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ON SOME GENERALIZATIONS OF BOOLEAN ALGEBRAS

Introduction

In this paper we considere some classes of algebras which are closely connected with Boolean algebras and theories with conditional definitions.

Recently we observe a great development of research of theories with conditional definitions. Some of these examinations are of logical, others are of algebraical character. The inspiration for algebraical direction of these studies was the observation that the operations on the sets determined by sententional formulas containing terms conditionally defined do not respect the laws of Boolean algebras.

The starting point for the considerations of this paper was the paper [5] in which the authors study some axiomatical theory T. This theory is the first trial of a general characterization of the properties of the set operations determined by sentential formulas containing terms conditionally defined.

In the paper [1] another approach to these questions is given. Studying from the logical point of view the set of sensible expressions of the theory containing conditional definitions one showed the existence of some class algebras playing the similar role as Boolean algebras.

This class called here the class S has not been studied yet in universal algebras.

So the purpose of this paper was to define the class S by means of algebraic constructions and to examine its properties, in particular its connection with the known classes of algebras and with the theory T from [5].

We tried to reach it in this paper. The essential role in these studies plays the construction of the sum of a direct system of algebras from [4].

1. Definitions of the classes L, M and S

First we give the definition of the class of algebras of the type (2,2,1), which we call the class L.

An algebra $\mathfrak{A}$ belongs to L iff there exists a non-degenerated Boolean algebra $\mathfrak{B} = (B; \cup, \cap, ', 0, 1)$ such that $\mathfrak{A} = (B \times B, \#, \circ, \urcorner)$ and

(1) $$\langle x_1,y_1\rangle \# \langle x_2,y_2\rangle = \langle x_1 \cap x_2, y_1 \cup y_2\rangle ,$$

(2) $$\langle x_1,y_1\rangle \circ \langle x_2,y_2\rangle = \langle x_1 \cap x_2, y_1 \cap y_2\rangle ,$$

(3) $$\urcorner \langle x_1,y_1\rangle = \langle x_1, y_1'\rangle ,$$

for arbitrary $x_1,x_2,y_1,y_2 \in B$.

In the semilattice $(B; \cap)$ we define the operations "+" and "-" as follows:

$$x + y = x \cap y$$
$$-x = x$$

for $x,y \in B$. In this way we obtain an algebra $\mathfrak{B}^{\cap} = (B; \cap, +, -)$ similar to the algebra $\mathfrak{B}$. We get a corollary: An algebra $\mathfrak{A}$ belongs to L iff there exists a non-degenerated Boolean algebra $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B}^{\cap} \times \mathfrak{B}$.

Thus any algebra of the class L is, roughly speaking, a direct product of a semilattice and a Boolean algebra. However, algebras of thise class are not Boolean algebras since the law $\urcorner x \circ x = \urcorner y \circ y$ does not (in general) hold in them.

Now we define the class M, and we shall show later that any algebra from M is a homomorphic image of some algebra from L. The type of algebras from M is (2,2,1). An algebra $\mathfrak{A}$ belongs to M iff there exists non-degenerated Boolean algebra $\mathfrak{B} = (B; \cup, \cap, ', 0, 1)$ such that $\mathfrak{A} = (B \times B, +, \cdot, \neg)$ and

$$(4)\quad \langle x_1,y_1\rangle + \langle x_2,y_2\rangle = \langle x_1 \cap x_2, (y_1 \cup y_2) \cap (x_1 \cap x_2)\rangle ,$$

$$(5)\quad \langle x_1,y_1\rangle \cdot \langle x_2,y_2\rangle = \langle x_1 \cap x_2, y_1 \cap y_2\rangle ,$$

$$(6)\quad \neg \langle x_1,y_1\rangle = \langle x_1, x_1 \cap y_1'\rangle ,$$

for $x_1,x_2,y_1,y_2 \in B$.

The operations in the algebras from the class M are defined by formulas (10')-(12') from [1] page 158.

Thus for any non-degenerated Boolean algebras we consider two algebras: one from the class L the other from the class M.

For any Boolean algebra $\mathfrak{B} = (B; \cup, \cap, ', 0, 1)$ we define a subset S(B) of the set $B \times B$ as follows

$$(7)\quad S(B) = \{\langle x_1, x_1 \cap y_1\rangle : x_1,y_1 \in B\}.$$

So we see that the set S(B) consists of all pairs of elements of the algebra $\mathfrak{B}$ such that the first element is not greater than the second one.

Observe that S(B) is not a subalgebra of the algebra $(B \times B; *, \circ, \urcorner)$ from the class L. For example $\langle 0,0\rangle \in S(B)$, $\urcorner\langle 0,0\rangle \notin S(B)$. However, S(B) is a subalgebra of the algebra $(B \times B; +, \cdot, \neg) \in M$ although $S(B) \notin M$.

