DEMONSTRATIO MATHEMATICA
Vol. X No 3-4 1977

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ON THE EQUALITY OF CLASSES OF CONTINUOUS FUNCTIONS WITH DIFFERENT TOPOLOGIES IN THE SET OF REAL NUMBERS

1. Introduction

E. Kocela in [1] has studied the classes $C(X,T)$ of all real continuous functions with some set X and different topologies T.

In particular, if X is the set of real numbers and T is a topology stronger than the natural topology, he obtained the following results:

Theorem 1. Let T be a topology stronger than the natural topology. Then the condition

$$(W) \qquad \left\{\lim_{n\to\infty} x_n = x_0 \wedge \left[\bigwedge_{n\in N} x_n \in (E_n)'_T\right]\right\} \Rightarrow x_0 \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T,$$

where $(E)'_T$ - the set of T-accumulation points of a set E is the sufficient condition for the equality $C(R,T) = C(R)$[1)]

Theorem 2. Let $\langle a,b\rangle \subset R$ and T be a topology stronger than the natural topology. If $C(\langle a,b\rangle,T) = C(\langle a,b\rangle)$ then the following two conditions must be satisfied:

1) The necessary and sufficient condition of $C(R,T) = C(R)$ is that every T-continuous function be bounded [1].

(1) Every interval $\langle c,d\rangle \subset \langle a,b\rangle$ is T-connected and

(2) Every T-neighbourhood of an arbitrary point $x_0 \in \langle a,b\rangle$ is a dense set in some interval $(x_0-\delta, x_0+\delta)$.

The aim of this paper is the analysis of the conditions (W), (1), (2).

2. Analysis of the condition (W)

First, we shall show that (W) is not the necessary condition for the equality $C(R,T) = C(R)$ to be satisfied. Construction of the appropriate example will be preceded by the following theorem, which is also a partial analysis of the condition (2).

T h e o r e m 3. Let T be a topology stronger than the natural topology, such that each T-neighbourhood of any point x_0 is a dense set on some interval $(x_0-\delta, x_0+\delta)$. Then, if f is a T-continuous function with a dense set of continuum points, it is continuous in the normal sense.

P r o o f . Suppose that f is not continuous at some point x_0, i.e. x_0 is an accumulation point of the set

$$\{x: |f(x) - f(x_0)| > \varepsilon_0\},$$

where ε_0 is some fixed positive number.

In view of T-continuity of the function there exists a T-neighbourhood U_0 of the point x_0 such that

$$|f(x) - f(x_0)| < \frac{\varepsilon_0}{3} \quad \text{for each} \quad x \in U_0.$$

Furthermore, there exists a number $\delta > 0$ such that the set U_0 is dense on the interval $(x_0-\delta, x_0+\delta)$, which means that the set

$$\left\{x: |f(x) - f(x_0)| < \frac{\varepsilon_0}{3}\right\}$$

is dense on this interval.

Since x_c is an accumulation point of the set

$$\{x: |f(x) - f(x_0)| \geqslant \varepsilon_0\}$$

there exists $x_1 \in (x_0-\delta, x_0+\delta)$ such that $|f(x_1) - f(x_0)| \geqslant \varepsilon_0$. x_1 is a T-continuity point of the function f, so there exists a T-neighbourhood U_1 of the point x_1 such that

$$|f(x) - f(x_1)| < \frac{\varepsilon_0}{3} \quad \text{for each} \quad x \in U_1.$$

There is also a number $\delta_1 > 0$ such that the set U_1 is dense on the interval $(x_1-\delta_1, x_1+\delta_1)$, so the set

$$\{x: |f(x) - f(x_1)| < \frac{\varepsilon_0}{3}\} \supset U_1$$

is dense on this interval.

Thus both sets

$$\{x: |f(x) - f(x_0)| < \frac{\varepsilon_0}{3}\} \quad \text{and} \quad \{x: |f(x) - f(x_0)| \geqslant \frac{2}{3}\varepsilon_0\}$$

are dense on the interval

$$(A,B) = (x_0- \;, x_0+\delta) \cap (x_1-\delta_1, x_1+\delta_1) \neq \emptyset$$

so the function f cannot have a continuity point on the interval (A,B). The above result is in contradiction with the assumption that a set of continuity points of the function f is dense. This concludes the proof of the theorem.

