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SOME f(3, ε) STRUCTURE MANIFOLDS

Introduction

The idea of an f structure on a differentiable manifold was initiated and developed by K. Yano [5]. Later Singh and Srivastava [4] studied few special structures on f structure manifolds with torsion. They established an inclusion relation between them and studied conformal diffeomorphisms between two such structures.

Present paper is a study of certain structures on a C^∞ real differentiable manifold equipped with (1,1) tensor field f satisfying $f^3 = \epsilon f$, $\epsilon = \pm 1$. We have defined some special structures on this manifold and established an inclusion relation between them.

1. Preliminaries

Let M^n be a C^∞ real differentiable manifold, $\mathcal{F}(M^n)$ be the ring of C^∞ real functions on it and $\mathcal{X}(M^n)$ be the $\mathcal{F}(M)$-module of vector fields on M^n.

An $f(3,\epsilon)$ manifold M^n is a differentiable manifold equipped with a C^∞-(1,1) tensor field f of constant rank r which satisfies

$$(1.1) \qquad f^3 = \epsilon f, \quad \epsilon = \pm 1.$$

If we put [6]

$$(1.2)\qquad \ell \overset{\text{def}}{=\!=} \epsilon f^2, \qquad m \overset{\text{def}}{=\!=} 1 - \epsilon f^2,$$

then

$$(1.3)\qquad \begin{cases} \ell + m = 1, & \ell m = m\ell = 0, \\ \ell^2 = 1, & m^2 = m, \\ \ell f = f\ell = f, & mf = fm = 0, \\ f^2\ell = \epsilon\ell, & mf^2 = f^2 m = 0. \end{cases}$$

Thus ℓ and m are complementary projection operators which determine two complementary distributions L and M.

It is known that M^n always admits a positive definite Riemannian metric g [2] such that

$$(1.4)\qquad g(X,Y) = g(fX,fY) + g(mX,Y).$$

A 2-co-tensor field F defined by [5]

$$(1.5)\qquad F(X,Y) = g(fX,Y)$$

always satisfies

$$(1.6)\qquad F(X,Y) = \epsilon F(Y,X).$$

Moreover

$$(1.7)\qquad g(fX,Y) = g(f^2X,fY).$$

Let ∇_X be the Riemannian connection corresponding to the metric g on M^n. Thus

(1.8) $$X(g(Y,Z)) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z),$$

(1.9) $$[X,Y] = \nabla_X Y - \nabla_Y X.$$

The Nijenhuis tensor N of type (1,2) is given by [5]

(1.10) $$N(X,Y) = [fX,fY] - f[fX,Y] - f[X,fY] + f^2[X,Y].$$

The derivatives of f and F are given by

(1.11) $$\nabla_X(f)(Y) = \nabla_X(fY) - f\nabla_X Y,$$

(1.12) $$\nabla_X(F)(Y,Z) = g(\nabla_X(f)(Y),Z).$$

2. Some theorems involving the structure tensor

We require the formula for the derivative of 2-co-tensor F [1] i.e.

(2.1) $$dS(X,Y,Z) \overset{\text{def}}{=\!=} \tau_{X,Y,Z}\, \nabla_X(F)(Y,Z),$$

where τ denotes the cyclic sum over X,Y,Z. We have the following

Theorem 2.1. The following relations hold:

(2.2) $$\nabla_X(F)(f^2Y,fZ) = -\epsilon\,\nabla_X(F)(fY,f^2Z),$$

(2.3) $$dS(fX,fY,fZ) - dS(fX,f^2Y,f^2Z) + g(fX,N(fY,f^2Z)) =$$
$$= 2\,\nabla_{fX}(F)(fY,fZ)+(1+\epsilon)\left[\nabla_{fY}(F)(fZ,fX)+ \nabla_{f^2Z}(F)(fY,f^2X)\right],$$

(2.4) $$dS(fX,fY,fZ) - dS(fX,f^2Y,f^2Z) + dS(fY,f^2Z,f^2X) +$$
$$+ dS(fZ,f^2X,f^2Y) = 2\nabla_{fX}(F)(fY,fZ)+(1-\epsilon)\,\nabla_{f^2Z}(F)(f^2Y,fZ)+$$
$$+ (1+\epsilon)\left[\nabla_{f^2Y}(F)(fZ,f^2X) - \nabla_{f^2Z}(F)(fX,f^2Y)\right],$$

(2.5) $$g(N(fX,f^2Y),fZ) - g(N(fX,fZ),f^2Y) - g(N(f^2Y,fZ) =$$
$$= (1-\epsilon)\ \nabla_{f^2X}(F)(f^2Y,fZ) + 2\epsilon\nabla_{fX}(F)(fY,fZ) +$$
$$+ (1+\epsilon)\left[\nabla_{f^2Z}(F)(f^2Y,fX) - \nabla_{fY}(F)(fZ,fX)\right].$$