Thus we define the class S of subalgebras of algebras of the class M as follows: an algebra $\mathfrak{A}_S$ belongs to S iff there exists an algebra $\mathfrak{A} \in M$ such that $\mathfrak{A} = (B \times B; +, \cdot, \neg)$ and $\mathfrak{A}_S = (A; +, \cdot, \neg)$ is a subalgebra of $\mathfrak{A}$ satisfying

$$(*) \qquad \bigwedge_{x,y\in B} (\langle x,y\rangle \in A \Rightarrow y = x \cap y).$$

Observe that the subalgebra with the carrier S(B) belongs to the class S although we have $A \neq S(B)$. From the formulas (*) and (7) it follows only that $A \subset S(B)$.

2. The connections between classes L, M and S

Now we give some theorems describing the connections between classes L, M and S.

Theorem 1. If $\mathcal{B} = (B; \cup, \cap, ', 0, 1)$ is a non-degenerated Boolean algebra then the algebra $\mathfrak{A}_1 = (B \times B; *, \circ, \urcorner\!\!\urcorner)$ of the class L is homomorphic to the algebra $\mathfrak{A}_2 = (B \times B; +, \cdot, \neg)$ of the class M.

Homomorphism is given by the màpping $h: B \times B \longrightarrow B \times B$

$$(8) \qquad h(\langle x_1,y_1\rangle) = \langle x_1, x_1 \cap y_1\rangle \quad \text{for} \quad x_1,y_1 \in B.$$

The easy proof we omit.

Theorem 2. Let an algebra $\mathfrak{A}$ belong to L and let $\mathfrak{A} = (B \times B; *, \circ, \urcorner\!\!\urcorner)$. Then the algebra $\mathfrak{A}_S = (S(B); +, \cdot, \neg)$ and $\mathfrak{A}_S \in S$ is a homomorphic immage of the algebra $\mathfrak{A} \in L$.

For the proof it is enaugh to check that the function h given by the formula (8) satisfies $h(B \times B) = S(B)$. This is a consequence of (7) and (8).

We have already observed that the subset S(B) of the set $B \times B$ is not a subalgebra of the algebra $(B \times B; *, \circ, \urcorner\!\!\urcorner) \in L$, however it follows from Theorem 2 that S(B) is a homomorphic immage of it.

Theorem 3. Let the algebra $\mathfrak{A}$ belong to M and $\mathfrak{A} = (B \times B; +, \cdot, \neg)$. The functions $h: B \times B \longrightarrow S(B)$ defined by (8) is a retraction of the algebra $\mathfrak{A}$ onto its subalgebra $\mathfrak{A}_S = (S(B); +, \cdot, \neg)$.

We omit the proof.

3. Problem of equational definiability of the class S

The classes L and M are not equationally definiable because subalgebras of algebras in these classes do not belong in general to them. We ask if the class S is equationally definiable? This class is hereditary with respect of subalgebras.

Denote by h(S) the class of all homomorphic images of algebras of S.

Theorem 1. The class S does not contain the class h(S).

To proof this we consider the following example. Let $\mathcal{B}_1 = (B_1; \cup, \cap, ', 0, 1)$ be a Boolean algebra isomorphic to $\mathcal{B} = (B; \cup, \cap, ', 0, 1)$ where $B_1 \cap B = \emptyset$. Let us denote by g the isomorphism of $\mathcal{B}$ onto $\mathcal{B}_1$.

Consider an algebra $(R; \oplus, \odot, *)$, defined as follows

$$\langle x_1, g(x_2)\rangle \in R \text{ iff } \langle x_1, x_2\rangle \in S(B),$$

$$\langle x_1, g(x_2)\rangle \oplus \langle x_3, g(x_4)\rangle = \langle x_1 \cap x_3, g(x_2 \cup x_4) \cap x_1 \cap x_3\rangle,$$

$$\langle x_1, g(x_2)\rangle \odot \langle x_3, g(x_4)\rangle = \langle x_1 \cap x_3, g(x_2 \cap x_4)\rangle,$$

$$\langle x_1, g(x_2)\rangle^* = \langle x_1, g(x_1 \cap x_2')\rangle$$

for any $\langle x_1, x_2\rangle, \langle x_3, x_4\rangle \in S(B)$.

In view of the assumptions the set R and the above operations are well defined.

It is easy to see that the function $f: S(B) \to R$ defined by the formula

$$f(\langle x_1, x_2\rangle) = \langle x_1, g(x_2)\rangle \quad \text{for} \quad \langle x_1, x_2\rangle \in S(B)$$

- is an isomorphism of the algebra $(S(B); +, \cdot, \neg)$ onto the algebra $(R; \oplus, \odot, *)$, where $(R; \oplus, \odot, *)$ does not belong to the class S. Thus the theorem 1 holds.

