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ON SOME (ϵf_1, ϵf_2) STRUCTURE MANIFOLDS

1. Introduction

In 1964, Hashimoto defined and studied (F,G) structures on a differentiable manifold [1]. In this paper we have obtained some interesting results connecting the structure tensor of an $(\epsilon f_1, \epsilon f_2)$ structure manifold [3]. Few special structures have been defined and certain results which bring out the interrelationship are proved.

Let M^n be a C^∞ real differentiable manifold and f_1, f_2 be two non null (1,1) tensor fields of constant rank r ($r \leq n$, n is dimension of M^n), satisfying [3]

$$(1.1)\qquad \begin{cases} f_1^3 - \epsilon_1 f_1 = 0, & f_2^3 - \epsilon_2 f_2 = 0 \\ \epsilon_1 f_1^2 = \epsilon_2 f_2^2, & f_1 f_2 = \epsilon f_2 f_1, \end{cases}$$

where $\epsilon_1^2 = \epsilon_2^2 = 1$ and $\epsilon = \epsilon_1 \epsilon_2 \epsilon_3$ and $\epsilon_3 = \pm 1$. Then if we put

$$f_1 f_2 = -\epsilon f_2$$

we get

$$f_3^3 - \epsilon_3 f_3 = 0.$$

T h e o r e m 1.1. The following relations hold on $(\epsilon f_1, \epsilon f_2)$ structure manifolds

(1.2) $$f_3^3 - \epsilon_3 f_3 = 0,$$

(1.3) $$f_1 f_2 = \epsilon f_2 f_1 = -\epsilon f_3,$$

(1.4) $$f_2 f_3 = \epsilon f_3 f_2 = -\epsilon_2 f_1,$$

(1.5) $$f_3 f_1 = \epsilon f_1 f_3 = -\epsilon_1 f_2,$$

(1.6) $$\epsilon_1 f_1^2 = \epsilon_2 f_2^2 = \epsilon_3 f_3^2.$$

Let ℓ and m be two projection operators defined by

(1.7) $$\ell \overset{\text{def}}{=} \epsilon_k f_k^2, \quad m = 1 - \epsilon_k f_k^2, \quad k = 1,2,3.$$

Then we note that

(1.8) $$\ell + m = 1, \quad m\ell = \ell m = 0, \quad \ell^2 = \ell \quad \text{and} \quad m^2 = m$$

and further

(1.9) $$\begin{cases} \ell f_k = f_k \ell = f_k, & m f_k = f_k m = 0 \\ \ell f_k^2 = \epsilon_k \ell, & m f_k^2 = f_k^2 m = 0. \end{cases}$$

Thus we obtain, corresponding to ℓ and m the two distributions L and M respectively.

R e m a r k s . 1) If $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$, then $(\epsilon f_1, \epsilon f_2)$ structure acts as an almost quaternion structure on the horizontal distribution, consequently in this case the dimension of L is a multiple of 4.

2) If $\epsilon_1 = -1$, $\epsilon_2 = \epsilon_3 = 1$ then $(\epsilon f_1, \epsilon f_2)$ structure is an almost complex structure of first kind on horizontal distribution [4].

3) If $\epsilon_1 = \epsilon_2 = -1$, $\epsilon_3 = 1$ then $(\epsilon f_1, \epsilon f_2)$ structure acts as an almost complex structure of second kind [4].

4) If $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ then each of the tensor fields f_1, f_2 and f_3 of $(\epsilon f_1, \epsilon f_2)$ structure is an almost product structure on the horizontal distribution.

If we take rank $(f_k) = n$, then the above remarks from (1) to (4) hold for the entire manifold.

Since M^n always admits a positive definite Riemannian metric "g" defined by

$$g_k(X,Y) \overset{def}{=\!=} g(f_kX, f_kY) + g(mX,Y), \quad k = 1,2,3, \tag{2.0}$$

hence we have

$$g_k(f_kX,Y) = g_k(f_k^2X, f_kY). \tag{2.1}$$

A two co-tensor F_k defined by [3]

$$F_k(X,Y) = g_k(f_kX,Y) \tag{2.2}$$

always satisfies

$$F_k(X,Y) = \epsilon_k F_k(Y,X). \tag{2.3}$$

For the tensor fields f_1, f_2 the Nijenhuis tensor is defined by [3]

$$\begin{aligned} [f_1,f_2]\,(X,Y) = {} & [f_1X,f_2Y] - f_1[f_2X,Y] - f_2[X,f_1Y] + \\ & + [f_2X,f_1Y] - f_2[f_1X,Y] - f_1[X,f_2Y] + \\ & + (f_1f_2 + f_2f_1)[X,Y]. \end{aligned} \tag{2.4}$$

Let ∇ be the Riemannian connection on M^n. Then the following equalities hold.

