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THE HAMILTONIAN BUNDLE OF A SYMPLECTIC LIE GROUP

Let G be a Lie group and K is a closed subgroup of G. A pair (M,Ω) consisting of a homogeneous space $M = G/K$ and a symplectic 2-form Ω on M is said to be a symplectic homogeneous space if the form Ω is G-invariant under the natural action of G on M. In the particular case of the trivial group K we call a pair (M,Ω) a symplectic Lie group. A symplectic homogeneous space (M,Ω) is said to be exact if Ω is of the form $d\beta$, where β is a G-invariant 1-form on M.

The study of symplectic homogeneous spaces was begun by Kostant [3] and Souriau [4]. Recently, remarkable results on the subject have been obtained by Chu [1] and Sternberg [6].

Let us assume $(G/K,\Omega)$ to be a symplectic homogeneous space and denote by p the natural projection $G \to G/K$. The form $\omega = p^*\Omega$ is a closed, left-invariant 2-form on G. Moreover, is $ad(K)$-invariant and if $X \in g$, where g is the Lie algebra of G, then the interior product $i_X\omega = 0$ if and only if X belongs to the Lie algebra k of K.

R e m a r k . Chu [1] proved that if ω is a closed, left-invariant 2-form on a Lie group G and G is simply connected or ω is exact, then there is a symplectic homogeneous space $(G/K,\Omega)$, where K is a closed subgroup of G, such that $\omega = p^*\Omega$.

An element X of g is called a Hamiltonian vector field iff the Lie derivative $\mathcal{L}_X\omega = 0$. From the equality

$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ it follows that Hamiltonian vector fields form a subalgebra of the Lie algebra g. Let H be the Hamiltonian subgroup of G, i.e. the connected subgroup of G with h as the Lie algebra.

L e m m a 1. H is a closed subgroup of G. The identity component K_0 of K is a normal subgroup of H.

P r o o f . The first part of the statement can be obtained as follows: H is the identity component of the closed subgroup of G consisting of all elements a of G such that $ad(a)^*\omega = \omega$. The second part of the lemma follows immediately from the equality

$$i_{[X,Y]} = [\mathcal{L}_X, i_Y]$$

which shows that $[k,h] \subset k$.

The above lemma allows to construct the fibre bundle $\pi: G/K \to G/H$ with the structure group and the fibre H/K, and the projection π defined by

$$\pi(aK) = aH$$

(comp. Steenrod [5], p. 30). This bundle will be called here a Hamiltonian bundle of the symplectic homogeneous space $(G/K,\Omega)$.

R e m a r k . $\mathcal{L}_X\omega = 0$ if and only if $i_X\omega|[g,g] = 0$. This fact implies that if $[g,g] = g$ (in particular, if g is semi-simple), then $H = K$ and the structure group of the Hamiltonian bundle is trivial. Similarly, if the group G is abelian, then $H = G$ and the Hamiltonian bundle is a trivial bundle over a point.

In the case of a symplectic Lie group the Hamiltonian bundle is a principal fibre bundle $G \to G/H$. This note is devoted to the investigation of connections in Hamiltonian bundles of symplectic Lie groups.

Let us take a subspace m of the Lie algebra g of a symplectic Lie group (G,ω) such that $g = [g,g] + m$ (direct sum). For any X of g there exists the only element Y of h such that

(1) $$i_Y\omega|m = i_X\omega|m.$$

Thus, the formula

$$\eta(X) = Y,$$

where Y satisfies (1), defines a linear mapping $\eta: g \to h$. Moreover, if $X \in h$, then $\eta(X) = X$. Thus, η is the identity mapping on h.

T h e o r e m 1. If $[h,m] \subset m$, then the formula

(2) $$\sigma|T_aG = \eta \circ L_{a^{-1}}, \quad a \in G,$$

defined a G-invariant connection form on the Hamiltonian bundle of the symplectic Lie group (G,ω).

P r o o f . Of course, the formula (2) defines a left-invariant 1-form on G. Taking an element X of h and denoting by X^* the fundamental vector field on G which respect to X (which is the left invariant vector field on G satisfying $X_e = X$) we have

$$\sigma(X^*_x) = \eta(L_{a^{-1}}L_xX) = \eta(X) = X$$

at any point x of G. Now it remains to prove that the form σ satisfies the equality

(3) $$R^*_a\sigma = ad(a^{-1})\sigma$$

for any a of H. Let $v = L_bX$, $X \in g$, be a vector tangent to G at a point b, $a \in H$. Then

$$R^*_a\sigma(v) = \sigma(R_av) = \sigma(R_aL_bX) = \eta(L_{(ba)^{-1}}R_aL_bX) = \eta(ad(a^{-1})X)$$

and

$$ad(a^{-1})\ \sigma(v) = ad(a^{-1})\ \eta(X).$$

Thus, the equality (3) is equivalent to the following one

$$\eta(ad(a)X) = ad(a)\ \eta(X). \tag{3'}$$

In order to prove (3') let us notice that $ad(H)m \subset m$ and that the form ω is $ad(H)$-invariant. Thus, if $\eta(X) = Y$, $\eta(ad(a)X) = Y'$, and $Z \in m$, then

$$i_{Y'}\omega(Z) = i_{ad(a)X}\omega(Z) = i_X\omega(ad(a^{-1})Z) =$$

$$= i_Y\omega(ad(a^{-1})Z) = i_{ad(a)Y}\omega(Z).$$

Consequently, $i_{Y'}\omega = i_{ad(a)Y}\omega$ and $Y' = ad(a)Y$. This ends the proof of the theorem.