The definitions of the classes L and M can be changed in such way that they S will be closed under homomorphic images. But it does not make the class S to be closed under homomorphic images. This remark is justified by the following example.

Consider the two-element Boolean algebra B_0. The algebra $\mathfrak{A}_S = (\{\langle 0,0\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}; +, \cdot, \neg)$ corresponds to B_0 and belongs to the class S. It is easy to check that the only non-trivial congruence in $\mathfrak{A}_S$ is the relation "$\sim$", defined by the following partition: $\{\langle 0,0\rangle\}, \{\langle 1,0\rangle, \langle 1,1\rangle\}$. The quotient algebra $\mathfrak{A}_S/\!\sim$ has two elements and satisfies the condition

(a) $$\neg x = x.$$

The algebra $\mathfrak{A}_S/\!\sim$ is not isomorphic to any algebra from the class S because the following theorem holds.

T h e o r e m 2. The only algebras of the class S which satisfy the equality (a) are the one-element algebras.

P r o o f . Any one-element algebra of the class S obviously satisfies (a). Let $\mathfrak{A} = (A; +, \cdot, \neg) \in S$. Suppose $\langle x_1, y_1\rangle \in A$ and satisfies (a). Thus

$$\neg \langle x_1, y_1\rangle = \langle x_1, y_1\rangle.$$

Hence $\langle x_1, x_1 \cap y_1'\rangle = \langle x_1, y_1\rangle$, and

(b) $$x_1 \cap y_1' = y_1.$$

From the last equality it follows that $(x_1 \cap y_1') \cap y_1 = y_1$. Thus $y_1 = 0$. From this together with (b) we get:

$$y_1 = x_1 \cap y_1' = x_1 \cap 0' = x_1 \cap 1 = x_1 = 0.$$

So $\langle x_1, y_1\rangle = \langle 0,0\rangle$ and $|A| = 1$.

4. Properties of some subalgebras of the algebras of the class S

We have seen that the class S is not equational. However, in view of applications of the algebras of this class in the theory of sets determined by the sentential expressions, where these expressions contain terms conditionally defined - it seems to be useful the examine the connection of this class with Boolean algebras.

It turns out that many equalites which hold in Boolean algebras hold also in the algebras of the class S, although not every equality of Boolean algebras holds in it.

L e m m a 1. Any algebra of the class S satisfies the following equalities:

(a) $x + x = x$ (a') $x \cdot x = x$

(b) $x + y = y + x$ (b') $x \cdot y = y \cdot x$

(c) $(x+y)+z = x+(y+z)$ (c') $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(d) $x \cdot (y+z) = x \cdot y + x \cdot z$ (d') $x+(y \cdot z) = (x+y) \cdot (x+z)$

We omit the proof.

L e m m a 2. The class S does not satisfy the equalities:

(a) $$x + xy = x$$

(b) $$(\neg x)x = (\neg y)y.$$

To prove this let us put $x = \langle 1,0 \rangle$, $y = \langle 0,0 \rangle$.

From Lemma 2 follows that algebras from the class S are not lattices.

T h e o r e m 1. Let $\sim$ be a relation in A defined as follows:

(**) $$\langle x_1,y_1 \rangle \sim \langle x_2,y_2 \rangle \quad \text{iff} \quad \neg \langle x_1,y_1 \rangle \cdot \langle x_1,y_1 \rangle = \neg \langle x_2,y_2 \rangle \cdot \langle x_2,y_2 \rangle .$$

Then $\sim$ is a congruence in the algebra $\mathfrak{A}_S = (A; +, \cdot, \neg) \in S$.

To prove this observe that

$$(***) \qquad \langle x_1,y_1\rangle \sim \langle x_2,y_2\rangle \quad \text{iff} \quad x_1 = x_2.$$

T h e o r e m 2. The congruence classes of the congruence $\sim$ are subalgebras. These subalgebras are Boolean algebras.

Observe that the operations in the subalgebra being the congruence class determined by the pair $\langle x_1,y_1\rangle$ are defined by the formulas:

$$(4') \quad \langle x_1,y_1\rangle + \langle x_1,y_2\rangle = \langle x_1, y_1 \cup y_2\rangle$$

$$(5') \quad \langle x_1,y_1\rangle \cdot \langle x_1,y_2\rangle = \langle x_1, y_1 \cap y_2\rangle$$

$$(6') \quad \neg\langle x_1,y_1\rangle = \langle x_1, y_1' \cap x_1\rangle \qquad \text{for} \quad \langle x_1,y_1\rangle, \langle x_2,y_2\rangle \in A.$$

The last formulas are simpler that the formulas (4)-(6), particularly in the case of the operation "+".

