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ON SOME SURFACES IMMERSED IN A MANIFOLD ADMITTING A SPECIAL CONCIRCULAR VECTOR FIELD

1. Introduction

Let M_n be an n-dimensional Riemannian manifold of class C^∞ covered by a system of coordinate neighbourhoods $\{U;x^h\}$ and let $g_{ji}, \left\{{h \atop j\ i}\right\}, \nabla_i$ denote the metric tensor, the Christoffel symbols formed with g_{ji} and the operator of covariant differentiation, respectively.

Let M_m $(1 < m < n)$ be an m-dimensional Riemannian manifold of class C^∞ covered by a system of coordinate neighbourhoods $\{V;y^a\}$, immersed in M_n, and let $x^h = x^h(y^a)$ be the local expression of the manifold M_m in M_n.

The indices h,i,j run over the range $\{1,2,\dots,n\}$ and the indices a,b,c,d,e over the range $\{1,2,\dots,m\}$.

If we put $B_a{}^h = \partial_a x^h$, where $\partial_a = \partial/\partial y^a$, then the fundamental metric tensor of M_m is given by

$$g_{ba} = g_{ji}\ B_a{}^j\ B_b{}^i.$$

We denote by

$$(1.1)\qquad \left\{{a \atop c\ b}\right\} = \left(\partial_c B_b{}^h + \left\{{h \atop j\ i}\right\} B_c{}^j\ B_b{}^i\right) B_d{}^k\ g^{da}\ g_{hk}$$

the Christoffel symbols determined by g_{ba}, by ∇_a the corresponding operator of covariant differentiation.

Let $\bar{R}_{abcd}$, $\bar{R}_{ad}$ and $\bar{R}$ denote the curvature tensor, Ricci tensor and the scalar curvature of M_m, respectively. Since the van der Waerden-Bortolotti covariant derivative of $B_a^{\ h}$ is given by

$$\nabla_b B_a^{\ h} = \partial_b B_a^{\ h} + \left\{ {h \atop j\ i} \right\} B_b^{\ j} B_a^{\ i} - B_c^{\ h} \left\{ {c \atop b\ a} \right\},$$

from (1.1) it follows that

$$g_{ji} (\nabla_b B_a^{\ j}) B_c^{\ i} = 0,$$

which states that $\nabla_b B_a^{\ j}$, viewed as a vector of M_n is orthogonal to M_m [4].

If we choose n-m mutually orthogonal unit vectors $N_x^{\ h}$ which are normal to M_m $(x,y = m+1,m+2,\dots,n)$, then we have

$$(1.2)\qquad \begin{cases} g_{ji} B_a^{\ j} N_x^{\ i} = 0, \\ g_{ji} N_x^{\ j} N_y^{\ i} = \delta_{xy} e_x \end{cases}$$

and

$$(1.3)\qquad g^{ab} B_a^{\ i} B_b^{\ j} = g^{ij} - \sum_x e_x N_x^{\ i} N_x^{\ j},$$

where e_x is the indicator of $N_x^{\ i}$.

Now $\nabla_b B_a^{\ h}$ can be expressed in the form

$$(1.4)\qquad \nabla_b B_a^{\ h} = \sum_x e_x H_{bax} N_x^{\ h},$$

i.e. in the form of the Gauss equations, where H_{bax} are the second fundamental tensors with respect to the directions $N_x^{\ j}$. The covariant derivative of the vectors $N_x^{\ h}$ can be written in the form

$$(1.5) \qquad \nabla_a N_x{}^h = - H_{bax}\, g^{bc}\, B_c{}^h + \sum_y e_y\, L_{axy}\, N_y{}^h,$$

where

$$L_{axy} = (\nabla_a N_x{}^i)\, N_y{}^j\, g_{ji}.$$

The equations (1.5) are the Weingarten equations for the surface M_m.

If Schouten's curvature tensor $H_{ba}{}^h = \nabla_b B_a{}^h$ of the manifold M_m [3] vanishes, then M_m is called a totally geodesic surface.

We consider now the normal bundle $N(M_m)$ of M_m. For $w^h \in N(M_m)$, we define a connection $\overset{*}{\nabla}$ on $N(M_m)$ as follows

$$\overset{*}{\nabla}_a w^h = (\nabla_a w^h)^N,$$

where $(\nabla_a w^h)^N$ denotes the normal component of $\nabla_a w^h = B_a{}^j \nabla_j w^h$ ([3], p. 254). When $\overset{*}{\nabla}_a w^h$ vanishes identically along M_m, we say that w^h is parallel with respect to the connection of the normal bundle $N(M_m)$ [2].

