+n+2\right]},$$

where C_5, C_6 are positive constans.

The inequality (12a) and the assumption b) imply uniform convergence of the integral $I^3_{pq}(X)$ in every set Q^2.

L e m m a 5. If the $F(Y)$ and $D_{y_i}F(Y)$ $(i=1,2,\dots,n)$ are bounded and continuous functions for $Y \in \bar{Z}$, then the integrals

$$I^4_{\alpha\beta}(X) = \int_{Z_t} F(Y)(t-s)D_t^{\beta}D^{\alpha}_{x_i}V(X;Y)dY, \qquad (i=1,2,\dots,n),$$

$$\alpha = 0,1,2,3,4; \qquad \beta = 0,1,2; \qquad \alpha + 2\beta \leq 4$$

are uniformly convergent in every set Q^2.

P r o o f . Indeed, if $\alpha + 2\beta \leq 3$, then from Lemma 4 the integral $I^4_{\alpha\beta}(X)$ is uniformly convergent. Consider that the case $\alpha + 2\beta = 4$.

Now

$$D_{x_i}V(X;Y) = -D_{y_i}V(X;Y) \qquad (i=1,2,\dots,n).$$

Applying then the formula for integration by parts (see [3] p.365) we obtain

$$\int_{Z_t} F(Y)(t-s)D_t^{\beta}D^{\alpha}_{x_i}V(X;Y)dY = \int_{Z_t} D_{y_i}F(y)(t-s)D_t^{\beta}D^{\alpha-1}_{x_i}V(X;Y)dY, \quad \alpha \neq 0,$$

i.e. from the lemma 4 we have that the integral $I^4_{\alpha\beta}(X)$ is uniformly convergent in every set Q^2, $\alpha \neq 0$. We observe that

$$D_tV(X;Y) = \sum_{i=1}^{n} D^2_{x_i}V(X;Y) - bV(X;Y).$$

Then for $\alpha = 0$ the proof is similar.

4. Consider the functions

$$u_1(X) = A\int_{S^0} f_0(y)G(X;Y)\Big|_{s=0} dy,$$

$$u_2(X) = At\int_{S^0}\left[f_1(y)-(\Delta_y-b)f_0(y)\right] G(X;Y)\Big|_{s=0} dy,$$

$$u_3^i(X) = -A\int_{S_t^i} g_0^i(y^i)G(X;Y)\Big|_{y^i=0} dy^i,$$

$$u_3(X) = \sum_{i=1}^{n} u_3^i(X),$$

$$u_4^i(X) = -A\int_{S_t^i}\left[bg_0^i(y^i) - g_1^i(y^i)\right] (t-s)G(X;Y)\Big|_{y^i=0} dy^i,$$

$$u_4(X) = \sum_{i=1}^{n} u_4^i(X),$$

$$u_5(X) = A\int_{Z_t} F(Y)(t-s)G(X;Y)dY,$$

where $A = (2\sqrt{\pi})^{-n}$.

We shall prove that the function

$$u(X) = \sum_{k=1}^{5} u_k(X) \tag{13}$$

is the solution of the problem $[C - N - n]$.

L e m m a 6. If the $F(Y)$ and $D_{y_i}F(Y)$ $(i=1,2,\dots,n)$ are bounded and continuous functions in the set Z, then the function $u_5(X)$ is the element of the class K.

P r o o f . Lemmas 4 and 5 and the formulas

$$\lim_{s\to t^-}\int_{S^0} F(Y)(t-s)^{\gamma} G(X;Y)dy = 0 \quad (\gamma \geqslant 0), \tag{14}$$

$$L^2\left[(t-s)G(X;Y)\right] = 0 \tag{15}$$

imply that

$$L^2u_5(X) = A\int_{S^o} F(Y)G(X;Y)\Big|_{s=t} dy,$$

where

$$A\int_{S^o} F(Y)G(X;Y)\Big|_{s=t} dy = \lim_{s\to t^-}\int_{S^o} F(Y)G(X;Y)dy.$$

Let

$$\bar{F}(Y) = \begin{cases} F(Y) & \text{for} \quad y \in Z^o,\ s \geqslant 0, \\ 0 & \text{for} \quad y \in E_n - Z^o,\ s \geqslant 0. \end{cases}$$