The 1-element and 0-element in the subalgebra under consideration are pairs $\langle x_1,x_1\rangle$ and $\langle x_1,0\rangle$, respectively. The existence of these pairs follows from the fact the congruence class is not empty. Denote $[\langle x_1,y_1\rangle]$ the class of the element $\langle x_1,y_1\rangle \in A$. Then

$$\neg\langle x_1,y_1\rangle \cdot \langle x_1,y_1\rangle = \langle x_1,0\rangle$$

$$\neg\langle x_1,y_1\rangle + \langle x_1,y_1\rangle = \langle x_1,x_1\rangle .$$

Thus the pairs $\langle x_1,0\rangle, \langle x_1,x_1\rangle$ exist in the set A and $\langle x_1,0\rangle \sim \langle x_1,y_1\rangle$, $\langle x_1,x_1\rangle \sim \langle x_1,y_1\rangle$.

From Theorem 2 it follows that the algebra $\mathfrak{A}_S \in S$ is not a Boolean algebra but it is decomposable into disjoint subalgebras being Boolean algebras. Thus in the algebra $\mathfrak{A}_S$ there exist local zeros and units although if $\sim$ has more than one class, there is neither 0 nor 1 in the whole algebra.

Observe that $\{y : \bigvee_x \langle x_1, y\rangle \in [\langle x_1, y_1\rangle]\}$ is an ideal in the algebra $\mathcal{B}$. Denote this ideal by $I_{\langle x_1,y_1\rangle}$. Let B is the carrier of $\mathcal{B}$ and $x_1 \in B$. Denote by (x_1) the principal ideal generated by x_1 in $\mathcal{B}$. Thus $I_{\langle x_1,y_1\rangle} \subset (x_1)$. In particular if $A = S(B)$ then $I_{\langle x_1,y_1\rangle} = (x_1)$.

5. <u>The sum of a direct system of Boolean algebras</u>

The problem arises, whether there exists a connection between subalgebras being congruence-classes of the congruence $\sim$ and the algebra $\mathfrak{A}_S \in S$. To find this connection we shall use the notion of the sum of a direct system of algebras given in [4]. We recal the definition of a direct system of algebras.

A direct system of algebras of the type τ is a triple $L = \langle I; \{\mathfrak{A}_i\}_{i\in I}, \{h_i^j\}_{i,j\in I,\ i\leq j}\rangle$ satisfying the following conditions:

(i) I is a non-empty set partially ordered by the relation $\leq$ with the upper bound property for any two element $i,j \in I$,

(ii) for any $i \in I$ the algebra $\mathfrak{A}_i = (X_i; F)$ of the type τ is given,

(iii) for any $i,j \in I$, $i \leq j$ the homomorphism $h_i^j : \mathfrak{A}_i \to \mathfrak{A}_j$ is given. The resulting set of homomorphisms satisfies the conditions:

(a) if $i \leq j \leq k$ then $h_j^k \circ h_i^j = h_i^k$, for $i,j,k \in I$,

(b) h_i^i is the identity mapping.

In [4] there was assumed additionally

(i') for any $i,j \in I$ the least upper bound in the set I exists,

(ii') if $i,j \in I$, $i \neq j$ then $X_i \cap X_j = \emptyset$.

In [4] the new algebra $S(L) = (\bigcup_{i\in I} X_i; F)$ was defined, and S(L) was called the sum of a direct system L. The operations in S(L) were defined as follows

$f(x_1,\dots,x_n) = f(h_{i_1}^{i_o}(x_1),\dots,h_{i_n}^{i_o}(x_n))$ when $i_o=\sup\{i_1,\dots,i_n\}$

and $x_j \in X_{i_j}$ $(j = 1,\dots,n)$.

The sets X_i $(i \in I)$ are called the components of the sum of a direct system L.

Let $\mathfrak{A}_S \in S$, $\mathfrak{A}_S = (A; +, \cdot, \neg)$ and $<x_1,y_1> \in A$. Because the class $[<x_1,x_1>]_{\sim,A}$ is well defined by the element x_1, it will be convenient to denote this class by A_{x_1}.

We denote by I the set of these elements which are the first elements of the pairs belonging to A.

Lemma 1. If $x_1,x_2 \in I$ and $x_1 \neq x_2$, then $A_{x_1} \cap A_{x_2} = \emptyset$.

We have also

Lemma 2. $\bigcup_{x_1 \in I} A_{x_1} = A$.