We say that the connection $\overset{*}{\nabla}$ of the normal bundle of M_m is trivial [2], if there exist vector fields $N_x{}^h$ satisfying the equation

$$(1.6) \qquad L_{axy} = 0.$$

An m-dimensional Riemannian manifold M_m $(m > 2)$ is said to be the Einstein manifold, if its Ricci tensor satisfies the condition

$$(1.7) \qquad \bar{R}_{ba} = \frac{\bar{R}}{m}\, g_{ba}.$$

An m-dimensional Riemannian manifold M_m $(m > 2)$ characterized by

$$\nabla_c \bar{R}_{ba} = 0 \tag{1.8}$$

will be called Ricci symmetric.

Thus each Einstein manifold ($m > 2$) is Ricci symmetric, but not conversely. If the condition $\bar{R}_{ba} = 0$ holds, then M_m is called Ricci flat.

The purpose of this paper is to obtain some results on surfaces immersed in M_n admitting a special concircular vector field v^h.

A vector field v^h is called special concircular [3], if its covariant derivative is of the form

$$\nabla_i v^h = C \delta_i^h , \tag{1.9}$$

where C is some non-constant function.

We denote by $v_a = v_h B_a{}^h$ the projection of $v_h = g_{jh} v^j$ onto M_m. The covariant derivative of v_a, in view of (1.9) and the equation

$$\nabla_b v_a = (\nabla_b v_h) B_a{}^h + v_h H_{ba}{}^h,$$

is given by

$$\nabla_b v_a = C g_{ba} + T_{ba}, \tag{1.10}$$

where ([3], p.254)

$$\nabla_b v_h = B_b{}^j \nabla_j v_h \quad \text{and} \quad T_{ba} = v_h H_{ba}{}^h.$$

2. Preliminary results

L e m m a 1. Let M_m be a connected surface immersed in a manifold M_n admitting a special concircular vector field v^h. If the scalar function $v^a v_a$ is everywhere non-zero, then the function v^aC_a cannot vanishes identically, where $C_a = \nabla_a C = B_a{}^j C_j$ and $C_j = \nabla_j C$.

P r o o f . For every manifold M_n admitting a special concircular vector field v^h we have [1]

(2.1) $$C_j\, v_h = C_h\, v_j.$$

Transvecting this with $B_a{}^j v^a$, we obtain $v^aC_a\, v_h = C_h\, v_a v^a$. Suppose now that $v^aC_a \equiv 0$. Then the last equation yields $v_a v^a C_h = 0$. Since oy assumption $v_a v^a \neq 0$ everywhere, $C_h = 0$ i.e. C is constant - a contradition. Our lemma is thus proved.

L e m m a 2. Let M_m be a surface immersed in a manifold M_n admitting a special concircular vector field v^h. Then the vector field $w^h \in N(M_m)$ given by

(2.2) $$w^h = \sum_x e_x v_j N_x{}^j\, N_x{}^h,$$

is parallel with respect to the connection $\overset{*}{\nabla}$ if and only if

(2.3) $$v^b H_{bax} = 0$$

holds. Moreover, if the connection $\overset{*}{\nabla}$ is trivial and w^h is parallel with respect to it, then

(2.4) $$\nabla_c(v_j N_x{}^j) = 0.$$

P r o o f . Transvecting (1.3) with v_j and using the definition of w^h and v_a, we find $w^j = v^j - v^b B_b{}^j$. Differentiating this covariantly and applying (1.9), (1.10), (1.4) and the equation

(2.5) $$\nabla_a v^j = B_a{}^i\, \nabla_i v^j,$$

we obtain

$$(2.6)\qquad \nabla_a w^j = -T^b{}_a B_b{}^j - v^b \sum_x e_x H_{bcx} N_x{}^j ,$$

where $T^b{}_c = g^{ba} T_{ac}$.

But (2.6) by the definition of $\overset{*}{\nabla}$, gives

$$\overset{*}{\nabla}_a w^j = - \sum_x e_x v^b H_{bax} N_x{}^j ,$$

hence by assumption that w^h is parallel with respect to $\overset{*}{\nabla}$ it follows (2.3), which completes the proof of the first part of our lemma.