Then

$$A\int_{\dot{S}^o} F(Y)V(X;Y)\Big|_{s=t} dy = A\int_{E^n} \bar{F}(Y)V(X;Y)\Big|_{s=t} dy.$$

Similarly to the [2] p.454, we obtain

$$\lim_{s\to t^-} A\int_{E_n} \bar{F}(Y)V(X;Y)dy = F(X) \quad \text{for} \quad X \in Z.$$

Besides for the $i = 2,3,\dots,2^n$ the points X_i are outside of domain Z. Then we have

$$\lim_{s\to t^-} A\int_{S^o} F(Y)V(X_i;Y)dy = 0 \quad (i=2,3,\dots,2^n).$$

Then the function $u_5(X)$ is an element of the class K.

T h e o r e m 1. If the $F(Y)$ and $D_{y_i}F(Y)$ $(i = 1,2,\dots,n)$ are bounded and continuous functions in the set

$\bar{Z}$ and the $D_y^{|w|} f_0(y), f_1(y)$, $g_k^i(y^i)$ $(k = 0,1;\ |w| \leq 2;$ $i = 1,2,\dots,n)$ are bounded and measurable functions in the sets Z^o, Z^i $(i = 1,2,\dots,n)$, then the function $u(X)$ given by formula (13) is the element of class K.

P r o o f . The function $u_1(X) + u_2(X)$ and its derivatives are linear combinations with constant coefficients of the integrals in the form $I^1_{pq}(X)$. By Lemma 2 the integral $I^1_{pq}(X)$ are almost uniformly convergent in the domain Z^o. Consequently the function $u_1(X) + u_2(X)$ is continuous together with the suitable derivatives and satisfying the equation

$$L^2\left[u_1(X) + u_2(X)\right] = 0.$$

Similarly, from Lemma 3 and from the conditions

$$(16)\quad \lim_{s\to t^-} \int_{E_{n-1}} g_0^i(y^i)G(X;Y)\Big|_{y_i=0} dy^i = 0 \qquad (i = 1,2,\dots,n),$$

$$(17)\quad \lim_{s\to t^-} \int_{E_{n-1}} \left[bg_0^i(y^i)-g_1^i(y^i)\right] G(X;Y)\Big|_{y_i=0} dy^i = 0 \quad (i=1,2,\dots,n),$$

we find, that the function $u_3(X) + u_4(X)$ is continuous together with the suitable derivatives and satisfying equation

$$L^2\left[u_3(X) + u_4(X)\right] = 0.$$

Then Lemma 6 implies that the function $u(X)$ is an element of class K.

5. We shall prove that the function $u(X)$ given by formula (13) satisfies the initial conditions (2).

L e m m a 7. If the $f_o(y)$ is bounded and measurable function for $y \in Z^o$ and continuous at the point x_o, then

$$\lim u_1(X) = f_o(x_o), \quad \text{when} \quad X \to X_o \in S^o,\ X \in Z.$$

P r o o f . Let

$$u_1(X) = J_1(X) + \sum_{i=2}^{2^n} J_i(X),$$

where

$$J_1(X) = A\int_{S^o} f_o(y)V(X;Y)\Big|_{s=0} dy,$$

$$J_i(X) = A\int_{S^o} f_o(y)V(X_i;Y)\Big|_{s=0} dy \quad (i = 2,3,\dots,2^n).$$

In view of the inequality (7) and the bound of the function $f_o(y)$ we obtain

$$|J_i(X)| \leq C_7 t^{\eta_2} \quad (i = 2,3,\dots,2^n),$$

where C_7, η_2 are the positive constans.

Then

$$\lim J_i(X) = 0 \quad (i = 2,3,\dots,2^n), \quad \text{when } X \to X_o \in S^o, \quad X \in Z.$$

Let

$$\bar{f}_o(y) = \begin{cases} f_o(y) & \text{for } y \in Z^o, \\ 0 & \text{for } y \in E_n - Z^o. \end{cases}$$