Denote by $\mathfrak{A}_{x_1}$ the algebra $<A_{x_1}; +, \cdot, \neg>$. From Theorem 1 of §3 and (9) it follows

Lemma 3. For any $x_1 \in I$ the algebra $\mathfrak{A}_{x_1}$ is a Boolean algebra.

In the set I we define a relation $\leqslant$ as follows:

(10) $$x_1 \leqslant x_2 \quad \text{iff} \quad x_2 = x_1 \cap x_2.$$

Lemma 4. The pair $(I; \leqslant)$ satisfies (i), (ii), (i'), (ii').

Proof. (i) and (i') are obvious. Putting $\sup\{x_1,x_2\} = x_1 \cap x_2$ we get (ii) and (ii') by Lemma 1, (9) and by the definition of the algebra $\mathfrak{A}_{x_1}$.

For $x_1 \leqslant x_2$, $x_1,x_2 \in I$, we define the mapping $h_{x_1}^{x_2}: A_{x_1} \to A_{x_2}$ as follows

(11) $$h_{x_1}^{x_2}(<x_1,y_1>) = <x_2, x_2 \cap y_1>.$$

L e m m a 5. If $x_1 \leq x_2$, $x_1, x_2 \in I$, then the function $h_{x_1}^{x_2}: A_{x_1} \longrightarrow A_{x_2}$ defined by (11) is a homomorphism satisfying conditions (a) and (b) from (iii).

P r o o f . First we show that $h_{x_1}^{x_2}: A_{x_1} \longrightarrow A_{x_2}$ is a homomorphism of $\mathfrak{A}_{x_1}$ into $\mathfrak{A}_{x_2}$ for $x_1 \leq x_2$, $x_1, x_2 \in I$.

By (4′)-(6′) and (11) we get

$$h_{x_1}^{x_2}(<x_1,y_1>+<x_1,y_2>)=h_{x_1}^{x_2}(<x_1,y_1\cup y_2>)=<x_2,x_2\cap(y_1\cup y_2)> =$$

$$=<x_2,x_2\cap y_1\cup x_2\cap y_2> = <x_2,x_2\cap y_1> + <x_2,x_2\cap y_2> =$$

$$= h_{x_1}^{x_2}(<x_1,y_1>) + h_{x_1}^{x_2}(<x_1,y_2>),$$

$$h_{x_1}^{x_2}(<x_1,y_1> \cdot <x_1,y_2>) = h_{x_1}^{x_2}(<x_1,y_1\cap y_2>) = <x_2,x_2\cap y_1\cap y_2> =$$

$$=<x_2,(x_2\cap y_1)\cap(x_2\cap y_2)> = <x_2,x_2\cap y_1> \cdot <x_2,x_2\cap y_2> =$$

$$= h_{x_1}^{x_2}(<x_1,y_1>) \cdot h_{x_1}^{x_2}(<x_1,y_2>),$$

$$h_{x_1}^{x_2}(\neg<x_1,y_1>) = h_{x_1}^{x_2}(<x_1,x_1\cap y_1'>) = <x_2,x_2\cap x_1\cap y_1'> =$$

$$=<x_1\cap x_2,\ x_1\cap x_2\cap(x_2\cap y_1)'> = \neg<x_1\cap x_2,\ x_2\cap y_1> =$$

$$=\neg<x_2,x_2\cap y_1> = \neg\, h_{x_1}^{x_2}(<x_1,y_1>).$$

Thus

$$h_{x_1}^{x_2}(<x_1,y_1> + <x_1,y_2>) = h_{x_1}^{x_2}(<x_1,y_1>) + h_{x_1}^{x_2}(<x_1,y_2>)$$

$$h_{x_1}^{x_2}(<x_1,y_1> \cdot <x_1,y_2>) = h_{x_1}^{x_2}(<x_1,y_1>) \cdot h_{x_1}^{x_2}(<x_1,y_2>)$$

$$h_{x_1}^{x_2}(\neg<x_1,y_1>) = \neg\, h_{x_1}^{x_2}(<x_1,y_1>)$$

so $h_{x_1}^{x_2}$ is a homomorphism.

Suppose that $x_1 \le x_2 \le x_3$, $x_1, x_2, x_3 \in I$. Hence there exist homomorphisms $h_{x_1}^{x_2}: \mathfrak{A}_{x_1} \to \mathfrak{A}_{x_2}$ and $h_{x_2}^{x_3}: \mathfrak{A}_{x_2} \to \mathfrak{A}_{x_3}$. For $<x_1, y_1> \in A_{x_1}$ we have by (11):

$$\left(h_{x_2}^{x_3} \circ h_{x_1}^{x_2}\right)(<x_1, y_1>) = h_{x_2}^{x_3}(<x_2, x_2 \cap y_1>) = <x_3, x_3 \cap y_1> =$$
$$= h_{x_1}^{x_3}(<x_1, y_1>).$$

Hence we get the equality

$$h_{x_2}^{x_3} \circ h_{x_1}^{x_2} = h_{x_1}^{x_3},$$

thus (a) holds.