3. Special $f(3,\epsilon)$ structures and their inclusion relation

Definitions. We shall call an $f(3,\epsilon)$ structure manifold an fK manifold iff

$$\nabla_{fX}(f) = 0,$$

an fAK-manifold iff

$$dS(fX,fY,fZ) = 0,$$

fNK-manifold iff

$$\nabla_{fX}(F)(fY,fZ) - \epsilon\nabla_{fY}(F)(fX,fZ) = 0,$$

fQK-manifold iff

$$2\nabla_{fX}(F)(fY,fZ) + (1-\epsilon)\ \nabla_{f^2X}(F)(f^2Y,fZ) =$$
$$= (1+\epsilon)\left[\nabla_{f^2Z}(F)(fX,f^2Y) - \nabla_{f^2Y}(F)(fZ,f^2X)\right],$$

fH-manifold iff

$$N(fX,fY) = 0$$

for any $X,Y,Z \in \mathcal{X}(M)$.

An inclusion relation between these special structures is given by the following theorem.

T h e o r e m 3.1. The following inclusions hold:

$$fK \begin{array}{l} \subseteq fAK \\ \qquad \subseteq fQK \\ \subseteq fNK \end{array} \quad \text{and} \quad fK \subseteq fH$$

P r o o f . $fK \subseteq fH$: proof follows from (1.9) and (1.10).

$fK \subseteq fAK$: proof follows from (1.12) and the definitions of fAK.

$fAK \subseteq fQK$: If a manifold is fAK then $dS(fX,fY,fZ) = 0$ consequently the left hand side of (2.4) vanishes and therefore the manifold is fQK.

$fK \subseteq fNK$: Proof follows from the definitions.

$fNK \subseteq fQK$: If the manifold is fNK then

$$\nabla_{fX}(F)(fY,fZ) = \epsilon \nabla_{fY}(F)(fX,fZ).$$

Consequently from (2.1)

$$dS(fX,fY,fZ) = 3\,\nabla_{fX}(F)(fY,fZ),$$

and thus

$$\begin{aligned} & dS(fX,fY,fZ) - dS(fX,f^2Y,f^2Z) + \\ & + dS(fY,f^2Z,f^2X) + dS(fZ,f^2X,f^2Y) = \\ & = 3\,\nabla_{fX}(F)(fY,fZ) - 3\,\nabla_{fX}(F)(f^2Y,f^2Z) + \\ & + 3\,\nabla_{fY}(F)(f^2Z,f^2X) + 3\,\nabla_{fZ}(F)(f^2X,f^2Y) = 0 \end{aligned}$$

in view of (1.6), (2.2) and definition of fNK and thus

$$\begin{aligned} & 2\,\nabla_{fX}(F)(fY,fZ) + (1-\epsilon)\,\nabla_{f^2X}(F)(f^2Y,fZ) = \\ & = (1+\epsilon)\left[\nabla_{f^2Z}(F)(fX,f^2Y) - \nabla_{f^2Y}(F)(fZ,f^2X)\right], \end{aligned}$$

i.e. the manifold is fQK.

Now we shall deal with different cases of such structures.

Case I: Let the structure tensor f satisfy

$$(3.1) \qquad \nabla_{fX}(f)(fY) = f\,\nabla_X(f)(fY).$$

Theorem 3.2. An $f(3,\epsilon)$ structure manifold satisfying (3.1) is always an fH manifold.

Proof. The proof easily follows from definition of fH and (1.9).

Theorem 3.3. An $f(3,\epsilon)$ structure manifold satisfying (3.1) and

$$(3.2) \qquad \underset{Y,Z}{\tau}\, \nabla_Z(f)(f^2Y) = 0$$

is an fQK manifold iff

$$f^2\,\nabla_X(f)(fY) = 0.$$

Proof. Using (2.2) and (1.12) we have

$$2\,\nabla_{fX}(F)(fY,fZ) + (1-\epsilon)\,\nabla_{f^2X}(F)(f^2Y,fZ) -$$
$$-(1+\epsilon)\left[\nabla_{f^2Z}(F)(fX,f^2Y) - \nabla_{f^2Y}(F)(fZ,f^2X)\right] =$$
$$= 2g(f\nabla_X(f)(fY),fZ) + (1-\epsilon)g(f^2\nabla_X(f)(fY),f^2Z) +$$
$$+ (1+\epsilon)\left[g(f^2\nabla_Z(f)(fY),X) + g(f^2\nabla_Y(f)(fZ),X)\right] =$$
$$= (3\epsilon-1)g(f^2\nabla_X(f)(fY,Z) + (1+\epsilon)g(f^2\,\underset{Y,Z}{\tau}\,\nabla_Z(f)(f^2Y),X) =$$
$$= (3\epsilon-1)g(f^2\nabla_X(f)(fY),Z),$$

and consequently the proof follows in view of the definition of fQK.