Theorem 2. Every G-invariant connection on the Hamiltonian bundle of a symplectic Lie group (G,ω) is determined by a decomposition of g into a direct sum $[g,g]+m$, where $[h,m] \subset m$, in a manner of Theorem 1.

Proof. We have to prove that for any linear mapping $\eta: g \to h$ satisfying conditions

$$\eta | h = id$$

and

$$\eta \circ ad(a) = ad(a) \circ \eta \quad \text{for} \quad a \in H$$

there exists a linear subspace m of g such that $g = [g,g] + m$, $[h,m] \subset m$, and $i_{\eta(X)}\omega|m = i_X\omega|m$ for every X of g.

Put $h^0 = \ker \eta$ and $m = \bigcap_{H \in h^0} \ker i_X\omega$. Of course, m is a linear subspace of g. If $X \in h^0$, $Y \in h$, and $Z \in m$, then

$$i_X\omega([Y,Z]) = \omega(X,[Y,Z]) = \mathcal{L}_Y\omega(X,Z) + \omega([X,Y],Z) = 0,$$

since $[h,h^0] \subset h^0$. Thus, $[h,m] \subset m$. If $Z \in m \cap [g,g]$, then $i_Z\omega|h^0 = 0$ and $i_Z\omega|h = 0$, that is $i_Z\omega = 0$ and $Z = 0$. For any Z of g there is an element Y of m such that $i_Y\omega|h = i_Z\omega|h$. Putting $X = Z - Y$ we see that $i_X\omega|h = 0$. Using the relation $[g,g] \subset \{A \in g;\ i_A\omega|h = 0\}$ and the equalities

$$\dim\{A \in g;\ i_A\omega|h = 0\} = \dim g - \dim h =$$

$$= \dim g - \dim\{A \in g;\ i_A\omega|[g,g]\} = 0 =$$

$$= \dim g - (\dim g - \dim [g,g]) = \dim [g,g]$$

we obtain the relation $X \in [g,g]$. Thus, $g = [g,g] + m$. Finally, if $X \in h$ (resp., $X \in h^0$), then $i_{\eta(X)}\omega = i_X$ for $\eta(X) = X$ (resp., $i_X\omega|m = 0$ and $i_{\eta(X)}\omega = 0$ for $\eta(X) = 0$). It proves the theorem.

It is easy to see that the principal fibre bundle $\pi: G \longrightarrow G/H$, where H is an arbitrary closed subgroup of G, admits a G-invariant connection if and only if the homogeneous space G/H is reductive. Comparing this fact and Theorem 2 we get the following results.

C o r o l l a r y 1. If the homogeneous space G/H, where H is the Hamiltonian subgroup of a symplectic Lie group (G,ω), is reductive, then

$$[h,g] = [h,[g,g]].$$

C o r o l l a r y 2. Let (G,ω) be an exact symplectic Lie group. The homogeneous space G/H is reductive if and only if $h = 0$.

P r o o f . Let us assume that the space G/H is reductive and let m be a subspace of g determined by a G-invariant connection in the bundle $\pi: G \longrightarrow G/H$ in a manner of

Theorem 2. Then $[h,m] = 0$ since $[h,m] \subset m \cap [g,g] = 0$. It yields Corollary 1. If, in addition, $\omega = d\beta$ and $X \in h$, then

$$i_X\omega(Y) = i_X d\beta(Y) = -\beta([X,Y]) = 0$$

for all Y of m. Thus, $i_X\omega = 0$ and $X = 0$.

E x a m p l e . Let g be a 4-dimensional nilpotent Lie algebra, $\dim[g,g] = 1$. Then there exists a left-invariant symplectic form on the connected Lie group G with g as the Lie algebra. In fact, $g = g_1 + g_2$, where g_1 is a 3-dimensional abelian ideal of g ([2], Ch. I, § 2). Then $[g,g] = [g_1,g_2] \subset g_1$. Thus, it is possible to choose a basis A_1,A_2,A_3,A_4 of g in such a manner that $A_1,A_2 \in g_1$, $A_3 \in [g,g]$, and $A_4 \in g_2$. The form ω defined by

$$\omega(A_i,A_j) = \delta_i^1\delta_j^2 + \delta_i^3\delta_j^4$$

for $i,j = 1,\dots,4$, $i \leq j$, is nondegenerate and $d\omega = 0$. The algebra h of Hamiltonian vector fields on the symplectic Lie group (G,ω) is equal to g_1. On the other hand, the Corollary 1 shows that if $[g,[g,g]] = 0$, then there are no left-invariant symplectic structures on G with $h = g_2$.

REFERENCES

[1] B.Y. C h u : Symplectic homogeneous spaces, Trans. Amer. Math. Soc. 197 (1974) 145 - 159.

[2] I. K a p l a n s k y : Lie algebras and locally compact groups. Chicago 1971.

[3] B. K o s t a n t : Lectures in Modern Analysis, III, Lecture Notes in Mathematics, 170, Berlin (1970) 87-208.

[4] J.M. S o u r i a u : Structure des systemes dynamiques, Maitrises de mathematiques, Paris 1970.

[5] N. S t e e n r o d : The topology of fibre bundles, Princeton 1951.

[6] S. S t e r n b e r g : Symplectic homogeneous spaces, Trans. Amer. Math. Soc. 212 (1975) 113-131.

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