To prove (b) let us assume that $<x_1, y_1> \in A_{x_1}$. Obviously $x_1 \le x_1$ thus $h_{x_1}^{x_1}$ exists. By (11) we have

$$h_{x_1}^{x_1}(<x_1, y_1>) = <x_1, x_1 \cap y_1> = <x_1, y_1>.$$

So $h_{x_1}^{x_1}$ is the identity on A_{x_1}.

T h e o r e m 1. The triple

$$L = <I; \{\mathfrak{A}_{x_1}\}_{x_1 \in I}, \{h_{x_1}^{x_2}\}_{x_1, x_2 \in I,\ x_1 \le x_2}>$$

is a direct system of Boolean algebras.

P r o o f . By lemma 4 the set I satisfies the conditions (i), (ii) from the definition of a direct system of algebras. By Lemma 3 the algebras $\{\mathfrak{A}_{x_1}\}_{x_1 \in I}$ are Boolean algebras. By Lemma 5 $\{h_{x_1}^{x_2}\}_{x_1, x_2 \in I, \epsilon x_1 \le x_2}$ is the required family of homomorphism.

T h e o r e m 2. There exists the sum S(L) of the direct system $L = <I; \{\mathfrak{A}_{x_1}\}_{x_1 \in I} \cdot \{h_{x_1}^{x_2}\}_{x_1, x_2 \in I,\ x_1 \le x_2}>$.

The proof follows from Theorem 2 §3 and Lemma 2 §5.

According to the definition of the sum of a direct system of algebras, the algebra S(L) is defined in the following way:

$S(L) = (\bigcup_{x_1 \in I} A_{x_1}; \#, \circ, \neg\!\!\neg)$ where for $x, y \in \bigcup_{x_1 \in I} A_{x_1}$, $x \in A_{x_1}$, $y \in A_{x_2}$.

(12) $$x \# y = h_{x_1}^{x_1 \cap x_2}(x) + h_{x_2}^{x_1 \cap x_2}(y)$$

(13) $$x \circ y = h_{x_1}^{x_1 \cap x_2}(x) \cdot h_{x_2}^{x_1 \cap x_2}(y)$$

(14) $$\neg\!\!\neg\, x = \neg x.$$

First we explain the formula (12). From the assumption that $x, y \in \bigcup_{x_1 \in I} A_{x_1}$ it follows that exist elements $x_1, x_2 \in I$ such that $x \in A_{x_1}$ and $y \in A_{x_2}$. Let $x = \langle x_1, y_1 \rangle$ and $y = \langle x_2, y_2 \rangle$. We already know that

$$\sup \{x_1, x_2\} = x_1 \cap x_2.$$

To get the value of "#" on the elements x, y in the algebra $S(L)$ one should give the images of these elements in the component $A_{x_1 \cap x_2}$ and find the value of "+" in the component $A_{x_1 \cap x_2}$ as in the algebra $\mathfrak{A}_{x_1 \cap x_2}$. Thus

$$x \# y = \langle x_1, y_1 \rangle \# \langle x_2, y_2 \rangle = h_{x_1}^{x_1 \cap x_2}(\langle x_1, y_1 \rangle) + h_{x_2}^{x_1 \cap x_2}(\langle x_2, y_2 \rangle) =$$
$$= h_{x_1}^{x_1 \cap x_2}(x) + h_{x_1}^{x_1 \cap x_2}(y).$$

Analogously we explain the formula (13). To explain (14) observe that the supremum of a one-element set is equal to this element. Hence from Lemma 5 we have

$$\neg\!\!\neg\, x = \neg\!\!\neg \langle x_1, y_1 \rangle = \neg\, h_{x_1}^{x_1}(\langle x_1, y_1 \rangle) = \neg \langle x_1, y_1 \rangle = \neg x.$$

So the operation $\urcorner\!\!\urcorner$ in the algebra S(L) coincides with the operation $\urcorner$ in the components of this algebra.

T h e o r e m 3. Every algebra $\mathfrak{A}_S = (A; +, \cdot, ')$ from the class S is the sum of a direct system of Boolean algebras.