C o r o l l a r y 1. If T is a topology stronger than the natural topology such that each T-neighbourhood of any point x_0 is a dense set on some interval $(x_0-\delta, x_0+\delta)$, then each Baire first class and T-continuous function is continuous in the normal sense.

Let us now pass on to the example announced earlier.

E x a m p l e 1. Let (R,T) be the space with the following topology: A set U is said to be T-open if for each $x_0 \in U$ the following two conditions are fulfilled:

1^0 x_0 is a density point of the set U,

2^0 $\bigvee_{\delta>0} (x_0-\delta,x_0+\delta)\cap(W+x_0) = U\cap(W+x_0)$,

where $W+x_0 = \{x: x-x_0 \in W\}$, and W is the set of rational numbers.

First, we shall show that $C(R,T) = C(R)$. Let f be any T-continuous function. From the condition 1^0 it follows that f is continuous approximately and as such it is of the first class of Baire. On the other hand it follows from the condition 2^0 that every T-neighbourhood of any point x_0 is a dense set in some interval $(x_0-\delta,x_0+\delta)$. Thus in accordance with Corollary 1, the function f is continuous in the normal sense, so that we have $C(R,T) = C(R)$.

We shall now show that the topology T does not satisfy the condition (W). In order to do this let us consider a sequence of sets

$$E_n = \langle 2^{-n} - 4^{-n}, 2^{-n}\rangle \setminus \bigcup_{w_k\in W} (w_k-\varepsilon_k, w_k+\varepsilon_k),$$

where $\{\varepsilon_k\}$ is the sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \varepsilon_k < 4^{-n}.$$

Let x_n $(n=1,2,\ldots)$ stand for an arbitrarily chosen point of the set $(E_n)'_T$; e.g. any density point of the set E_n. Clearly $x_n \to 0$. To show that T does not meet the condition (W) it is enough to demonstrate that $0 \notin \left(\bigcup_{n=1}^{\infty} E_n\right)'_T$. We shall show this by proving that $R\setminus\bigcup_{n=1}^{\infty} E_n$ is a T-neighbourhood of the point 0, i.e., that $R\setminus\bigcup_{n=1}^{\infty} E_n \in T$, or that for any $x_0 \in R \setminus \bigcup_{n=1}^{\infty} E_n$ the conditions 1^o and 2^o are satisfied ($0 \in R \setminus \bigcup_{n=1}^{\infty} E_n$, what is obvious).

In fact, if $x_0 \in R \setminus \bigcup_{n=1}^{\infty} E_n$ and $x_0 \neq 0$, then x_0 is the internal point of this set in ordinary sense, so that the conditions 1^o and 2^o are satisfied. It remains to prove the case $x_0 = 0$. The fulfillment of the condition 2^o can be obtained from the inclusion $R \setminus \bigcup_{n=1}^{\infty} E_n \supset W = W + 0$.

It remains to be demonstrated that 0 is a density point of the set $R \setminus \bigcup_{n=1}^{\infty} E_n$, or that 0 is a point of dispersion of the set $\bigcup_{n=1}^{\infty} E_n$ (point of right-hand dispersion, considering that $\bigcup_{n=1}^{\infty} E_n \subset (0,+\infty)$.

For $h > 0$ let n_h represent the smallest natural numbers satisfying the inequality

$$2^{-n} - 4^{-n} < h.$$

Thus we have $n_h \to \infty$, where $h \to 0$, and

$$\left|\langle 0,h) \cap \bigcup_{n=1}^{\infty} E_n\right| = \left|\langle 0,h) \cap \bigcup_{n=n_h}^{\infty} E_n\right| \leqslant \sum_{n=n_h}^{\infty} \frac{1}{4^n} = \frac{4}{3} \cdot \frac{1}{4^{n_h}} .$$