(2.5) $$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

(2.6) $$[X,Y] = \nabla_X Y - \nabla_Y X$$

(2.7) $$\nabla_X(f_k)(Y) = \nabla_X(f_k Y) - f_k \nabla_X Y$$

(2.8) $$\nabla_X F_k(Y,Z) = g_k(\nabla_X(f_k)(Y), Z).$$

T h e o r e m 1.2. The following results hold in an $(\epsilon f_1, \epsilon f_2)$ structure manifold.

(2.9) $$f_1 \nabla_X(f_2)(f_3 Y) = \epsilon_2 \nabla_X(f_1)(f_1 Y) - \epsilon \nabla_X(f_3)(f_3 Y),$$

(3.0) $$f_2 \nabla_X(f_3)(f_1 Y) = \epsilon_1 \nabla_X(f_2)(f_2 Y) - \epsilon_2 \nabla_X(f_1)(f_1 Y),$$

(3.1) $$f_3 \nabla_X(f_1)(f_2 Y) = \epsilon \nabla_X(f_3)(f_3 Y) - \epsilon_1 \nabla_X(f_2)(f_2 Y),$$

(3.2) $$f_1 \nabla_X(f_3)(f_2 Y) = \epsilon_1 \epsilon_3 \nabla_X(f_1)(f_1 Y) - \epsilon_2 \epsilon_3 \nabla_X(f_2)(f_2 Y),$$

(3.3) $$f_2 \nabla_X(f_1)(f_3 Y) = \epsilon_2 \epsilon_3 \nabla_X(f_2)(f_2 Y) - \nabla_X(f_3)(f_3 Y),$$

(3.4) $$f_3 \nabla_X(f_2(f_1 Y) = \nabla_X(f_3)(f_3 Y) - \epsilon_1 \epsilon_3 \nabla_X(f_1)(f_1 Y).$$

T h e o r e m 1.3. In an $(\epsilon f_1, \epsilon f_2)$ structure manifold the following hold

(3.5) $$f_i \nabla_X(f_j)(f_k Y) = -\epsilon f_k \nabla_X(f_j)(f_i Y),$$

(3.6) $$f_i \nabla_X(f_j)(f_k Y) + f_j \nabla_X(f_k)(f_i Y) + f_k \nabla_X(f_i)(f_j Y) = 0,$$

(3.7) $$\nabla_X(F_k)(f_i Y, f_j Z) = -\epsilon_k \nabla_X(F_k)(f_j Y, f_i Z)$$

i,j,k being all possible different permutations of 1,2,3 $(i \neq j \neq k)$.

T h e o r e m 1.4. In an $(\epsilon f_1, \epsilon f_2)$ structure manifold the following equalities always hold

(3.8) $$f_1 \nabla_X(f_2)(X) + \nabla_X(f_1)(f_2Y) = -\epsilon\nabla_X(f_3)(Y),$$

(3.9) $$f_2 \nabla_X(f_3)(Y) + \nabla_X(f_2)(f_3Y) = -\epsilon_2 \nabla_X(f_1)(Y),$$

(4.0) $$f_3 \nabla_X(f_1)(Y) + \nabla_X(f_3)(f_1Y) = -\epsilon_1 \nabla_X(f_2)(Y).$$

Theorem 1.5. In an $(\epsilon f_1, \epsilon f_2)$ structure manifold the 2-co-tensor F_k with respect to the Riemannian connection ∇_X, always satisfies the identity

(4.1) $$\epsilon_k \nabla_X(F_i)(f_jY, f_kZ) + \epsilon_i \nabla_X(F_j)(f_kY, f_iZ) + \epsilon_j \nabla_X(F_k)(f_iY, f_jZ) = 0$$

for all possible permutations of i, j, k.

Theorem 1.6. In an $(\epsilon f_1, \epsilon f_2)$ structure manifold the following hold

(4.2) $$\frac{1}{2}\left[f_i, f_i\right](X,Y) = \nabla_{f_iX}(f_i)(Y) - \nabla_{f_iY}(f_i)(X) + f_i \nabla_Y(f_i)(X) - f_i \nabla_X(f_i)(Y),$$

(4.3) $$\left[f_i, f_j\right](X,Y) = \nabla_{f_iX}(f_j)(Y) + \nabla_{f_jX}(f_i)(Y) - f_i \nabla_X(f_j)(Y) - f_j \nabla_X(f_i)(Y) - \nabla_{f_iY}(f_j)(X) - \nabla_{f_jY}(f_i)(X) + f_i \nabla_Y(f_j)(X) + f_j \nabla_Y(f_i)(X).$$

Definition: We shall call an $(\epsilon f_1, \epsilon f_2)$ structure manifold to be an f_{ijk}-K-manifold iff