P r o o f . To prove the theorem it is enough to show that the algebra $\mathfrak{A}_S$ is equal to the algebra S(L). From Lemma 2 it follows that

$$A = \bigcup_{x_1 \in I} A_{x_1}.$$

So we should prove the equalities

$$x \# y = x + y, \quad x \circ y = x \cdot y, \quad \urcorner\!\!\urcorner x = \urcorner x.$$

The last equality holds by (14). We show the first one also holds. Let us accept $x = \langle x_1, y_1 \rangle$, $y = \langle x_2, y_2 \rangle$. Hence we get by (12), (11)

$$x \# y = h_{x_2}^{x_1 \cap x_2}(x) + h_{x_2}^{x_1 \cap x_2}(y) =$$

$$= \langle x_1 \cap x_2, x_1 \cap x_2 \cap y_1 \rangle + \langle x_1 \cap x_2, x_1 \cap x_2 \cap y_2 \rangle =$$

$$= \langle x_1 \cap x_2, x_1 \cap x_2 \cap y_1 \cup x_1 \cap x_2 \cap y_2 \rangle =$$

$$= \langle x_1 \cap x_2, y_1 \cap x_2 \cup x_1 \cap y_2 \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle =$$

$$= x + y.$$

Similarly using (13) we show that $x \circ y = x \cdot y$. Thus the theorem is proved.

The theorem 3 allows us to explain what distinguishes the algebra $\mathfrak{A} \in S$ from other known algebras. This algebra is in general not a lattice, so it is not a Boolean algebra either. However, it is composed from the disjoint Boolean algebras and the operations in the algebra $\mathfrak{A}_S$ work also in the sum of a direct system of its subalgebras.

It seems that the algebras of the class S are natural examples of sums of direct systems.

6. Connections of the system T with the classes L, M, S

As we mentioned in the introduction in [5] there is given an axiomatic system T whose fundamental symbols are:

(a) $$\cup, \cap, ', V, \approx, S.$$

The first five symbols are the symbols of the classical algebra of sets. One accepts here that the symbol "≈" instead of "=". By means of "=" one denotes the identity in [5]. The symbol "S" is a term of a unary function mapping 2^V into 2^V. The variables of the system T are symbols X, Y,...,.

The theorems of T are all laws of the first-order calculus of quantifiers with identify and the classical algebra of sets. The characteristic axioms of the system T are:

A1. $S(X') = S(X)$

A2. $S(X \cup Y) = S(X) \cap S(Y)$

A3. $S(X \cap Y) = S(X) \cap S(Y)$

A4. $S(S(X)) = V$

A5. $S(V) = V$

A6. $X \approx Y \wedge S(X) \approx S(Y) \Rightarrow X = Y.$

The following definitions are included into the system:

D1. $X^o = X \cap S(X)$

D2. $X^* = X' \cap S(X)$

D3. $X + Y = (X \cup Y) \cap S(X) \cap S(Y)$

D4. $X \cdot Y = (X \cap Y) \cap S(X) \cap S(Y)$

D5. $A(X) \Leftrightarrow X = X \cap S(X).$

To explain the connection of the system T with algebras with L, M, S, we introduce the following two definitions of the symbols "s" and "$\sim$".

$$s(\langle x_1,x_2\rangle) = \begin{cases} \langle x_1,x_1\rangle & \text{that} \quad x_1 \neq x_2 \\ \langle 1, 1\rangle & \text{that} \quad x_1 = x_2 \end{cases}$$

for $x_1,x_2 \in B$, where $\mathcal{B} = (B; \cup, \cap, ', 0, 1)$ is a Boolean algebra.

$$\langle x_1,x_2\rangle \sim \langle y_1,y_2\rangle \quad \text{iff} \quad x_2 = y_2 \quad \text{for} \quad x_1,x_2,y_1,y_2 \in B.$$

Now let us adjoin the symbols

(b) $*\ , \circ\ , \urcorner\ , \langle 1,1\rangle\ , \sim,\ s$

to the symbols (a) respectively and the symbols $\langle x_1,x_2\rangle$, $\langle y_1,y_2\rangle$ to the symbols "X", "Y" respectively. Substituting the symbols occuring in the axioms A1-A6 and the definitions D1-D5 by the symbols (b), respectively, we get

A1'. $s(\urcorner\langle x_1,x_2\rangle) = s(\langle x_1,x_2\rangle)$

A2'. $s(\langle x_1,x_2\rangle * \langle y_1,y_2\rangle) = s(\langle x_1,x_2\rangle) \circ s(\langle y_1,y_2\rangle)$

A3'. $s(\langle x_1,x_2\rangle \circ \langle y_1,y_2\rangle) = s(\langle x_1,x_2\rangle) \circ s(\langle y_1,y_2\rangle)$

A4'. $s(s(\langle x_1,x_2\rangle)) = \langle 1,1\rangle$

A5'. $s(\langle 1,1\rangle) = \langle 1,1\rangle$

A6'. $\langle x_1,x_2\rangle \sim \langle y_1,y_2\rangle \wedge s(\langle x_1,x_2\rangle) \sim s(\langle y_1,y_2\rangle) \Rightarrow$