If the connection $\overset{*}{\nabla}$ is trivial, then the equations (1.5) and (1.6) yield

$$(2.7)\qquad \nabla_c N_x{}^j = -H_{cdx} g^{bd} B_b{}^j .$$

Differentiating now $v_j N_x{}^j$ covariantly and using (1.9), (2.5), (2.7), (1.2) and (2.3) we obtain (2.4), as desired.

L e m m a 3. Let M_m $(m > 2)$ be a Ricci symmetric surface immersed in a manifold M_n admitting a special concircular vector field v^h. Then the relation

$$(2.8)\qquad -C_a \bar{R}_{bc} - \nabla_a T^e{}_b \bar{R}_{ec} = (m-1)\nabla_a \nabla_b C_c + \nabla_a \nabla_b (\nabla_c T - \nabla_e T^e{}_c)$$

holds, where $T = T_{ba} g^{ba}$.

P r o o f . Differentiating (1.10) covariantly, we have $\nabla_c \nabla_b v_a = C_c g_{ba} + \nabla_c T_{ba}$, so in view of the Ricci identity, it follows that

$$(2.9)\qquad - v_e \bar{R}^e{}_{abc} = C_c g_{ab} - C_b g_{ac} + \nabla_c T_{ab} - \nabla_b T_{ac} .$$

This by contraction with g^{ab} gives

(2.10) $$- v_e \bar{R}^e{}_c = (m-1)C_c + \nabla_c T - \nabla_e T^e{}_c$$

which by covariant differentiation together with (1.8), (1.10) implies that

$$- C\bar{R}_{bc} - T^e{}_b \bar{R}_{ec} = (m-1)\nabla_b C_c + \nabla_b(\nabla_c T - \nabla_e T^e{}_c).$$

But the last equation gives (2.8), which completes the proof.

L e m m a 4. Let M_m $(m > 2)$ be a Ricci symmetric surface immersed in a manifold M_n admitting a special concircular vector field v^h. If T_{ab} satisfies the equation

(2.11) $$\nabla_c T_{ab} - \nabla_b T_{ac} = p_c g_{ab} - p_b g_{ac}$$

for some vector p_c, then the condition

(2.12) $$t_d\left[v_a v^a \bar{R}_{bc} + (m-1)v^a t_a g_{bc}\right] = 0$$

holds, where

(2.13) $$t_c = C_c + p_c.$$

P r o o f . Substituting (2.11) in (2.9) we have

(2.14) $$- v_e \bar{R}^e{}_{abc} = t_c g_{ab} - t_b g_{ac},$$

whence by contraction with g^{ab} and transvection with v^a, we obtain

(2.15) $$- v_e \bar{R}^e{}_c = (m-1)t_c$$

and

(2.16) $$t_c v_b = t_b v_c,$$

respectively. Substituting now (2.11) in (2.8) we get

(2.17) $$-C_a\bar{R}_{bc} - \nabla_a T^e{}_b \bar{R}_{ec} = (m-1)\nabla_a \nabla_b t_c,$$

and consequently

$$-C_a\bar{R}_{bc} + C_b\bar{R}_{ac} - \nabla_a T^e{}_b\bar{R}_{ec} + \nabla_b T^e{}_a \bar{R}_{ec} =$$

$$= (m-1)(\nabla_a \nabla_b t_c - \nabla_b \nabla_a t_c).$$

This, in view of (2.11), (2.13) and the Ricci identity yields

$$-t_a\bar{R}_{bc} + t_b\bar{R}_{ac} = -(m-1)t_e\bar{R}^e{}_{cba}.$$

Multiplying the last equation by v_d and using (2.16) and (2.14), we find

$$t_d\left[-v_a\bar{R}_{bc} + v_b\bar{R}_{ac} - (m-1)(t_a g_{bc} - t_b g_{ac})\right] = 0,$$

from which the condition (2.12) follows by transvection with v^a and on account of (2.15). Our lemma is thus proved.

Since the tensor T_{ab} is symmetric, we have the following

R e m a r k . If M_m satisfies either of the following conditions

(2.18) $$\nabla_a T_{bc} + \nabla_b T_{ac} = K_c g_{ab},$$

(2.19) $$\nabla_a T_{bc} + \nabla_b T_{ac} = 2u_c g_{ab} - u_a g_{bc} - u_b g_{ac},$$

(2.20) $$\nabla_c T_{ab} = \frac{1}{m} T_c g_{ba},$$

then the equation (2.11) holds, where $p_b = -K_b$, $p_b = -3u_b$ and $p_c = \frac{1}{m} T_c$, $T_c = \nabla_c T$, respectively.