Case II. In this case the structure tensor f of $f(3,\epsilon)$ manifold satisfies

$$\nabla_{fX}(f)(fY) = \epsilon f \nabla_X(f)(fY). \tag{3.3}$$

Theorem 3.4. An $f(3,\epsilon)$ structure manifold satisfying (3.2) and (3.3) is an fQK manifold if

$$f^2\nabla_X(f)(fY) = 0.$$

In this case

$$N(fX,fY) = (\epsilon-1)\ell\left[\nabla_X(f)(fY) - \nabla_Y(f)(fX)\right].$$

The converse also holds in the particular case when $\epsilon = -1$.

Proof.

$$2\nabla_{fX}(F)(fY,fZ) + (1-\epsilon)\nabla_{f^2X}(F)(f^2Y,fZ) +$$

$$+ (1+\epsilon)\left[\nabla_{f^2Y}(F)(fZ,f^2X) - \nabla_{f^2Z}(F)(fX,f^2Y)\right] =$$

$$= (1+\epsilon)g(f^2\nabla_X(f)(fY),Z) + g(f^2 \underset{Y,Z}{\tau} \nabla_Z(f)(f^2Y),X) = 0,$$

and

$$N(fX,fY) = f^2\nabla_X(f)(fY) - f^2\nabla_Y(f)(fX) -$$

$$- \epsilon f^2\nabla_X(f)(fY) + f^2\nabla_Y(f)(fX) =$$

$$= \ell\left[(\epsilon-1)\nabla_X(f)(fY) - \nabla_Y(f)(fX)\right].$$

Theorem 3.5. An $f(3,\epsilon)$ structure manifold satisfying (3.3) is an fH manifold if

$$\ell\nabla_X(f)(fY) - \ell\nabla_Y(f)(fX) = 0.$$

P r o o f . If

$$\ell\nabla_X(f)(fY) - \ell\nabla_Y(f)(fX) = 0,$$

then using Theorem 3.4 manifold is fH. Conversely if the manifolds is fH then

$$N(fX,fY) = 0.$$

Case III. Lastly we consider the case when the structure tensor f satisfies

$$\nabla_{fX}(f)(fY) = f\nabla_X(f)(Y). \tag{3.4}$$

T h e o r e m 3.6. An $f(3,\epsilon)$ structure manifold satisfying (3.4) and

$$f\nabla_X(f)(Y) = 0 \tag{3.5}$$

is an fQK manifold.

P r o o f . By virtue of (2.4) we have

$$2\nabla_{fX}(F)(fY,fZ) + (1-\epsilon)\nabla_{f^2X}(F)(f^2Y,fZ) +$$

$$+ (1+\epsilon)\left[\nabla_{f^2X}(F)(f^2Y,fZ) - \nabla_{f^2Z}(F)(fX,f^2Y)\right] =$$

$$= 2\epsilon\, g(f\nabla_X(f)(Y),fZ) - (1-\epsilon)g(f^2\nabla_X(f)(Y),fZ) +$$

$$+ (1+\epsilon)\left[g(f\nabla_X(f)(Y),fZ) + \epsilon g(f^2\nabla_Z(f)(X),fY)\right] =$$

$$= (3\epsilon-1)g(f\nabla_X(f)(Y),Z) + (1+\epsilon)g(f\nabla_X(f)(Y),Z) +$$

$$+ g(f\nabla_Z(f)(X),Y) = 0.$$

REFERENCES

[1] N.J. Hicks: Notes on differential geometry. Princeton, New Jersey 1964.

[2] K. Yano, S. Ishihara: Structures defined by f satisfying $f^3 + f = 0$. Proc. of U.S. Japan Seminar in Diff. Geometry, Kyoto, Japan (1965) 153-166.

[3] K. Yano: Differential geometry on complex and almost complex spaces. New York 1965.

[4] K.D. Singh, Nilima Srivastava: On some f structure manifolds with torsion. Thesis.Dep. of Math. Lucknow University (1976).

[5] K. Yano: On a structure defined by tensor field f of type (1,1) satisfying $f^3 + f = 0$. Tensor 14 (1963) 99-109.

[6] K.D. Singh, R.K. Vohra: $(\epsilon f_{(1)}, \epsilon f_{(2)})$ manifolds. Thesis submitted to the Dep. of Math. Lucknow University (1974).

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