Hence

$$\frac{\left|\langle 0,h) \cap \bigcup_{n=1}^{\infty} E_n\right|}{h} \leqslant \frac{\frac{4}{3}\frac{1}{4^{n_h}}}{h} < \frac{\frac{4}{3}\frac{1}{4^{n_h}}}{\frac{1}{2^{n_h}} - \frac{1}{4^{n_h}}} < \frac{4}{3}\,\frac{\frac{1}{4^{n_h}}}{\frac{1}{2^{n_h}} - \frac{1}{2}\frac{1}{2^{n_h}}} = \frac{8}{3}\,\frac{1}{2^{n_h}},$$

and this means that 0 is a dispersion point of the set $\bigcup_{n=1}^{\infty} E_n$.

Modifying the condition (W) somewhat we obtain another condition (W_1) (weaker than (W)), which is necessary and sufficient for the equality $C(R,T) = C(R)$. In the formulation of the condition (W_1) the notion of a totally open set is used, the definition of which is presented below.

Definition 1. A subset E of a topological space X is called a totally open set if it is of the form

$$E = f^{-1}(G),$$

where f is a real function continuous on X, and G is an open subset of R.

Definition 2. We say that a topology T fulfills the condition (W_1), if

$$\left\{\lim_{n\to\infty} x_n = x_0 \wedge \left[\bigwedge_{n\in N} (x_n \in E_n \wedge E_n \text{ - totally } T\text{-open})\right]\right\} \Rightarrow$$

$$\Rightarrow x_0 \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T .$$

Theorem 4. Let T be a topology in R, which is stronger than the natural topology. The condition (W_1) is necessary and sufficient condition for $C(R,T) = C(R)$.

Proof. Suppose that (W_1) is not satisfied, i.e., there is a sequence $\{E_n\}$ of sets and a sequence $\{x_n\}$ of points such that

$$\lim_{n\to\infty} x_n = x_0 \wedge \bigwedge_{n\in N} (x_n \in E_n \wedge E_n \text{ - totally } T\text{-open}),$$

but

$$x_0 \notin \left(\bigcup_{n=1}^{\infty} E_n\right)'_T .$$

This means that there exists a T-neighbourhood E of the point x_0 such that $E \cap \bigcup_{n=1}^{\infty} E_n = \phi$. We can assume that $x_i \neq x_j$, when $i \neq j$. Let I be an interval such that

$$x_0, x_1, x_2, \ldots \in \text{Int } I.$$

We shall show that there is an unbounded function continuous on (I,T). Taking a suitable sequence $\{\delta_n\}$, $\delta_n > 0$ (such that the intervals $(x_n-\delta_n, x_n+\delta_n)$ are disjoint), we are forming the sets

$$U_n = E_n \cap (x_n-\delta_n, x_n+\delta_n) \cap \text{Int } I$$

also totally T-open, but disjoint.

Moreover, we have

$$x_n \in U_n \quad \text{for} \quad n=1,2,\ldots \quad \text{and} \quad x_0 \notin \left(\bigcup_{n=1}^{\infty} U_n\right)'_T.$$

Next, we are forming a sequence of T-continuous functions such that

$$f_n(x) > 0 \quad \text{for} \quad x \in U_n, \quad f_n(x) = 0 \quad \text{for} \quad x \notin U_n, \; n=1,2,\ldots$$

Multiplying, if necessary, the functions of the sequence $\{f_n\}$ by appropriate constants λ_n (e.g. $\lambda_n = \frac{n}{f_n(x_n)}$, where $x_n \in U_n$) we can assume that

$$\sup_{x \in I} f_n(x) \geqslant n.$$

We put

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Considering the selection of the sets U_n, for every x there exists a T-neighbourhood U_x of x in which all, except perhaps two, of the functions f_n vanish. Thus the function f, as the sum of uniformly converging series of T-continuous functions on U_x, is T-continuous on U_x, and in view of the option of x, it is T-continuous on (I,T).

It follows, however, from the condition $\sup_{x\in I} f_n(x) \geqslant n$ that f is not bounded and it contradicts the equality $C(R,T) = C(R)$.

Conversely, let now the condition (W_1) be satisfied. We shall show that each T-continuous function is continuous in the ordinary sense.