(4.4) $$f_i \nabla_X(f_j)(f_kY) = 0;$$

f_{ijk}-AK-manifold iff

$$\underset{X,Y,Z}{\tau} \nabla_X(F_i)(f_jY, f_kZ) = 0, \tag{4.5}$$

where τ denotes the cyclic sum over X, Y, Z;

f_{ij}-NK manifold iff

$$\nabla_{f_jX}(f_i)(Y) - \epsilon_j \nabla_Y(f_i)(f_jX) = 0 \tag{4.6}$$

and

$$\nabla_X(f_i)(f_jY) + \nabla_Y(f_i)(f_jX) = 0;$$

f_{ij}-QK manifold iff

$$\nabla_{f_jX}(f_iY) + \nabla_{f_j^2X}(f_i)(f_jY) = 0; \tag{4.7}$$

f_{ijk}-H manifold iff

$$\left[f_i, f_j\right] (f_kX, f_kY) = 0. \tag{4.8}$$

T h e o r e m 1.7. An f_{ijk}-K-manifold is also an f_{kji}-K-manifold.

The proof follows from (3.8) and (4.4).

T h e o r e m 1.8. If an $(\epsilon f_1, \epsilon f_2)$ structure manifold is any two of the six types f_{123}-K, f_{132}-K, f_{231}-K, f_{213}-K, f_{312}-K and f_{321}-K then it is also of the remaining types.

P r o o f . If the manifold is f_{ijk}-K and f_{jki}-K then $f_i \nabla_X(f_j)(f_kY) = 0$ and $f_j \nabla_X(f_k)(f_iY) = 0$. Now using (3.6) we obtain

$$f_k \nabla_X(f_i)(f_jY) = 0$$

and therefore manifold is f_{kij}-K. Moreover using Theorem 1.7 the proof follows.

T h e o r e m 1.9. An f_{ijk}-AK manifold is also an f_{kji}-AK manifold.

T h e o r e m 1.10. If $(\epsilon f_1, \epsilon f_2)$ structure manifold is any two of the six types f_{123}-AK, f_{132}-AK, f_{231}-AK, f_{213}-AK, f_{312}-AK, f_{321}-AK then it is essentially of the remaining types.

P r o o f . Let M^n be f_{123}-AK and f_{231}-AK then from (4.5) we have

$$(4.9)\quad \nabla_X(F_1)(f_2Y,f_3Z) + \nabla_Y(F_1)(f_2Z,f_3X) + \nabla_Z(F_1)(f_2X,f_3Y)=0,$$

$$(5.0)\quad \nabla_X(F_2)(f_3Y,f_1Z) + \nabla_Y(F_2)(f_3Z,f_1X) + \nabla_Z(F_2)(f_3X,f_1Y)=0.$$

Now adding (4.9) and (5.0) after multiplying them by ϵ_3 and ϵ_1 respectively, we obtain

$$\epsilon_3 \nabla_X(F_1)(f_2Y,f_3Z) + \epsilon_1 \nabla_X(F_2)(f_3Y,f_1Z) +$$

$$+ \epsilon_3 \nabla_Y(F_1)(f_2Z,f_3X) + \epsilon_1 \nabla_Y(F_2)(f_3Z,f_1X) +$$

$$+ \epsilon_3 \nabla_Z(F_1)(f_2X,f_3Y) + \epsilon_1 \nabla_Z(F_2)(f_3X,f_1Y) = 0.$$

And using the identity (4.1) we obtain

$$\nabla_X(F_3)(f_1Y,f_2Z) + \nabla_Y(F_3)(f_1Z,f_2X) + \nabla_Z(F_3)(f_1X,f_3Y) = 0$$

i.e. the manifold is f_{312}-AK. Now with the help of Theorem 1.9 the theorem is completely established.

T h e o r e m 1.11. An f_{ij}-NK manifold is also f_{ij}-QK manifold.

P r o o f . If M^n is f_{ij}-NK then

$$(5.1)\quad \nabla_{f_jX}(f_i)(Y) - \epsilon_j \nabla_X(f_i)(f_jX) = 0,$$

$$(5.2)\quad \nabla_X(f_i)(f_jY) + \nabla_Y(f_i)(f_kX) = 0.$$

Now

$$\nabla_{f_j X}(f_i)(Y) + \nabla_{f_j^2 X}(f_i)(f_j Y) = \nabla_{f_j X}(f_i)(Y) - \nabla_Y(f_i)(f_j^3 Y) =$$

$$= \nabla_{f_j X}(f_i)(Y) - \epsilon_j \nabla_Y(f_i)(f_j X) = 0$$

consequently M^n is f_{ij}-QK.

T h e o r e m 1.12. An f_{ijk}-H manifold is also an f_{jik}-H manifold.

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