$\Rightarrow \langle x_1,x_2\rangle = \langle y_1,y_2\rangle$

D1. $\langle x_1,x_2\rangle^{o} = \langle x_1,x_2\rangle \circ s(\langle x_1,x_2\rangle)$

D2. $\langle x_1,x_2\rangle^{*} = \urcorner\langle x_1,x_2\rangle \circ s(\langle x_1,x_2\rangle)$

D3'. $\langle x_1,x_2\rangle + \langle y_1,y_2\rangle = (\langle x_1,x_2\rangle \# \langle y_1,y_2\rangle) \circ s(\langle x_1,x_2\rangle)\circ$

$\circ\, s(\langle y_1,y_2\rangle)$

D4'. $\langle x_2,x_2\rangle \cdot \langle y_1,y_2\rangle = \langle x_1,x_2\rangle \# \langle y_1,y_2\rangle \circ s(\langle x_1,x_2\rangle)\circ$

$\circ\, s(\langle y_1,y_2\rangle)$

D5'. $A(\langle x_1,x_2\rangle) \Leftrightarrow \langle x_1,x_2\rangle = \langle x_1,x_2\rangle \circ s(\langle x_1,x_2\rangle)$

T h e o r e m 1. The expressions A1'-A6' hold in every algebra of the class L.

P r o o f . For example we shall prove that A6' is satisfied in every $\mathfrak{A} \in L$. The proof of the others is similar.

Assume that

$$\langle x_1,x_2\rangle \sim \langle y_1,y_2\rangle \quad \text{and} \quad s(\langle x_1,x_2\rangle) \sim s(\langle y_1,y_2\rangle).$$

Hence by the definition of the symbol "s" we get

$$x_2 = y_2 \quad \text{and} \quad \langle x_1,x_1\rangle \sim \langle y_1,y_1\rangle$$

thus $\qquad x_2 = y_2$ and $x_1 = y_1$.

So $\langle x_1,x_2\rangle = \langle y_1,y_2\rangle$. Thus the expression A6 is true in every $\mathfrak{A} \in L$.

Now let us adjoin

(c) $\qquad +,\ \cdot,\ \neg,\ \langle 1,1\rangle,\ \sim,\ s,\ \langle x_1,x_2\rangle,\ \langle y_1,y_2\rangle$

to the symbols (a) and X, Y occuring in the system T, respectively.

T h e o r e m 2. The expressions obtained from A1-A6 by substituting the symbols (a) and X, Y by the symbols (c) are true in any algebra from the class M.

T h e o r e m 3. The expressions obtained from A1-A6 by substituting the symbols (a) and X, Y by the symbols (c) are true in any algebra from the class S.

T h e o r e m 4. The function "o" defined by the expression D1 is a homomorphism of an algebra fr m the class L into the corresponding algebra from the class M.

From the theorems 1-4 and from the definition of the system T it follows that the algebras of the classes L and S are models of the system T.

Final remarks

In connection with the examination of the class S we have obtained one more class of algebras considered in [2]. This class is equationally definiable. The type of this class is $<2,2>$ with the fundamental operation symbols $+$ and $\cdot$ The axioms of this class are expressions (a)-(d), (a')-(c') from Lemma 1 §4.

It is known that from these expressions and from (a) (Lemma 2 §4) there fillows (d') (Lemma 1) [comp. [6], pp.27]. Observe, that (d') does not follow from (a)-(d) and (a')-(c') Lemma 1.

Namely there exists the following algebra

$$\mathfrak{A} = (\{a_1,a_2,a_3\}, \Delta, \square), \quad \text{where}$$

$$a_i \,\Delta\, a_j = a_{\max\{i,j\}} \quad \text{for} \quad i,j \in \{1,2,3\}$$

$$a_i \,\square\, a_j = \begin{cases} a_{\min\{i,j\}} & \text{if} \quad \{i,j\} \neq \{2,3\} \\ a_3 & \text{if} \quad \{i,j\} = \{2,3\}. \end{cases}$$

In this algebra the equalities (a)-(d) and (a')-(c') hold but (d') is false since

$$a_2 \,\Delta\, (a_1 \,\square\, a_3) = a_2 \,\Delta\, a_1 = a_2$$

$$(a_2 \,\Delta\, a_1) \,\square\, (a_2 \,\Delta\, a_3) = a_2 \,\square\, a_3 = a_3.$$

In [2] the smallest congruence was found having the property that the quotient algebra is a distributive lattice.

But finding the representation of this class remains as an open problem.

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