From the definition of the function $f_o(y)$ we have

$$J_1(X) = Ae^{-bt}\int_{E_n} \bar{f}_o(y)U(X;Y)\Big|_{s=0} dy.$$

Since

$$A\int_{E_n} U(X;Y)\Big|_{s=0} dy = 1$$

we obtain

$$J_1(X) = e^{-bt} f_o(x_o) + \bar{J}_1(X),$$

where

$$\bar{J}_1(X) = A \int_{E_n} \left[\bar{f}_o(y) - f_o(x_o) \right] V(X;Y) \Big|_{s=0} dy .$$

As in [2] p.449 we find

$$\lim \bar{J}_1(X) = 0, \quad \text{when} \quad X \to X_o \in S^o, \ X \in Z.$$

Finally we obtain

$$\lim u_1(X) = f_o(x_o), \quad \text{when} \quad X \to X_o \in S^o, \ X \in Z.$$

L e m m a 8. If the $D_y^{|w|} f_o(y)$, $f_1(y)$ ($w \leq 2$) are bounded and measurable functions for $y \in Z^o$, the $f_1(y)$ is continuous function at the point x_o and the function $f_o(y)$ is of class C^2 in the neighbourhood of the point x_o, then

$$\lim D_t \left[u_1(X) + u_2(X) \right] = f_1(x_o), \quad \text{when} \quad X \to X_o \in S^o, \quad X \in Z.$$

P r o o f . The definition of the function $u_1(X) + u_2(X)$ and Lemma 2 imply

$$D_t \left[u_1(X) + u_2(X) \right] = A \int_{S^o} f_o(y) D_t G(X;Y) \Big|_{s=0} dy +$$

$$+ A \int_{S^o} \left[f_1(y) - (\Delta_y - b) f_o(y) \right] G(X;Y) \Big|_{s=0} dy +$$

$$+ At \int_{S^o} \left[f_1(y) - (\Delta_y - b) f_o(y) \right] D_t G(X;Y) \Big|_{s=0} dy .$$

Now

$$D_t G(X;Y)\Big|_{s=0} = \Delta_x G(X;Y)\Big|_{s=0} - bG(X;Y)\Big|_{s=0}.$$

As in the [2] p. 452 we have

$$\lim A \int_{S^o} f_o(y) D_t G(X;Y)\Big|_{s=0} dy = \Delta_x f_o(x_o) - bf_o(x_o),$$

as $X \to X_o \in S^o$, $X \in Z$.

Moreover, it is easy to prove that Lemma 7 implies the following equality

$$\lim A \int_{S^o} \left[f_1(y) - (\Delta_y - b)f_o(y)\right] G(X;Y)\Big|_{s=0} dy = f_1(x_o) - (\Delta_x - b)f_o(x_o)$$

and

$$\lim At \int_{S^o} \left[f_1(y) - (\Delta_y - b)f_o(y)\right] D_t G(X;Y)\Big|_{s=0} dy = 0,$$

when $X \to X_o$ S^o, $X \in Z$.

From the above formula and from Lemma 7 we obtain

$$\lim D_t\left[u_1(X) + u_2(X)\right] = f_1(x_o), \quad \text{when} \quad X \to X_o \in S^o, \ X \in Z.$$

L e m m a 9. If the $D_y^{|w|} f_o(y)$, $f_1(y)$, $(|w| \leq 2)$ are bounded and measurable functions for $y \in Z^o$, then

$$\lim u_2(X) = 0, \quad \text{when} \quad X \to X_o \in S^o, \ X \in Z.$$

P r o o f . Because the functions $D_y^{|w|} f_o(y)$, $f_1(y)$ $(|w| \leq 2)$ are bounded we have

$$\left|u_2(X)\right| \leq 3AMt \int_{S^o} G(X;Y)\Big|_{s=0} dy,$$

where $M = \left\{ \sup\left(\left|f_1(y)\right| \ (|w| \leq 2) \ \left|D_y^{|w|} f_o(y)\right|\right) : y \in Z^o \right\}$.