L e m m a 5. Let M_m $(m > 2)$ be a Ricci symmetric, connected surface, immersed in a manifold M_n admitting a special concircular vector field v^h. If the condition (2.18) or (2.19) is satisfied, $v_a v^a \neq 0$ everywhere and $t_d \equiv 0$, then M_m is Ricci flat.

P r o o f . Since $t_d \equiv 0$ the conditions (2.15) and (2.17) give

$$v_e \bar{R}^e{}_c = 0 \tag{2.21}$$

and

$$- C_a \bar{R}_{bc} + \nabla_a T^e{}_b \bar{R}_{ec} = 0,$$

whence

$$- C_b \bar{R}_{ac} + \nabla_b T^e{}_a \bar{R}_{ec} = 0,$$

and consequently

$$- C_a \bar{R}_{bc} - C_b \bar{R}_{ac} + (\nabla_a T^e{}_b + \nabla_b T^e{}_a) \bar{R}_{ec} = 0. \tag{2.22}$$

In the first case, if $C_a = K_a$, the equation (2.22) yields

$$- C_a \bar{R}_{bc} - C_b \bar{R}_{ac} + z_{ab} C_e \bar{R}^e{}_c = 0. \tag{2.23}$$

Transvecting (2.23) with v^a and using (2.21) we have

$$v^a C_a \bar{R}_{bc} + v_b C_e \bar{R}^e{}_c = 0. \tag{2.24}$$

and transvecting (2.1) with $B_a{}^j B_b{}^h$ we obtain

$$C_a v_b = C_b v_a. \tag{2.25}$$

Thus from (2.24) and (2.21) we have $v^aC_a\bar{R}_{bc} = 0$, whence, in view of Lemma 1 and the assumption that the Ricci tensor is parallel, it follows that M_m is Ricci flat.

In the remaining case, if $C_a = 3u_a$, the equation (2.22) yields (2.23), from which, in the same way as in the first case, it follows, that M_m. is Ricci flat. Our lemma is thus proved.

3. On totally geodesic surfaces

T h e o r e m 3.1. Let M_m be a connected surface immersed in a manifold M_n admitting a special concircular vector field v^h. Moreover, let the connection $\overset{*}{\nabla}$ be trivial and $v_av^a \neq 0$ everywhere. Then M_m is totally geodesic if and only if the vector field w^h defined by (2.2) is parallel with respect to the connection $\overset{*}{\nabla}$ and the condition

$$\nabla_c H_{abx} = 0 \tag{3.1}$$

holds.

P r o o f . From (1.4), (2.4) and (3.1) it follows that

$$\nabla_c T_{ab} = 0. \tag{3.2}$$

Differentiating (2.3) covariantly and using (1.10) we get

$$CH_{acx} + T^d{}_c H_{dax} + v^d \nabla_c H_{dax} = 0, \tag{3.3}$$

hence

$$CH_{acx} + T^d{}_c H_{dbx} = 0.$$

Differentiating this covariantly and applying (3.1) and (3.2), we find $C_eH_{acx} = 0$, whence

$$v^eC_eH_{acx} = 0. \tag{3.4}$$

But from the last equation in view of Lemma 1 and (3.1) it follows that M_m is totally geodesic.

Conversely, if M_m is totally geodesic, then obviously the condition (3.1) is satisfied, and from (2.6) it follows that w^h is parallel with respect to the connection $\overset{*}{\nabla}$. Our theorem is proved.

T h e o r e m 3.2. Let M_m be a connected surface immersed in a manifold M_n admitting a special concircular vector field v^h. Moreover, let the connection $\overset{*}{\nabla}$ be trivial and $v_a v^a \neq 0$ everywhere. Then M_m is totally geodesic if and only if the vector field w^h is parallel with respect to $\overset{*}{\nabla}$ and the condition

$$(3.5) \qquad \nabla_c H_{abx} = \rho_{xc} g_{ab}$$

holds, where

$$\rho_{xc} = \nabla_c \rho_x, \quad \rho_x = \frac{1}{m} g^{ab} H_{abx}.$$

P r o o f . Substituting (3.5) in (3.3) we get $CH_{acx} + T^d{}_c H_{dax} + v_a \rho_{xc} = 0$, from which, by transvection with v^a and by (2.3), we obtain $v_a v^a \rho_{xc} = 0$. Thus the condition (3.5) reduces to (3.1) which in view of Theorem 3.1 completes the proof of the theorem.