Suppose that there exists a T-continuous function f, but f is not continuous in the point x_0. Then there is a sequence $\{x_n\}$ and $\varepsilon_0 > 0$ such that

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad |f(x_n) - f(x_0)| > \varepsilon_0.$$

Put

$$E_n = \left\{x: |f(x) - f(x_n)| < \frac{\varepsilon_0}{3}\right\}.$$

Since f is T-continuous, the sets E_n are totally T-open, then by (W_1)

$$x_0 \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T.$$

Next, there exists a T-neighbourhood E of the point x_0 such that

$$|f(x) - f(x_0)| < \frac{\varepsilon_0}{3} \quad \text{for every} \quad x \in E.$$

Since $x_0 \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T$, we have

$$E \cap \left(\bigcup_{n=1}^{\infty} E_n\right) \neq \phi .$$

So, there is $x \in R$ and $n \in N$ such that

$$\left| f(x) - f(x_0) \right| < \frac{\varepsilon_0}{3},$$

and

$$\left| f(x) - f(x_n) \right| < \frac{\varepsilon_0}{3}.$$

Hence

$$\left| f(x_n) - f(x_0) \right| < \frac{2}{3} \varepsilon_0.$$

The above contradiction with the inequality $\left|f(x_n)-f(x_0)\right| \geqslant \varepsilon_0$ concludes the proof of the theorem.

3. Analysis of the conditions (1) - (2)

The following theorem is due to Zahorski [3].

T h e o r e m 5. If f is a function of the first class of Baire and f has the property of Darboux, then also a function $f+g$ has the property of Darboux for every continuous function g. T.Świątkowski raises the following hypothesis relative to the possibility of the inversion of this theorem.

H y p o t h e s i s . For every function f which is not of the first class of Baire there exists a continuous function g such that $f+g$ has not the property of Darboux.

This hypothesis is connected with the problem of Kocela [1], relative to the sufficiency of the conditions (1) - (2) for the equality $C(R,T) = C(R)$. Namely, in the connection with the Corollary 1, if this hypothesis should be true then the conditions (1) - (2) would imply the equality $C(R,T)=C(R)$.

However, we shall show an example of the function which is not of the first class of Baire such that it will have the property of Darboux after adding of an arbitrary continuous function. This example abolishes the hypothesis of

Świątkowski, suggesting that it is double full that Kocela's problem has negative solution.

Example 2. In virtue of Sierpiński's Theorem on the distribution of a metric space into disjoint sets [2], the set of real numbers can be represented as the continuum sum of disjoint sets, each of which has a common part, of continuum power, with an arbitrary interval.

The set of the pairs $(\langle a,b\rangle ,f)$, where f is a continuous function, is of continuum power. Thus there is a one-one mapping of the set these pairs on the family of the sets forming of the above mentioned the distribution.

Let E_{abf} denote the set associated with pair $(\langle a,b\rangle ,f)$. The sets E_{abf} are, of course, disjoint and $E_{abf} \cap \langle a,b\rangle$ is of continuum power. Hence, there exists a function φ_{abf} on R such that

$$\varphi_{abf}(E_{abf} \cap \langle a,b\rangle) = R.$$

Now let g denote the function on R which for every continuous function f and for pairs of numbers a,b $(a<b)$ satisfies the condition

$$\text{(i)} \qquad g(x) = \varphi_{abf}(x) - f(x) \quad \text{for} \quad x \in E_{abf}.$$

The existence of such a function follows from the fact that the sets E_{abf} are disjoint.

We shall show that the function g has the announced properties. Because the function g assumes every value on every interval (it follows from the condition (i) for $f \equiv 0$), thus g is not of the first class of Baire.

Also, the sum $g+f$ assumes every real value on every interval $\langle a,b\rangle$ for every continuous function f.

In fact, we have

$$(g+f)(\langle a,b\rangle) \supset (g+f)(E_{abf} \cap \langle a,b\rangle) = \varphi_{abf}(E_{abf} \cap \langle a,b\rangle) = R.$$

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Received June 12, 1976.