Similarly to the proof of Lemma 7 we obtain the following equality

$$\lim u_2(X) = 0, \quad \text{when} \quad X \to X_o \in S^o,\ X \in Z.$$

From inequality (12) we find

L e m m a 10. If the $g(y^i)$ are bounded and measurable functions for $y^i \in Z^i$ $(i = 1,2,\dots,n)$, then

$$\lim I_{pq}^{2,i}(X) = 0 \quad (i = 1,2,\dots,n), \quad \text{when } X \to X_o \in S^o,\ X \in Z.$$

From Lemma 10 we obtain

L e m m a 11. If the $g_k^i(y^i)$ $(k = 0,1;\ i = 1,2,\dots,n)$ are bounded and measurable functions for $y^i \in Z^i$ $(i=1,2,\dots,n)$ then

$$\lim D_t^k u_3^i(X) = 0 \quad (k = 0,1;\ i = 1,2,\dots,n),$$

$$\lim D_t^k u_4^i(X) = 0 \quad (k = 0,1;\ i = 1,2,\dots,n),$$

when $X \to X_o \in S^o$, $X \in Z$.

From inequality (12a) we have

L e m m a 12. If the $F(Y)$ is bounded and measurable function for $Y \in \bar{Z}$ and $\sum_{i=1}^{n}(p_i - 2q_i) + n + 2 > 0$, then

$$\lim I_{pq}^3(X) = 0, \quad \text{when} \quad X \to X_o \in S^o,\ X \in Z.$$

It is easy to prove, that the lemma 12 and the inequality (7) imply the following lemma.

L e m m a 13. If the $F(Y)$ is bounded and measurable function for $Y \in \overline{Z}$, then

$$\lim D_t^k u_5(X) = 0, \quad \text{when} \quad X \to X_0 \in S^0,\ X \in Z,\ (k=0,1).$$

Lemmas 7 - 13 yield the following theorem.

T h e o r e m 2. If the $D_y^{|w|} f_0(y)$, $f_1(y)$, $g_k^i(y^i)$ $(k = 0,1;\ |w| \leqslant 2;\ i = 1,2,\dots,n)$ are bounded and measurable functions property for $y \in Z^0$, $y^i \in Z^i$, $Y \in \overline{Z}$, the function $f_1(y)$ is continuous at the point x_0 and if the function $f_0(y)$ is of class C^2 in the neighbourhood of the point x_0, then the function $u(X)$ is satisfying the initial conditions (2).

6. We shall prove that the function $u(X)$ given by formula (13) satisfies the boundary conditions (3).

Let

$$W_k(X_i;Y) = \begin{cases} x_k(t-s)^{-1}V(X_i;Y)\Big|_{y_k=0} & \text{for } s < t \ (k=1,2,\dots,n), \\ 0 & \text{for } s \geqslant t,\ X \neq Y (i=1,2,\dots,2^n). \end{cases}$$

Similarly then to the [1] p.5 we find

L e m m a 14. If the $g^i(y^i)$ is bounded and measurable function for $y^i \in Z^i$ and $g^i(y^i)$ is continuous at the point $x_0^i (i=1,2,\dots,n)$, then

$$\lim A \int_{-\infty}^{t} \int_{E_{n-1}} g^i(y^i) W_i(X;Y)\,dy^i = g^i(x_0^i) \ (i = 1,2,\dots,n),$$

when $X \to X_0^i \in S^i$, $X \in Z$.

L e m m a 15. If $g_0^i(y^i)$ is bounded and measurable function for $y^i \in Z^i$ and $g_0^i(y^i)$ is continuous at the point x_0^i $(i = 1,2,\dots,n)$, then

$$\lim D_{x_k} u_3^i(X) = \begin{cases} g_0^k(x_0^k) & \text{for } i = k;\ i,k = 1,2,\dots,n, \\ 0 & \text{for } i \neq k;\ i,k = 1,2,\dots,n, \end{cases}$$

when $X \to X_0^k \in S^k$, $X \in Z$.

P r o o f . From lemma 3 we have

$$D_{x_k} u_3^i(X) = A \int_{S_t^i} g_0^i(y^i) D_{x_k} G(X;Y)\Big|_{y_i=0} dy^i \quad (i,k=1,2,\dots,n).$$

Moreover from Lemma 3 we obtain, that the $D_{x_k} u_3^i(X)$ is continuous function at the point X_0^k $(i \neq k;\ i,k=1,2,\dots,n)$. Then from the following formula

$$D_{x_k} G(X_0^k;Y)\Big|_{y_i=0} = 0 \quad \text{for} \quad s < t$$

we find

$$\lim D_{x_k} u_3^i(X) = 0 \quad (i \neq k;\ i,k=1,2,\dots,n), \quad \text{when} \quad X \to X_0^k \in S^k.$$