T h e o r e m 3.3. Let M_m be a connected surface immersed in a manifold M_n admitting a special concircular vector field v^h. Moreover, let the connection $\overset{*}{\nabla}$ be trivial and let $v_a v^a \neq 0$ everywhere. Then M_m is totally geodesic if and only if the vector field w^h is parallel with respect to $\overset{*}{\nabla}$ and the condition

$$(3.6) \qquad \nabla_c H_{abx} + \nabla_b H_{acx} = K_{xa} g_{bc}$$

holds for some vector fields K_{xa}.

P r o o f . Transvecting (3.6) with v^a and v^c we obtain

$$(3.7) \qquad v^a \nabla_c H_{abx} + v^a \nabla_b H_{acx} = v^a K_{xa} g_{bc},$$

and

$$v^{c}\nabla_{c}H_{acx}+v^{c}\nabla_{b}H_{acx}=K_{xa}v_{b}, \tag{3.8}$$

respectively. Multiplying both sides of (3.3) by $e_x v_j N_x{}^j$, summing over x and using (2.4) and the definition of T_{ab}, we get

$$CT_{ab}+T_{da}T^{d}{}_{b}+v^{d}\nabla_{b}T_{da}=0, \tag{3.9}$$

whence

$$v^{d}\nabla_{b}T_{da}=v^{d}\nabla_{a}T_{db}. \tag{3.10}$$

In the same way, the conditions (3.7) and (3.8) give

$$v^{d}\nabla_{c}T_{bd}+v^{d}\nabla_{b}T_{dc}=v^{d}K_{d}g_{bc} \tag{3.11}$$

and

$$v^{d}\nabla_{d}T_{bc}+v^{d}\nabla_{b}T_{cd}=K_{c}v_{b} \tag{3.12}$$

respectively, where

$$K_{c}=\sum_{x}e_{x}v_{j}N_{x}{}^{j}K_{xc}.$$

From (3.10) and (3.12) it follows that

$$K_{a}v_{b}=K_{b}v_{a}. \tag{3.13}$$

Applying (3.10) in (3.11) we find

$$v^{d}\nabla_{c}T_{bd}=\frac{1}{2}v^{d}K_{d}g_{bc}.$$

Substituting now the last equation in (3.9) we obtain

$$CT_{ab}+T_{ea}T^{e}{}_{b}+\frac{1}{2}v^{e}K_{e}g_{ab}=0, \tag{3.14}$$

whence by transvection with v^b and in virtue of (2.3) and (3.13) we have $v_a v^a K_b = 0$ and consequently $K_b = 0$. Thus multiplying (3.6) by $e_x v_j N_x{}^j$, summing over x and using the definition of K_b we get

$$\nabla_c T_{ab} + \nabla_b T_{ac} = 0,$$

whence

$$(3.15) \qquad \nabla_c T_{ab} = 0.$$

Differentiating now (3.14) covariantly and substituting (3.15) and $K_b = 0$ we obtain $C_e T_{ab} = 0$ hence transvecting this with v^e we find $v^e C_e T_{ab} = 0$. From Lemma 1 and (3.15) we conclude that $T_{ab} = 0$. Thus the equation (3.3) is reduced to $CH_{acx} + v^d \nabla_c H_{dax} = 0$. But from the last relation and (3.7) it follows that $2CH_{acx} + v^d K_{xd} g_{ac} = 0$, from which $-2C\varrho_x = v^d K_{xd}$, and, consequently, $H_{acx} = \varrho_x g_{ac}$. Differentiating this covariantly we obtain (3.5) which in view of Theorem 2 completes the proof.

T h e o r e m 3.4. Let M_m be a connected surface immersed in a manifold M_n admitting a special concircular vector field v^h such that $v^a C_a \neq 0$ everywhere. Then M_m is totally geodesic if and only if the vector field w^h is parallel with respect to the connection $\overset{*}{\nabla}$ and the following conditions

$$(3.16) \qquad \nabla_d \nabla_c H_{abx} - \nabla_c \nabla_d H_{abx} = 0,$$

$$(3.17) \qquad \nabla_b T_{ad} = \nabla_d T_{ab}$$

hold.