For $i = k$ we have

$$D_{x_k} u_3^k(X) = A \sum_{l=1}^{2^{k-1}} \sum_{j=0}^{2^{n-k}-1} \int_{S_t^k} g_0^k(y^k) W_k(X_{j2^k+l};Y) dy^k \quad (k=1,2,\dots,n).$$

Now from inequality (7) we obtain

$$(18) \quad \Big| A \int_{S_t^k} g_0^k(y^k) W_k(X_{j2^k+l};Y) dy^k \Big| \leq C_8 x_k \int_0^t (t-s)^{\eta_3} ds,$$

for $l=2,3,\dots,2^{k-1}$ and $j=0,1,\dots,2^{n-k}-1$ or $l=1$ and $j = 1,2,\dots,2^{n-k}-1$; $k=1,2,\dots,n$, where C_8 and η_3 are positive constans.

Let

$$\bar{g}_o^k(y^k) = \begin{cases} g_o^k(y^k) & \text{for } y^k \in Z^k, \\ 0 & \text{for } y^k \in E_n - Z^k \quad (k = 1,2,\dots,n). \end{cases}$$

Then

$$S\int_{S_t^k} g_o^k(y^k) W_k(X;Y) dy^k = A\int_{-\infty}^{t}\int_{E_{n-1}} \bar{g}_o^k(y^k) W_k(X;Y) dy^k \quad (k=1,2,\dots,n).$$

The above equality, Lemma 14 and the inequalities (18) imply the proposition of Lemma 15.

L e m m a 16. If the $g_m^i(y^i)$ $(m = 0,1;\ i=1,2,\dots,n)$ are bounded and measurable functions for $y^i \in Z^i$ $(i = 1,2,\dots,n)$, then

$$\lim D_{x_k} u_4^i(X) = 0 \quad (i,k=1,2,\dots,n), \text{ when } X \to X_o^k \in S^k,\ X \in Z.$$

P r o o f . Lemma 15 implies

$$(19)\quad \lim D_{x_k} u_4^i(X) = 0 \quad (i \neq k;\ i,k:1,2,\dots,n), \text{ when } X \to X_o^k \in S^k, X \in Z.$$

Besides for $i = k$ we have inequality

$$(20)\quad \left| D_{x_k} u_4^k(X) \right| \le C_9 x_k \int_0^t (t-s)^{-\frac{1}{2}} ds = 2C_9 x_k t^{\frac{1}{2}} \quad (k = 1,2,\dots,n),$$

where C_9 is a arbitrary positive number.

The inequality (20) and the equality (19) imply the proposition of Lemma 16.

L e m m a 17. If the $f_o(y)$ is bounded and measurable function in the set Z^o, then

$$\lim D_{x_k} u_1(X) = 0, \quad \text{when} \quad X \to X_0^k \in S^k,\ X \in Z.$$

P r o o f . From Lemma 2 we obtain

$$D_{x_k} u_1(X) = A \int_{S^0} f_0(y) D_{x_k} G(X;Y)\Big|_{s=0} dy \quad (k=1,2,\dots,n).$$

Moreover Lemma 2 implies, that the function $D_{x_k} u_1(X)$ $(k = 1,2,\dots,n)$ is continuous at the point $X_0^k \in S^k$ $(k = 1,2,\dots,n)$. Then the formula

$$D_{x_k} G(X_0^k;Y)\Big|_{s=0} = 0 \quad \text{for} \quad t_0 > 0$$

implies the proposition of Lemma 17.

Similarly we find the following

L e m m a 18. If the $D_y^{|w|}(y)$, $f_1(y)$ $(|w| \leq 2)$, $F(Y)$ are bounded and measurable functions in the sets Z^0 and $\overline{Z}$ respectively, then

$$\lim D_{x_k} u_m(X) = 0 \ (m = 2,5), \quad \text{when} \quad X \to X_0^k \in S^k \quad (k=1,2,\dots,n)$$

L e m m a 19. If the $g_m^i(y^i)$ $(m = 0,1;\ i = 1,2,\dots,n)$ are bounded and measurable functions for $y^i \in Z^i$ and the $g_m^i(y^i)$ are continuous at the points x_0^i $(i = 1,2,\dots,n)$ respectively, then

$$\lim D_{x_k} P\left[u_3^i(X) + u_4^i(X)\right] = \begin{cases} g_1^k(x_0^k) & \text{for } i = k \ (i,k=1,2,\dots,n), \\ 0 & \text{for } i \neq k \ (i,k=1,2,\dots,n), \end{cases}$$

when $X \to X_0^k \in S^k$, $X \in Z$.