P r o o f . In view of the Ricci identity, the equation (3.16) yields

$$H_{ebx} \bar{R}^e{}_{acd} + H_{aex} \bar{R}^e{}_{bcd} = 0,$$

whence by transvection with v^c we obtain

$$H^e{}_{bx}v^c\bar{R}_{cdea} + H^e{}_{ax}v^c\bar{R}_{cdeb} = 0. \tag{3.18}$$

On the other hand substituting (3.17) in (2.9), we have $-v_e\bar{R}^e{}_{abc} = C_c g_{ab} - C_b g_{ac}$. The last equation together with (3.18) gives

$$C_a H_{dbx} - g_{ad}C_e H^e{}_{bx} + C_b H_{dax} - g_{bd}C_e H^e{}_{ax} = 0.$$

Transvecting this relation with v^a and using (2.25) and (2.3) we get (3.4), from which it follows that M_m is totally geodesic.

Conversely if M_m is totally geodesic then the conditions (3.16) and (3.17) are satisfied and since by (2.6) w^h is parallel with respect to $\overset{*}{\nabla}$. Hence our theorem is proved.

4. <u>On Ricci symmetric surfaces</u>

T h e o r e m 4.1. Let M_m $(m > 2)$ be a Ricci symmetric connected surface immersed in a manifold M_n admitting a special concircular vector field v^h, such that everywhere $v_a v^a \neq 0$. If T_{ab} satisfies the condition (3.17) then M_m is an Einstein manifold.

P r o o f . Substituting (3.17) in (2.8) we obtain

$$-\nabla_a T_{eb}\bar{R}^e{}_c - C_a\bar{R}_{bc} = (m-1)\nabla_a \nabla_b C_c,$$

whence

$$-\nabla_b T_{ea}\bar{R}^e{}_c - C_b\bar{R}_{ac} = (m-1)\nabla_b \nabla_a C_c.$$

These equations, together with (3.17) and the Ricci identity, yield

$$-C_a\bar{R}_{bc} + C_b\bar{R}_{ac} = -(m-1)C_e\bar{R}^e{}_{cba}.$$

Multiplying this by v_d and using (2.25), (2.9) and (3.17), we find

$$C_d\left[- v_a\bar{R}_{bc} + v_b\bar{R}_{ac} - (m-1)(C_a g_{bc} - C_b g_{ac})\right] = 0$$

so transvecting this with $v^a v^d$ and using (2.10) we have

$$(4.1)\quad v^d C_d\left[- v_a v^a \bar{R}_{bc} - (m-1)v^a C_a g_{bc} - (\nabla_c T - \nabla_e T^e{}_c)\right] = 0.$$

But from (3.17) it follows that $\nabla_c T - \nabla_e T^e{}_c = 0$, thus (4.1) give

$$(4.2)\qquad v^d C_d\left[- v_a v^a \bar{R}_{bc} - (m-1)v^a C_a g_{bc}\right] = 0.$$

Contracting (4.2) with g^{bc}, we obtain

$$v^d C_d\left[- v_a v^a \frac{\bar{R}}{m}\right] = v^d C_d (m-1) v^a C_a.$$

Substituting this in (4.2) we obtain

$$v^d C_d\left[\bar{R}_{bc} - \frac{\bar{R}}{m} g_{bc}\right] = 0.$$

Hence, in view of Lemma 1 and (1.8) we get (1.7) which completes the proof.

T h e o r e m 4.2. Let M_m $(m > 2)$ be a Ricci symmetric, connected surface immersed in a manifold M_n admitting a special concircular vector field v^h such that everywhere $v_a v^a \neq 0$. If T_{ab} satisfies (2.11) and the vector field t_c defined by (2.13) is non-zero, then M_m is an Einstein manifold.

P r o o f . Transvecting (2.12) with g_{bc} we obtain

$$-v_a v^a \frac{\bar{R}}{m} t_d = (m-1) v^a t_a t_d.$$

Substituting this into (2.12) we get

$$t_d\left[\bar{R}_{bc} - \frac{\bar{R}}{m} g_{bc}\right] = 0.$$

Since t_d is non-zero and the condition (1.8) holds the last equation gives (1.7). Our theorem is thus proved.

Theorem 4.2, Lemma 4, Remark and Lemma 5 imply together the following corollary.

C o r o l l a r y . Let M_m $(m > 2)$ be a Ricci symmetric, not Ricci flat, connected surface immersed in a manifold M_n admitting a special concircular vector field v^h such that everywhere $v_a v^a \neq 0$. If one of the conditions (2.18), (2.19), (2.20) is satisfied, then M_m is an Einstein manifold.

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