P r o o f . From Lemma 3, the inequalities (16), (17) and from the formulas

$$P_X\left\{U(X_j;Y)\Big|_{y_i=0}\right\} = 0 \quad (j = 1,2,\dots,2^n;\ i = 1,2,\dots,n),$$

we obtain

$$P\left[u_3^i(X) + u_4^i(X)\right] = bu_3^i(X) + bu_4^i(X) +$$

$$+ A\int_{S_t^i}\left[bg_0^i(y^i) - g_1^i(y^i)\right] G(X;Y)\Big|_{y_i=0} dy^i \quad (i = 1,2,\dots,n).$$

Then Lemmas 15 and 16 imply the proposition of Lemma 19.

L e m m a 20. If the $f_0(y)$ is bounded and measurable function for $y \in Z^0$, then

$$\lim D_{x_k} Pu_1(X) = 0, \quad \text{when} \quad X \to X_0^k \in S^k,\ X \in Z\ (k=1,2,\dots,n).$$

P r o o f . From Lemma 2 we obtain

$$Pu_1(X) = bu_1(X).$$

Then Lemma 17 implies the proposition of Lemma 20.

L e m m a 21. If the $D_y^{|w|} f_0(y)$, $f_1(y)$ ($|w| \leq 2$) are bounded and measurable function for $y \in Z^0$, then

$$\lim D_{x_k} Pu_2(X) = 0, \quad \text{when} \quad X \to X_0^k \in S^k,\ X \in Z.$$

P r o o f . Lemma 2 implies the following equality

$$Pu_2(X) = bu_2(X) - A\int_{S^0}\left[f_1(y) - (\Delta_y - b)f_0(y)\right] G(X;Y)\Big|_{s=0} dy.$$

Then from Lemmas 17 and 18 we obtain the proposition of Lemma 21.

L e m m a 22. If the $F(Y)$ is bounded and measurable function for $Y \in \bar{Z}$, then

$$\lim D_{x_k} Pu_5(X) = 0, \quad \text{when} \quad X \to X_0^k \in S^k,\ X \in Z \ (k=1,2,\dots,n).$$

P r o o f . The Lemma 4 implies the following formula

$$Pu_5(X) = bu_5(X) - A \int_{Z_t} F(Y)G(X;Y)dY.$$

Moreover from Lemma 4 and from the equality

$$D_{x_k} G(X_0^k;Y) = 0$$

we obtain

$$D_{x_k}\left\{ A \int_{Z_t} F(Y)G(X;Y)dY \right\} = 0 \quad (k = 1,2,\dots,n), \text{ when } X \to X_0^k \in S^k,\ X \in Z.$$

The above equalities and Lemma 18 imply the proposition of Lemma 22.

Lammas 14 - 22 yield.

T h e o r e m 3. If the $D_y^{|w|} f_0(y)$, $f_1(y)$, $g_k^i(y^i)$ $(k = 0,1;\ |w| \leqslant 2;\ i = 1,2,\dots,n)$ are bounded and measurable functions for $y \in Z^0$, $y^i \in Z^i$ and the $g_k^i(y^i)$ are continuous functions at the points x_0^i $(k = 0,1;\ i = 1,2,\dots,n)$, then the function $u(X)$ given by formula (13) satisfies the boundary conditions (3).

Theorems 1 - 3 imply

T h e o r e m 4. If the $f_k(y)$, $g_k^i(y^i)$ $(k = 0,1;\ i = 1,2,\dots,n)$, $F(Y)$, $D_{y_i} F(Y)$ $(i = 1,2,\dots,m)$ are continuous and bounded functions for $y \in Z^0$, $y^i \in Z^i$ $(i = 1,2,\dots,n)$ and the function $f_0(y)$ is of class C^2 in the set S^0, then the function u(X) given by formula (13) is the solution of the problem [C - N - n].

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, KRAKOW
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